Basic Algebraic Number Theory

Thomas Poguntke

18.10.2013

Abstract

Algebraic number theory is the study of number fields, i.e. finite extensions $K \mid \mathbb{Q}$, and their rings of integers $\mathbb{Z} \subseteq \mathcal{O}_K \subseteq K$. In general, $\mathcal{O}_K$ is not a unique factorization domain, so the usual primes in $\mathbb{Z}$ are replaced by prime ideals $\mathfrak{p} \subseteq \mathcal{O}_K$, which allow unique factorization of ideals $\mathfrak{a} \subseteq \mathcal{O}_K$.
We will discuss basic definitions and state some of the fundamental results, such as the finiteness of the class group $\mathrm{Cl}(K)$ and Dirichlet's Unit Theorem, which tells us the structure of the group of units $\mathcal{O}_K^\times$.
We will see a simple example of calculating a class group using Minkowski's bound on ideal norms. Another important result explains how the factorization of principal ideals $(p) \subseteq \mathcal{O}_K$ generated by primes is (more or less) given by factorizing the minimal polynomial of $K = \mathbb{Q}(\alpha)$ over $\mathbb{Q}$ modulo p.
Finally, a proof will be given of one of the most astounding results of number theory, the Quadratic Reciprocity Law, by applying the above result on the splitting of primes in quadratic and cyclotomic number fields.