Kleine AG
Degeneration of the Hodge-to-de Rham spectral sequence

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1. Introduction

The aim of this workshop is the algebraic proof of the degeneration of Hodge-de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_{X/K}) \Rightarrow H^{p+q}_{DR}(X/K) = \mathbb{H}^{p+q}(X, \Omega^\bullet_{X/K}) \tag{1}$$

at E_1 for a smooth and proper variety X over a field K of characteristic zero and also for a vast class of varieties over fields of positive characteristic. Deligne and Illusie [1] established this result by "spreading out" X, so that one can study reductions over finite places of K where one can use characterstic p techniques based on the magic formula

$$(a+b)^p = a^p + b^p. \tag{2}$$

Once the analogous result is available over some appropriate finite places, one can lift it back to the generic fibre, which is X, by base change techniques.

The degeneration result is worthwhile studying for many reasons. First of all there is an easy relation between important geometric invariants, namely due to degeneration the Hodge numbers $h^{p,q}(X/K) := \dim H^q(X, \Omega^p_{X/K})$ are related to the Betti numbers $b_n(X/K) := \dim \mathbb{H}^{p+q}(X, \Omega^\bullet_{X/K})$ by the simple formula

$$b_n = \sum_{p+q=n} h^{p,q}.$$

Second, it is a purely algebraic proof of the same statement well known in complex geometry. Moreover this topic deserves study for the generality of arguments involved as well as the use of the special features only available where the above formula (2) makes sense. Last but not least by an idea of Raynaud the degeneration entails the famous Kodaira-Akizuki-Nakano vanishing theorem: for smooth and projective X/K and an ample invertible sheaf L one has

$$H^i(X, L \otimes \Omega^j_{X/K}) = 0 \quad \forall i+j > \dim X$$
$$H^i(X, L^{-1} \otimes \Omega^j_{X/K}) = 0 \quad \forall i+j < \dim X$$

The perquisites of this meeting will be some basic knowledge on algebraic geomerty in the language of schemes, and some homological algebra. It would not be a bad idea to know some derived categories and spectral sequences.

2. The program

1. Talk (45 Min.): Differentials, Deformations and the de Rham complex. ([2, section 1]) The aim of twofold. It will introduce the main object of study, namely the de Rham complex, and the spectral sequence arising from its hypercohomology. The second objective is to give a reminder on the basic of deformation theory of smooth schemes. All references refer to [2].

The definition of the de Rham complex for a morphism $X \to S$ is found in 1.7. Hypercohomology and the associated spectral sequence come up in 4.8. In this context, one should repeat the naive and the natural filtration of complexes.

In case of the de Rham complex, the hypercohomology is denoted by $H^*_{DR}(X/S)$. If equality holds in

$$\sum_{i+j=n} \dim H^j(X, \Omega^i_{X/K}) \geq \dim H^n_{DR}(X/K)$$

we get degeneration of the Hodge to de Rham spectral sequence (1). This will be used later on.

Give a quick overview on the situation over $\mathbb{C}$, as in the introduction of [2].

For the second part of the talk, we repeat the smoothness definition from 2.2 via the local existence of infinitesimal extensions. The propositions 2.11 and 2.12 make use of this to give a global theory of extensions i.e. deformations by establishing a cohomological obstruction theory. Give the outline of the proof in 2.13.

2. Talk (45 Min.): Differential calculus in positive characteristic: Frobenius and Cartier isomorphism. Let $X \to S$ be a morphism of schemes of characteristic p, i.e. $p\mathcal{O}_X = 0 = p\mathcal{O}_S$. First of all we will recall definition and basic properties of Frobenius $F_S : S \to S$, the Frobenius twist $X' = X \times_S (S, F_S)$ and the relative Frobenius $F : X \to X'$. This is [2, 3.1 - 3.4], see also [1, 1.1]. It was nice if we could see much of this, in particular it would be very instructive to do some calculations for $X = \mathbb{A}^n_T$, which are often also part of the proof.

The main aim is to understand the proof of Theorem 3.5 in [2] (cf. also [1, Thm 1.2]): For smooth $X \to S$ there is an isomorphism of $\mathcal{O}_{X'}$-algebras

$$C^{-1} : \bigoplus_i \Omega^i_{X'/S} \to \bigoplus_i \mathcal{H}^i F_* \Omega^\bullet_{X'/S},$$

the Cartier isomorphism. We would like to see the full proof of this, which is found in [2], 3.6 or a bit more in detail in [3], section 7.2.

Finally explain [2], Corollary 3.6: the Frobenius pushforwards of the relative de Rham complex, its cycles, boundaries and cohomology are locally free, coherent $\mathcal{O}_{X'}$-modules.

3. Talk (45 Min.): Decomposition in positive Characteristic I. ([2, 3.7 - 3.8]) We are now ready to state the main result:

Theorem 2.1 ([2] Theorem 5.1)**.** *Let S be a scheme of characterstic p. Assume S has a lifting T over $\mathbb{Z}/p^2\mathbb{Z}$. Let X be a smooth S scheme, and denote by $F\colon X \to X'$ the relative Frobenius.*

Assume X admits a lifting Z over T. If also X' admits a Z' lifting over T, then the complex of $\mathcal{O}_{X'}$-modules $\tau_{<p} F_* \Omega^{\bullet}_{X/S}$ is decomposable in the derived category $D(X')$.

Explain that if $S = \mathrm{Spec}(k)$ where k is a perfect field of positive characterstic, one obtains the lifting T via the Witt vectors of length two (section 3.9). Spend a second, just like it is done there, on reminding us of what the Witt vectors are.

As a direct application of Theorem 5.1 give the proof of the Kodaira-Akizuki-Nakano vanishing theorem (5.8).

Now we turn to the proof of 5.1. The first step is to give a proof under the additionally assumption that we had a *global* Frobenius lifting $\widetilde{G}\colon X' \to Z$. (compare Proposition 5.3). The required material is given in section 3.7. Give a careful explanation of the arguments, and proof 3.8.

To finish, point out that in general lifting the Frobenius is obstructed (end of 3.9), i.e. there is no global lifting. This sets the stage for the next talk.

4. Talk (45 Min.): Decomposition in positive Characteristic II. ([2, section 5]) Here, we finish the proof of 5.3. Since a lifting of Frobenius exists affine locally, and this case was treated in the previous talk, we have to explain how to glue together a quasi-isomorphism in $D(X')$:

$$\bigoplus_{i<p} \Omega^i_{X'/S}[-i] \to F_* \Omega^{\bullet}_{X/S}$$

The main part is Step B in [2]. After proving 5.3, the prove of degeneration of (1) which is Corollary 5.6 provides no further problems.

5. Talk (45 Min.): From Characteristic p to zero. ([2], 6.) This talk will be concerned with lifting the result of the previous talks from characteristic p to zero. So we first consider the passage in the other direction. Given a variety X over a field K of characteristic zero, we write K as an inductive limit over its subalgebras which are of finite type over $\mathbb{Z}$. Then X is obtained from a corresponding scheme over one such algebra B by extension of scalars. The algebra B allows to pass to positive characteristic and apply what we learned so far.

To this end, [2], section 6, begins with some general results on inductive and direct limits (6.1 and 6.2), which should merely be mentioned at the absolutely necessary stages of the proof. The more geometric statements in connection with limit issues (6.3, 6.4 and 6.5) on projectivity, smoothness and closed points deserve the attention of the audience and should be presented with an indication of proof.

Proposition 6.6 on base change questions is the key technical ingredient of the proof and should be explained carefully together with convincing arguments for its validity.

The statement we all have been waiting for is Theorem 6.9. It claims the degeneration of Hodge de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_{X/K}) \Rightarrow H^{p+q}_{DR}(X/K) = \mathbb{H}^{p+q}(X, \Omega^\bullet_{X/K})$$

at E_1 for a proper smooth K-scheme X, where K is a field of characteristic zero. It should be explained with proof as the highlight of this workshop in great detail. If anything of this talk has to be chopped, don't let it be this wonderful collection of arguments.

References

[1] Pierre Deligne and Luc Illusie. Relèvements modulo p^2 et décomposition du complexe de de Rham. *Invent. Math.*, 89(2):247–270, 1987.

[2] Luc Illusie. Frobenius and hodge degeneration. In *Introduction to Hodge theory*, volume 8 of *SMF/AMS Texts and Monographs*, pages x+232. American Mathematical Society, Providence, RI, 2002. Translated from the 1996 French original by James Lewis and Peters.

[3] Nicholas M. Katz. Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. *Inst. Hautes Études Sci. Publ. Math.*, (39):175–232, 1970.