**ARGOS: CYCLES ON SHIMURA VARIETIES VIA GEOMETRIC SATAKE UNIVERSITY OF BONN, WINTERSEMESTER 2017**

The aim of this seminar is to understand the article [LX17]. Let $(G, X)$ be a Shimura datum of Hodge type, and $S_K\colon = S_K(G, X)$ be the associated Shimura variety of some level $K$ defined over its reflex field $E$. Fix a rational prime $p \geq 3$, and let $v$ be a $p$-adic place of $E$. Assume that $G$ is unramified at $p$, and $K$ is hyperspecial at $p$. Then Vasiu and Kisin showed that $S_K$ admits a canonical integral model $\mathscr{S}_K$, which is a quasi-projective and smooth scheme over $\mathcal{O}_{E,v}$. We denote by $\mathrm{Sh}_K\colon = \mathscr{S}_K \otimes_{\mathcal{O}_{E,v}} \mathbb{F}_v$ the special fiber of $\mathscr{S}_K$ over the residue field of $v$. One has a natural stratification by Newton polygons on $\mathrm{Sh}_K$, and there is a unique closed stratum called the basic locus $\mathrm{Sh}_K^b \subseteq \mathrm{Sh}_K$. We are interested in the geometric structure of $\mathrm{Sh}_K^b$ and its contribution of to the $\ell$-adic étale cohomology $H^i_c(\mathrm{Sh}_{K,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$ for some prime $\ell \neq p$.

On the other hand, let $\hat{G}$ be the Langlands dual group of $G$, equipped with an action by $\mathrm{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p)$ for some integer $m \geq 1$. The Hodge cocharacter $\mu : \mathbb{G}_{m,E} \to G_E$ gives rise to an irreducible algebraic representation $V_{\mu^*}$ of $\hat{G}$ dominant of weight $\mu^* = -w_0(\mu) \in X^\bullet(\hat{T})$ for some maximal torus and Borel subgroups $\hat{T} \subseteq \hat{B} \subseteq \hat{G}$. Then the authors of [LX17] define a certain subset $\Lambda^{\mathrm{Tate}_p} \subseteq X^\bullet(\hat{T})$ depending on the natural action of $\mathrm{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p)$ on $X^\bullet(\hat{T})$, and put

$$V_{\mu^*}^{\mathrm{Tate}_p}\colon = \bigoplus_{\lambda \in \Lambda^{\mathrm{Tate}_p}} V(\lambda),$$

where $V(\lambda)$ is the weight space of $V_{\mu_*}$ on which $\hat{T}$ acts via $\lambda$.

The main result of [LX17] says that, under the assumption $V_{\mu_*}^{\mathrm{Tate}_p} \neq 0$, then

- $\mathrm{Sh}_K$ is even dimensional, say of dimension $d$, and the basic locus $\mathrm{Sh}_K^b$ is purely of half dimension $d/2$;
- one has an explicit description of (the perfection of) $\mathrm{Sh}_K^b$ in terms of affine Deligne-Lusztig varieties of $G$;
- if $\pi_f$ is an irreducible module of the Hecke algebra $\overline{\mathbb{Q}}_\ell[K\backslash G(\mathbb{A}_f)/K]$, then the contribution of cycle classes of the irreducible components of $\mathrm{Sh}_K^b \otimes \overline{\mathbb{F}}_p$ to the $\pi_f$-isotypical component of $H^d_c(\mathrm{Sh}_{K,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(d/2))$ is as large as possible provided that the Satake parameter of $\pi_{f,p}$ is general with respect to $V_{\mu^*}$;
- in the case of Kottwitz' simple Shimura varieties, then the cycle classes of the irr educible components of $\mathrm{Sh}_K^b \otimes \overline{\mathbb{F}}_p$ actually generate all Tate classes in the $\pi_f$-isotypic component of $H^d_c(\mathrm{Sh}_{K,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(d/2))$ if the Satake parameter of $\pi_{f,p}$ is strongly general with respect to $V_{\mu^*}$.

Note that $V_{\mu^*}^{\mathrm{Tate}_p}$ is defined purely in terms of representation theory of algebraic groups. However, by (unramified) Langlands conjecture, the étale cohomology $H^i_c(\mathrm{Sh}_{K,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$ is controlled by $V_{\mu^*}$, and the assumption $V_{\mu^*}^{\mathrm{Tate}_p} \neq 0$ is equivalent to saying that $H^d_c(\mathrm{Sh}_{K,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(d/2))$ contain plenty of geometric Tate classes (i.e. the elements invariant under some power of the Frobenius at $v$). The main theorem of [LX17] thus roughly says that *if some Tate classes show up in* $H^d_c(\mathrm{Sh}_{K,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(d/2))$ *for some representational theoretic reasons, then those Tate classes should mostly come from the geometric irreducible components of the basic locus* $\mathrm{Sh}_K^b$.

Our seminar can be roughly divided into three parts. The first four lectures is devoted to the preliminaries on affine Grassmannians and geometric Satake equivalence, as well as their applications to the study of affine Deligne-Lusztig varieties. The main result of this part is explained in Lecture 4, where one gives a description of irreducible components of affine Deligne-Lusztig varieties for unramified $\sigma$-conjugacy classes. This is a purely local result, and will be applied later to the basic locus of Shimura varieties. The second part of the seminar consists of Lectures 5-8, where we explain the main conceptual innovation of [LX17], namely the moduli space of local shtukas and the category of perverse sheaves on such moduli spaces. The key result of this part in given in Lecture 8 which defines the spectral action on the category of perverse sheaves on moduli of locus shtukas. The compatiblity of the spectral action with the usual Hecke action via geometric Satake is the analog of V. Lafforgue's $S = T$ Theorem. The last part of the seminar (Lecture 9-12) is on the global applications of the previous results, and finishes the proof the main result of [LX17] mentioned above.

The contents of each talk are as follows:

**Lecture 1: Affine Grassmannian and geometric Satake.** Give a quick introduction to affine Grassmannians and the geometric Satake equivalence. Let $\mathcal{O}$ be the ring of integers of a non-archimedean local field, and $G$ be a reductive group scheme over $\mathcal{O}$. Define the affine Grassmannian $\mathrm{Gr}_G$ and its Schubert cells, and discuss their basic properties (cf. [Zhu16+] for the equal characteristic case, and [BS15] and [Zhu17] for the mixed characteristic case). State the geometric Satake equivalence, and explain its relationship with the classical Satake isomorphism. (cf. [Zhu16+], [Zhu17] and [LX17, Section 3.4-3.5]).

**Lecture 2: Affine Deligne-Lusztig Varieties: dimension formulae**. Define affine Deligne-Lusztig varieties (ADLV) $X_\mu(b)$ in $\mathrm{Gr}_G$, explain their dimension formulae and its relationship with Rapoport-Zink spaces (cf. [Zhu17, §3]). Assume that $G$ is an unramified reductive group scheme over $\mathcal{O}$. Define unramified elements of $B(G, \mu)$ and discuss some criteria for the basic element of $B(G, \mu)$ to be unramified. (cf. [LX17, §4.1-4.2]).

**Lecture 3: Semi-infinite orbits and Mirković-Vilonen Cycles.** Define Satake cycles and MV-cycles, explain their relationship. Explain the crystal structure on the set of MV-cycles and its relation with Littelmann paths. (cf. [LX17, 3.1.8-§3.3])

**Lecture 4: Applications of MV-cycles to ADLVs.** Apply the construction in previous lecture to ADLVs. Describe the irreducible components of an ADLV $X_\mu(b)$ for an unramified element $b \in B(G, \mu)$. (cf. [LX17, §4.3-4.4], and the key results are Theorems 4.4.5 and 4.4.14.)

**Lecture 5: Local Hecke stack and moduli of local shtukas.** Introduce the local Hecke stack and the moduli of local shtukas, partial Frobenius maps, Hecke correspondences and local shtukas and their compositions. (cf. [LX17, §5.1-5.2]).

**Lecture 6. Moduli of restricted local shtukas.** Introduce the restricted version of moduli stack of local shtukas, partial Frobenius map, and Hecke correspondence. Define the category of perverse sheaves on the moduli of local shtukas. [LX17, §5.3].

**Lecture 7: Cohomological correspondences and perverse sheaves on the moduli of local shtukas.** Give a review on the generalities of cohomological correspondences on perfect algebraic stacks, and basic properties (cf. [LX17, §A.2.1-A.2.17]). Define the category $P^{\mathrm{Corr}}(\mathrm{Sht}_{\overline{k}})$

and compute the endomorphism ring of the basic object (cf. [LX17, §5.4]).

**Lecture 8: Spectral action on perverse sheaves on the moduli of local shtukas.** Explain Theorem 6.0.1 of [LX17] and its proof. (cf. [LX17, §6]. This lecture may need more than one session.)

**Lecture 9: The special fiber of Shimura varieties of Hodge type.** Introduce the local model diagram for the Shimura variety of Hodge type $\mathscr{S}_K$, and relate the special fiber $\mathrm{Sh}_K$ of $\mathscr{S}_K$ to the moduli space of local shtukas. Describe explicitly (the perfection of) the basic locus $\mathrm{Sh}_K^b \subseteq \mathrm{Sh}_K$ in terms of affine Deligne-Lusztig varieties (cf. [LX17, §7.1-7.2]).

**Lecture 10. Correspondence between Shimura varieties and cycle classes of the basic locus.** Use cohomological correspondences of moduli of local shtukas to construct correspondences between (the perfection) of different Shimura varieties in characteristic $p$. Give an explicit description of the cycle classes of the basic locus $\mathrm{Sh}_K^b$, and interpret the intersection matrix in terms of the spectral action. (cf. [LX17, §7.3-7.4]).

**Lecture 11. Generalized Chevalley restriction map and determinant of the intersection matrix.** Justify [LX17, Theorem 1.4.1], and use it to deduce that the $\pi$-isotypical component of hte intersection matrix of the cycle classes of the basic Newton locus in the previous lecture is non-degenerate as long as the Satake parameter of $\pi_p$ are sufficiently generic. (cf. [LX17+])

**Lecture 12. Applications to Kottwitz' simple Shimura varieties.** Apply the results to the case of Kottwitz' simple Shimura varieties, and prove [LX17, Theorem 1.1.4 (3) and 1.5.1, Corollary 1.5.2]. (cf. [LX17, §2.2])

## References


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