ARithmetische Geometrie OberSeminar

Bonn, Wintersemester 2024/25, Time and place: Mondays, 14-16h, Lipschitzsaal

Programmvorschlag: P. Scholze

Berkovich Motives, and Motivic Geometrization of Local Langlands

In [1], a general construction of (semisimple) Langlands parameters was given. However, like all applications of étale cohomology, it a priori depends on a choice of an auxiliary prime $\ell \neq p$. Grothendieck's vision was that there should exist a theory of motives interpolating between étale cohomology for all primes ℓ. The work of Voevodsky and others has produced an approximation to Grothendieck's vision: Most notably, it gives a full 6-functor formalism that specializes to the ℓ-adic étale formalism for any ℓ. However, often some obstacles are present in order to execute all expected arguments at the motivic level and deduce independence of ℓ statements.

However, it turns out that in the context of [1], the motivic formalism works very well, and can be used to prove independence of ℓ. One key point is that local Galois groups are, away from the residual characteristic p, already essentially independent of ℓ (contrary to global Galois groups); this is reflected in a similar assertion relating rigid-analytic motives over local fields to motives over the residue field in an explicit manner. This relies on the theory of rigid-analytic motives initiated by Ayoub [4] and then developed further notably by Ayoub–Gallauer–Vezzani [5] and Binda–Gallauer–Vezzani [6].

It turns out that if one restricts to étale "overconvergent" motives – which we will call Berkovich motives, and which are sufficient for the intended applications – it is possible to develop the theory from scratch, with little effort. The resulting categories of motivic sheaves are in fact better behaved than previously known versions: While not compactly generated, they are rigid dualizable (just like sheaves on a compact Hausdorff space).

The first half of the seminar will be about the paper [2] that develops the theory of (étale) Berkovich motives. The second half of the seminar will be about [3] that applies these ideas in the context of the geometrization of the local Langlands correspondence [1].

Talks

1. Talk: Overview (7.10.2024)

Overview talk and distribution of the talks. Please attend this meeting if you want to give a talk.

2. Talk: Banach rings and Berkovich spectrum (14.10.2024)

Introduce the category of Banach rings and the Berkovich spectrum functor to compact Hausdorff spaces. We will always work with Banach rings admitting a topologically nilpotent unit. Discuss the completed residue fields, the Gelfand transform, and uniform Banach rings. Show that the category of uniform Banach rings has all colimits. Recall the Gelfand-Mazur theorem.

3. Talk: Perfectoid rings and strictly totally disconnected rings (21.10.2024)

Introduce the v-topology on Banach rings, and study the local nature of Banach rings in the v-topology. Over the complex numbers, discuss the resulting relation to compact

Hausdorff spaces, and p-adically to perfectoid rings. In general, define strictly totally disconnected Banach rings and show that they form a base for the v-topology.

4. Talk: Finitely presented v-sheaves (28.10.2024)

Define the derived category of finitely presented v-sheaves of abelian groups, and give some examples. Prove proper base change and construct a functor $f_!$ with right adjoint $f^!$. Give some background on 6-functor formalisms and their construction. Discuss the failure of the projection formula.

5. Talk: Ball-invariant sheaves (4.11.2024)

Introduce the subcategory of ball-invariant sheaves. Show that with torsion coefficients or over archimedean fields, these reduce to usual étale sheaves. Discuss the Tate twist $\mathbb{Z}(1)$ and its relation to $\mathbb{G}_m$.

6. Talk: Higher weight motivic cohomology, algebraic K-theory (11.11.2024)

Give a description of higher weight rational motivic cohomology $\mathbb{Q}(n)$ in terms of a version of algebraic K-theory.

7. Talk: Motivic sheaves and the full 6 functors (18.11.2024)

By inverting the Tate twist, construct the full category of (étale) motivic sheaves, and all 6 functors on them.

8. Talk: Motivic sheaves over a complete algebraically closed field (25.11.2024)

Show that over a complete algebraically closed field C, the category of Berkovich motives is generated by compact dualizable objects, given as the motives of projective smooth varieties over C. Using a spreading out argument, deduce that in general, the category of Berkovich motives is rigid dualizable.

9. Talk: Relation to algebraic motives (2.12.2024)

Compare Berkovich motives over C with algebraic motives over its residue field k. Observe that the latter category could also be (re)defined by evaluating Berkovich motives on a suitable Berkovich stack. Make the category of mixed Tate motives over $\overline{\mathbb{F}}_p$ explicit.

10. Talk: Motivic sheaves on Bun_G (9.12.2024)

Recall the stack Bun_G of G-bundles on the Fargues–Fontaine curve, its Harder–Narasimhan stratification, and the strata. Describe the category of motivic sheaves on Bun_G (relatively over étale motives over $\overline{\mathbb{F}}_p$).

11. Talk: Motivic sheaves on Div^1 (16.12.2024)

Recall the moduli space of degree 1 divisors Div^1, and describe the category of motivic sheaves on Div^1 (relatively over étale motives of $\overline{\mathbb{F}}_p$). Make explicit the resulting motivic Hopf algebra in Tate motives over $\overline{\mathbb{F}}_p$ and its relation to the classical Weil–Deligne group. Use this to define moduli spaces of L-parameters canonically over $\mathbb{Z}[\frac{1}{p}]$.

12. Talk: Geometric Satake (13.1.2025)

Following the ideas of [7] adapted to the Fargues–Fontaine curve, discuss the upgrade of the geometric Satake equivalence to motivic sheaves.

13. Talk: Synopsis (20.1.2025)

Combine all the previous work to define excursion operators and/or the spectral action motivically and deduce independence of ℓ of Langlands parameters.

References

[1] L. Fargues and P. Scholze, *Geometrization of the local Langlands correspondence*, arXiv:2102.13459 (2021).

[2] P. Scholze, *Berkovich motives*, in preparation.

[3] P. Scholze, *Geometrization of local Langlands, motivically*, in preparation.

[4] J. Ayoub, *Motifs des variétés analytiques rigides*, Mém. Soc. Math. Fr. (N.S.) (2015), no.140-141, vi+386 pp.

[5] J. Ayoub, M. Gallauer and A. Vezzani, *The six-functor formalism for rigid analytic motives*, Forum Math. Sigma 10 (2022), Paper No. e61, 182 pp.

[6] F. Binda, M. Gallauer and A. Vezzani, *Motivic monodromy and p-adic cohomology theories*, arXiv:2306.05099 (2023).

[7] T. Richarz and J. Scholbach, *The motivic Satake equivalence*, Math. Ann. 380 (2021), no.3-4, 1595–1653.