# HOMOTOPY QUOTIENTS OF MAPPING SPACES AND THEIR STABLE SPLITTING

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## 1. Introduction

**1.1.** THIS paper gives stable splittings for homotopy orbit spaces of certain mapping spaces (and spaces of sections) with Lie group actions. For example we show (in 3.5 below).

THEOREM *Let $G$ be a compact Lie group, which acts on a closed manifold $M$ and on a based, connected space $X$. Suppose $M$ is $G$-parallelizable, $TM = M \times V$ for some $G$-representation $V$. Then there are $G$-spaces $D_n(M; X)$ such that*

$$\Omega^\infty S^\infty(EG \ltimes_G \mathrm{map}\,(M; S^V X)) \simeq \prod_{n \geq 1} \Omega^\infty S^\infty(EG \ltimes_G D_n(M; X))$$

Here $G$ acts on map $(M; S^V X)$ via conjugation. In general, if $M$ is not parallelizable, or not closed, then the theorem remains true with the mapping space being replaced by a certain section space or, a space of based maps or sections, cf. 3.4. The spaces $D_n(M; X)$ are known as the $n$-adic construction on $M$ with labels in $X$. For example, $D_1(M; X) = M \ltimes X = M_+ \wedge X$. For $n \geq 2$ and $X$ a sphere, $D_n(M; X)$ is the Thom space a vector bundle over space of unlabeled configurations of $M$, cf. 2.1.

**1.2.** EXAMPLE. For the natural rotation action of $G = SO(2)$ on $M = S^1$, $V$ is the trival representation $\mathbb{R}$. For a trivial $G$-space $X$ the conjugation action on map $(S^1; S^V X)$ is the rotation action $g \cdot \lambda = \lambda \circ g^{-1}$ on the free loop space $\Lambda SX$. In 4.1 we will construct an $SO(2)$ homotopy equivalence $D_n(S^1; X) \simeq S^1_+ \wedge_{\mathbb{Z}_n} X^{(n)}$. Thus 1.1 specializes to the theorem of Carlsson and Cohen, cf. [**5**],

$$\Omega^\infty S^\infty(ESO(2) \ltimes_{SO(2)} \wedge SX) \simeq \Pi \Omega^\infty S^\infty(E\mathbb{Z}_n \ltimes_{\mathbb{Z}_n} X^{(n)}).$$

**1.3.** EXAMPLE. We can vary 1.3 by regarding $M = S^1$ as an $O(2)$-manifold. Then $V$ is the non-trivial representation $\mathbb{R}^-$ induced via the determinant $O(2) \to \mathbb{Z}_2$. The $O(2)$-conjugation on $\Lambda S^V X$ extends the former $SO(2)$-conjugation. Let $\nabla_n$ be the dihedral group of order $2n$, acting on $X^{(n)}$ as a subgroup of the symmetric group $\Sigma_n$. Specifically, $\nabla_n$ is generated by the $n$-cycle and by the permutation which maps $i$ to $n - i + 1$. One proves (cf. 4.1) that $D_n(S^1; X)$ is $O(2)$-homotopy equiv-

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alent to $S^1 \ltimes_{\nabla_n} X^{(n)}$, and obtains

$$\Omega^\infty S^\infty(EO(2) \ltimes_{O(2)} \Lambda S^V X) \simeq \prod_{n \geq 1} \Omega^\infty S^\infty(E\nabla_n \ltimes_{\nabla_n} X^{(n)}).$$

This was found previously by J. Lodder **[14]**. These two examples have connections to cyclic and dihedral homology **[9]**, **[12]**, **[13]**, and to pseudo isotopy theory **[4]**, **[6]**.

**1.4.** Our proof follows the lines of **[2]** and are based upon configuration space models for mapping and section spaces, and their splittings, **[1]**, **[15]**. These models are recalled in Section 2 (as $G$-spaces). Section 3 contains our splitting theorem and Section 4 gives examples.

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## 2. The combinatorial models

**2.1.** This section recalls the combinatorial models for various spaces of sections. The reader is referred to **[1]**, **[8]**, **[15]** for further details.

Let $N$ be a smooth compact manifold, $N_0$ a compact submanifold, and let $X$ be a $CW$-complex, with basepoint $x_0$. The configuration space of particles in $N$ modulo $N_0$ with labels in $X$ is defined as

$$C(N, N_0; X) = \left( \coprod_{k \geq 1}^{\infty} \tilde{C}^k(N) \times_{\Sigma_k} X^k \right) \Big/ \approx.$$

Here $\tilde{C}^k(N)$ is the space of $k$-tuples of distinct points in $N$, and the relation $\approx$ identifies $(z_1, \ldots, z_k; x_1, \ldots, x_k)$ with $(z_1, \ldots, z_{k-1}; x_1, \ldots, x_{k-1})$ if $z_k \in N_0$ or $x_k = x_0$.

We write elements of $C(N, N_0; X)$ as formal sums $\xi = \sum z_i x_i$. The obvious basepoint 0 is represented by any such sum with $z_i \in N_0$ or $x_i = x_0$, all $i$. The length of configurations induces a filtration

$$C_n(N, N_0; X) = \left( \coprod_{k=1}^{n} \tilde{C}^k(N) \times_{\Sigma_k} X^k \right) \Big/ \approx$$

of $C(N, N_0; X)$ with quotients $D_n(N, N_0; X) = C_n(N, N_0; X)/C_{n-1}(N, N_0; X)$.

**2.2.** Suppose that a compact Lie group $G$ acts smoothly on $N$ leaving $N_0$ invariant, and that it acts on $X$ leaving $x_0$ fixed. There is an induced action on $C(N, N_0; X)$,

$$g \cdot \xi = g \cdot \left(\sum z_i x_i\right) = \sum (g z_i)(g x_i).$$

The filtration is preserved so $C_n(N, N_0; X)$, $C_n(N, N_0; X) - C_{n-1}(N, N_0; X)$ and $D_n(N, N_0; X)$ are $G$-spaces; and $C^n(N) = \tilde{C}^n(N)/\Sigma_n$ are $G$-manifolds.

**2.3.** Next we recall the connection between configuration spaces and spaces of sections, **[1]**, **[15]**, **[16]**. Let $W$ be a smooth manifold without boundary, containing $N$ as a codimension zero submanifold (e.g. $W = N \cup \partial N \times [0, 1)$). The fibrewise one point compactification $\dot{T}W$ of the tangent bundle and the trivial bundle $W \times X \to W$ both have preferred sections, namely the point at infinity and the basepoint $x_0$, in each fibre. Their fibrewise smash product $\dot{\tau}_X = \dot{T}W \wedge_W (W \times X)$ has a preferred section 0. We denote by $\Gamma(W - N_0, W - N; X)$ the space of sections of $\dot{\tau}_X$ which are defined outside of $N_0$ and agree with 0 outside of $N$. There is a natural map of based spaces,

$$\gamma\colon\ C(N, N_0; X) \to \Gamma(W - N_0, W - N; X).$$

For $\xi \in C(N, N_0; X)$ and $z \in W - N_0$, $\gamma(\xi)(z)$ is the image of $\xi$ under the composition

$$C(N, N_0; X) \to C(N, N_0 \cup (N - \operatorname{int} D(z)); X) \cong C(D(z), \partial D(z); X)$$
$$\cong C(D_zW, \partial D_zW; X) \xrightarrow{R} (D_zW/\partial D_zW) \wedge X \to \dot{T}W \wedge_W (W \times X).$$

The first map is the natural quotient, the second map is excision and the third is induced by the 'exponential' map from the unit disc $D_zW$ in $T_zW$ onto the neighbourhood $D(z)$ of $z$ in $W - N_0$. The map $R$ is a deformation retraction of the inclusion $(D_zW/\partial D_zW) \wedge X \to C(D_zW, \partial D_zW; X)$. The last map is the fibre inclusion. See **[15]**, **[16]** for details, and for a proof of

PROPOSITION. *Suppose $(N, N_0)$ or $X$ is connected. Then $\gamma$ is a homotopy equivalence.*

**2.4.** If $G$ acts on $X$ and on $W$ leaving $N$ and $N_0$ invariant we have an induced action on $\Gamma(W; X)$: for $s \in \Gamma(W; X)$ define

$$g \cdot s = (\dot{T}g \wedge_W (g \times g)) \circ s \circ g^{-1}.$$

The section 0 is fixed, and $\Gamma(W - N_0, W - N; X)$ inherits the action. It is immediate from the definitions that $\gamma$ is equivariant. It is not, however an equivariant homotopy equivalence. For example if $N = S^1$, $N_0 = \varnothing$ and $G = SO(2)$ acting trivially on $X$ then $C(S^1; X)^G$ is just the basepoint, but $\Gamma(S^1; X)^G = (\Lambda SX)^G = X$, c.f. 1.2.

**2.5.** If $W$ is parallelizable, $TW \cong W \times \mathbb{R}^m$, then $\dot{\tau}_X = W \times (S^m \wedge X)$ and

$$\Gamma(W - N_0, W - N; X) = \operatorname{map}(W - N_0, W - N; S^mX)$$

with the induced action of $G$. Even better, suppose that $W$ is $G$-parallelizable, $TW \cong W \times V$ for some $G$-representation $V$. Then the $G$-action on map $(W - N_0, W - N; S^V X)$ is via conjugation, $g \cdot s = (\dot{g} \wedge g) \circ s \circ g^{-1}$. As a special case, let $N$ be the disc $DV$ in a $G$-representation $V$, and $N_0 = \varnothing$. Then $\gamma$: $C(DV; X) \to \Omega^V S^V X$ is the well-known approximation map. It is $G$-equivariant, and for connected $X$ a non-equivariant homotopy equivalence. Also $C(DV; X) \simeq C(V; X)$ as $G$-spaces. For $G$-equivariant approximation results we refer to **[7]**, **[11]**, **[16]** and **[17]**.

**2.6.** Finally, recall from **[1]** or **[8]** the power set maps

$$\sigma_k\colon\ C(N, N_0; X) \to C(C^k(N); D_k(N, N_0; X)).$$

Given $\xi = \sum_{i \in I} z_i x_i$ in $C(N, N_0; X)$ consider all subsets $\alpha \subset I$ of cardinality $k$. Set $z_\alpha = \sum_{i \in \alpha} z_i \in C^k(N)$ and let $\bar{\xi}_\alpha$ be the image of $\xi_\alpha = \sum_{i \in \alpha} z_i x_i$ under the quotient map $C_k(N, N_0; X) \to D_k(N, N_0; X)$. Define $\sigma_k(\xi) = \sum_\alpha z_\alpha \bar{\xi}_\alpha$. The $G$-action on $C^k(N)$ and on $D_k(N, N_0; X)$ induce a $G$-action on $C(C^k(N); D_k(N, N_0; X))$, and by 2.2, $\sigma_k$ is $G$-equivariant.

## 3. Homotopy orbit spaces

**3.1.** We shall consider ex-spaces over a fixed space $B$, that is maps $\pi$: $A \to B$ together with a section $\iota$: $B \to A$. For a based space $S$ the product $B \times S$ with the obvious section is an ex-space over $B$. For a vector bundle $\eta$ we regard its fibrewise one-point compactification $\dot{\eta}$ with the section at infinity as an ex-space. We can form the fibrewise smash product $\pi_1 \wedge_B \pi_2$ of ex-spaces and we can form the Thom space $\mathrm{Th}(A) = A/\iota(B)$.

Note that $(\eta_1 \oplus \eta_2)^{\cdot} = \dot{\eta}_1 \wedge_B \dot{\eta}_2$ for vector bundles $\eta_1$, $\eta_2$, and that $\mathrm{Th}((B \times S) \wedge_B A) = S \wedge \mathrm{Th}(A)$ for a based space $S$ and an ex-space $A$. In particular $\mathrm{Th}(B \times S) = B_+ \wedge S = B \ltimes S$.

**3.2.** Let $G$ be a compact Lie group, $C$ and $D$ based $G$-spaces, $V$ a $G$-representation and $p$: $S^V C \to S^V D$ a based $G$-map. Consider a principal $G$-bundle $E \to B$ such that the vector bundle $E \times_G V \to B$ has an inverse $\eta$, $(E \times_G V) \oplus \eta \cong B \times \mathbb{R}^n$ for some $n$. There is an induced map of ex-spaces

$$(id \times_G p) \wedge_B id\colon\ (E \times_G S^v C) \wedge_B \dot{\eta} \to (E \times_G S^V D) \wedge_B \dot{\eta}$$

Let $q$ be the induced map of Thom spaces $q$: $S^n(E \ltimes_G C) \to S^n(E \ltimes_G D)$.

**3.3.** Let $N$, $N_0$ and $X$ be as in 2.2. Fix $k$, and set $C = C(N, N_0; X)$ and $D_k = D_k(N, N_0; X)$. Choose a $G$-representation $V = V_k$ which contains

$C^k(N)$, and let $p = p_k$ be the adjoint of the composition

$$C \xrightarrow{\sigma_k} C(C^k(N); D_k) \subseteq C(V_k; D_k) \xrightarrow{\gamma_k} \Omega^{V_k} S^{V_k} D_k$$

with $\sigma_k$ from 2.6 and $\gamma_k$ from 2.5. Let $EG \to BG$ be the universal principal $G$-bundle, and let $B_rG \subset BG$ be the usual finite $CW$-complexes which filter $BG$. If $E_rG \to B_rG$ denotes the restriction of the universal $G$-bundle, then the vector bundle $E_rG \times_G V_k \to B_rG$ has an inverse $\eta_{k,r}$, say

$$(E_rG \times_G V_k) \oplus \eta_{k,r} \cong B_rG \times \mathbb{R}^{n(k,r)}$$

for some $n(k, r)$. Moreover, we may choose the $\eta_{k,r}$ compatible in the sense that the restriction of $\eta_{k,r}$ to $B_{r-1}G$ is the direct sum of $\eta_{k,r-1}$ and the trivial bundle of dimension $n(k, r) - n(k, r-1)$. By 3.2 we have maps

$$q_{k,r}\colon\ S^{n(k,r)}(E_rG \ltimes_G C) \to S^{n(k,r)}(E_rG \ltimes_G D_k)$$

and commutative diagrams

$$\begin{array}{ccc}
\Omega^{n(k,r)}S^{n(k,r)}(E_rG \ltimes_G C) & \xrightarrow{\Omega^{n(k,r)}q_{k,r}} & \Omega^{n(k,r)}S^{n(k,r)}(E_rG \ltimes_G D_k) \\
\uparrow & & \uparrow \\
\Omega^{n(k,r-1)}S^{n(k,r-1)}(E_{r-1}G \ltimes_G C) & \xrightarrow{\Omega^{n(k,r-1)}q_{k,r-1}} & \Omega^{n(k,r-1)}S^{n(k,r-1)}(E_{r-1}G \ltimes_G D_k)
\end{array}$$

The vertical maps are induced by the inclusions $E_{r-1}G \subset E_rG$ and by the inclusion $\eta_{k,r-1} \subset \eta_{k,r}$. Passing to the limit over $r$ gives a map

$$q_k\colon\ \Omega^\infty S^\infty(EG \ltimes_G C) \to \Omega^\infty S^\infty(EG \ltimes_G D_k).$$

**3.4.** The maps $q_k$ will serve as the components of a decomposition of $EG \ltimes_G \Gamma(W - N_0, W - N; X)$, where $N$, $N_0$, $W$ and $X$ are as in 2.2 and 2.3.

THEOREM. *There is a homotopy equivalence*

$$\Omega^\infty S^\infty(EG \ltimes_G \Gamma(W - N_0, W - N; X)) \simeq \prod_{n=1}^{\infty} \Omega^\infty S^\infty(EG \ltimes_G D_n(N, N_0; X)).$$

*Proof.* From the Proposition 2.3 and 2.4 we have the homotopy equivalence

$$\mathrm{id} \ltimes_G \gamma\colon\ EG \ltimes_G C(N, N_0; X) \to EG \ltimes_G \Gamma(W - N_0, W - N; X).$$

We can replace $\Gamma(W - N_0, W - N; X)$ by $C = C(N, N_0; X)$ for our assertion. Write $D_k = D(N, N_0; X)$ and $C_n = C_n(N, N_0; X)$, and get

$$q = \prod_{k=1}^{\infty} q_k\colon\ \Omega^\infty S^\infty(EG \ltimes_G C) \to \prod_{k=1}^{\infty} \Omega^\infty S^\infty(EG \ltimes_G D_k).$$

The restrictions of $q$ give commutative diagrams

$$\begin{array}{ccc}
\Omega^\infty S^\infty(EG \ltimes_G C_{n-1}) & \longrightarrow & \prod_{k=1}^{n-1} \Omega^\infty S^\infty(EG \ltimes_G D_k) \\
\downarrow & & \downarrow \\
\Omega^\infty S^\infty(EG \ltimes_G C_n) & \longrightarrow & \prod_{k=1}^{n} \Omega^\infty S^\infty(EG \ltimes_G D_k) \\
\downarrow & & \downarrow \\
\Omega^\infty S^\infty(EG \ltimes_G C_n/C_{n-1}) & \longrightarrow & \Omega^\infty S^\infty(EG \ltimes_G D_n)
\end{array}$$

The lower horizontal map is obviously homotopic to the identity. Starting with $C_1 = D_1$, it follows by induction on $n$ that each restriction of $q$ and hence $q$ itself is a homotopy equivalence.

**3.5.** *Proof of Theorem* 1.2.

Suppose $TM \cong M \times V$ for some $G$-representation $V$. Let $M_0$ be an invariant submanifold, $X$ be a based, connected $G$-space. We put $W = M \cup \partial M \times [0, 1)$ and apply Proposition 2.3 to $N = M - M_0$, $N_0 = \partial M - M_0$. This gives a $G$-map and homotopy equivalence

$$C(M - M_0, \partial M - M_0; X) \xrightarrow{\simeq} \Gamma(W - (\partial M - M_0), W - (M - M_0)).$$

(Actually, we must replace $M_0$ be an open tubular neighbourhood, in order to have $M - M_0$ compact; but this leaves the $G$-homotopy type of both sides unchanged.) By excision and parallelizability

$$\Gamma(W - (\partial M - M_0), W - (M - M_0)) \approx \operatorname{map}(M, M_0; S^V X)$$

as $G$-spaces, where the action on the mapping space is via conjugation, cf. 2.5. By Theorem 3.4

$$\Omega^\infty S^\infty(EG \ltimes_G \operatorname{map}(M, M_0; S^V X)) \simeq \prod_{n=1}^{\infty} \Omega^\infty S^\infty(EG \ltimes_G D_n(M - M_0, \partial M - M_0; X)).$$

The case $M_0 = \varnothing$, $\partial M = \varnothing$ is Theorem 1.2; the spaces $D_n(M; X)$ are the filtration quotients of $C(M; X)$.

## 4. Examples

**4.1.** The space $\tilde{C}^n(S^1) \subset S^1 \times \cdots \times S^1$ has an $O(2)$-$\Sigma_n$ bi-action with $O(2)$ acting (diagonally) from the left and $\Sigma_n$ acting from the right, permuting factors. Let $\Delta^{n-1}$ denote the open $(n-1)$-simplex of points $(d_1, \ldots, d_n) \in \mathbb{R}^n$ with $d_i > 0$ and $d_1 + \cdots + d_n = 2\pi$. On $S^1 \times \Delta^{n-1}$ consider the $\mathbb{Z}_n$-action given by

$$(z; d_1, \ldots, d_n) \cdot T = (e^{id_1} \cdot z; d_2, \ldots, d_n, d_1),$$

where $T$ is a generator of $\mathbb{Z}_n$.

Define a right $\Sigma_n$-homeomorphism

$$h\colon \tilde{C}^n(S^1) \to (S^1 \times \Delta^{n-1}) \ltimes_{\mathbb{Z}_n} \Sigma_n$$

as follows. For $\xi = (z_1, \ldots, z_n)$ choose $\sigma \in \Sigma_n$ such that the sequence $z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(n)}$ defines the standard orientation of $S^1$; then set $h(\xi) = [z_{\sigma^{-1}(1)}; d_1, \ldots, d_n; \sigma]$, where $d_i$ is the distance between $z_{\sigma^{-1}(i)}$ and $z_{\sigma^{-1}(i+1)}$ (with $\sigma^{-1}(n+1) = \sigma^{-1}(1)$).

Obviously, $h$ preserves the $SO(2)$-actions. Let $c \in O(2)$ denote complex conjugation, acting on $(S^1 \ltimes_G \Delta^{n-1}) \ltimes_{\mathbb{Z}_n} \Sigma_n$ by

$$c \cdot [z; d_1, \ldots, d_n; \sigma] = [\bar{z}; d_{\iota(1)}, \ldots, d_{\iota(n)}; T^{-1}\iota\sigma].$$

Here $T$ is the generator of $\mathbb{Z}_n$, $T(i) = i + 1 (\mathrm{mod}\, n)$ and $\iota \in \Sigma_n$ is the permutation with $\iota(j) = n - j + 1$. Since $\iota T \iota^{-1} = T^{-1}$, $\iota$ and $T$ generate the dihedral subgroup $\nabla_n \subset \Sigma_n$ of order $2n$. The space $(S^1 \ltimes \Delta^{n-1}) \ltimes_{\mathbb{Z}_n} \Sigma_n$ is retractible onto the subspace of equidistant configurations, which is homeomorphic to $S^1$. The retraction is given by

$$\Phi_t[z; d_1, \ldots, d_n; \sigma] = [ze^{\varphi_t(d_1, \ldots, d_n)}; d_1^t, \ldots, d_n^t; \sigma]$$

with $d_i^t = d_i + \left(\dfrac{2\pi}{n} - d_i\right)t$ and

$$\varphi_t(d_1, \ldots, d_n) = \frac{t}{n} \sum_{j=1}^{n-1} \left(\frac{2\pi(n-j)}{n} - \sum_{i=j+1}^{n} d_i\right),$$

for $0 \leq t \leq 1$. $\Phi_t$ is $O(2)$-equivariant, and gives an equivalence $\tilde{C}^n(S^1) \simeq S^1 \ltimes_{\mathbb{Z}_n} \Sigma_n$.

Let $X$ be a based, connected $O(2)$-space. Then

$$D_n(S^1; X) = \tilde{C}^n(S^1) \ltimes_{\Sigma_n} X^{(n)} \simeq S^1 \ltimes_{\mathbb{Z}_n} X^{(n)}$$

as $O(2)$-spaces. In particular, if $X$ has trivial $O(2)$ action,

$$EO(2) \ltimes_{O(2)} D_n(S^1; X) \simeq EO(2) \ltimes_{O(2)} (S^1 \ltimes_{\mathbb{Z}_n} X^{(n)}) = E\nabla_n \ltimes_{\nabla_n} X^{(n)}.$$

From 3.4 we get

$$\Omega^\infty S^\infty(EO(2) \ltimes_{O(2)} \Lambda S^{\mathbb{R}^-} X) \simeq \prod_{n=1}^{\infty} \Omega^\infty S^\infty(E\nabla_n \ltimes_{\nabla_n} X^{(n)}).$$

Similarly, restricting to $G = SO(2)$ we get

$$\Omega^\infty S^\infty(ESO(2) \ltimes_{SO(2)} \Lambda SX) \simeq \prod_{n=1}^{\infty} \Omega^\infty S^\infty(EZ_n \ltimes_{\mathbb{Z}_n} X^{(n)}).$$

This proves the special cases of the splitting theorem listed in 1.2 and 1.3.

**4.2.** Let $V=\mathbb{R}^-$ be the non-trivial representation of $\mathbb{Z}/2$. The space $\Omega SX$ of based loops may be regarded as the space of all maps $\lambda\colon V\to S^VX$ such that $\lambda(z)=*$ if $z\notin M=[-1,1]$. It has an obvious involution, and $\Omega SX\simeq C(M;X)$. Note that $\tilde{C}^n(M)=M\times\Delta^{n-1}\times\Sigma_n$ as $\mathbb{Z}_2-\Sigma_n$ spaces, so in 3.4

$$D_n(M;X)=\tilde{C}^n(M)\ltimes_{\Sigma_n}X^{(n)}\simeq\Sigma_n\ltimes_{\Sigma_n}X^{(n)}=X^{(n)}$$

as $\mathbb{Z}_2$-spaces. Here the involution $\iota$ on $X^{(n)}$ is given by $\iota(x_1\wedge\cdots\wedge x_n)=\bar{x}_n\wedge\cdots\wedge\bar{x}_1$, with $\bar{x}$ the involution on $X$.

This gives the equivariant version of the James-Milnor splitting

$$\Omega^\infty S^\infty(E\mathbb{Z}_2\ltimes_{\mathbb{Z}_2}\Omega S^VX)\simeq\prod_{n=1}^{\infty}\Omega^\infty S^\infty(E\mathbb{Z}_2\ltimes_{\mathbb{Z}_2}X^{(n)}).$$

It fits into the corresponding splitting of the space of free loops $\Lambda S^VX$ via the evaluation fibration

$$\Omega S^VX\to\Lambda S^VX\xrightarrow{\varepsilon}S^VX,\qquad \varepsilon(\lambda)=\lambda(-1).$$

**4.3.** In general it is not easy to determine explicitly the spaces $D_n(N,N_0;X)$, or their homotopy orbit spaces, when $n\geqslant 2$. We list a few examples for $D_2$.

Let $G=O(3)$ with its standard action on $N=S^2$ and any action on $X$. Then $D_2(S^2;X)\simeq S^2\ltimes_{\Sigma_2}X^{(2)}$ with $\Sigma_2$ acting antipodally on $S^2$, so

$$EO(3)\ltimes_{O(3)}D_2(S^2;X)\simeq EK\ltimes_K(S^2\ltimes X^{(2)})$$

where $K=O(3)\times\Sigma_2$.

Let $G$ be arbitrary compact Lie, acting on $N=G$ by left translation. Then $\tilde{C}^n(G)=G\times\tilde{C}^{n-1}(G-\{1\})$. For $G=S^3$,

$$EG\ltimes_G D_2(S^3;X)=E\Sigma_2\ltimes_{\Sigma_2}X^{(2)}.$$

Finally, let $G=\Sigma_r$, $N=\{1,\ldots,r\}$ and let $X$ have trivial action. Then $\Gamma(N;X)=X^r$ and $D_n(N;X)=\bigvee X_I$ with $X_I=X_{i_1}\wedge\cdots\wedge X_{i_n}$; $X_j=X$. Here $I$ ranges over the subsets of $N$ of cardinality $n$. There are $\Sigma_r$-equivalences

$$D_1(N;X)=X\vee\cdots\vee X,\qquad D_r(N;X)=X^{(r)}.$$

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