ON THE FAST ROTATION ASYMPTOTICS OF A NON-HOMOGENEOUS MAGNETOHYDRODYNAMIC SYSTEM

Thesis written for the obtention of a Master's diploma in Mathematics

Dimitri Cobb,
under the supervision of
Francesco Fanelli

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Abstract

This text is a Master's thesis written at the end of a year term at the *ENS de Lyon* and the *Université de Lyon 1.* It is the result of a four-month-long internship work done under the supervision of Francesco FANELLI. The aim is to study the fast rotating asymptotics of a magnetohydrodynamic (MHD) model for density-dependent incompressible conducting fluids.

The PDE system we study bears strong resemblance with the rotating inhomogeneous incompressible NAVIER-STOKES whose asymptotic study has been performed by FANELLI and GALLAGHER [8]. In fact, many of the methods therein will be used in this thesis, especially the compensated compactness argument which is instrumental in dealing with the convective term.

Two different regimes are considered: that of a quasi-homogeneous fluid whose density is a perturbation of a constant density state, and a general non-homogeneous fluid. In the first case, the limit system is a homogeneous MHD system, coupled with a transport equation which solves by the perturbation of the constant density. In the second case, the target density is non constant but entirely given by the form of the initial data. The limit system is linear, under-determined and made of equations which involve the magnetic field, a perturbation function of the limit density profile and the vorticity: sufficient regularity is lacking on the velocity field to prove convergence on the momentum equation.

Our present work brings two contributions to the problem. Firstly, we consider more general density-dependent viscosity and resistivity. The non-linearity these terms introduce in the equations require additional strong convergence of the fluid density, which the methods of [8] are insufficient to obtain. Instead, we use an argument based on the well-posedness of the linear transport equation for low regularity velocity fields.

Secondly, in the case of a quasi-homogeneous fluid, we obtain quantitative approximation inequalities by means of relative entropy estimates. This provides a structure theorem for the solutions of the rotating equations: at any given rotation speed, the solutions are the sum of the limit profile and a small remainder. The precision achieved on these estimates is $O(\epsilon)$.

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1 Introduction

1.1 General Physical Remarks

The purpose of this thesis is the mathematical study of a model for conducting fluids in a rotating frame of coordinates. The equations involved are those of two-dimensional incompressible and density-dependent magnetohydrodynamics (MHD for short) to which a term representing the CORIOLIS force has been added[1]:

$$
(1) \qquad \begin{cases} \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \frac{1}{\epsilon}\nabla\pi + \frac{1}{\epsilon}\rho u^\perp = \mathrm{div}(\nu(\rho)\nabla u) + \mathrm{div}(b \otimes b) - \frac{1}{2}\nabla\big(|b|^2\big) \\ \partial_t\rho + \mathrm{div}(\rho u) = 0 \\ \partial_t b + \mathrm{div}(u \otimes b - b \otimes u) = \mathrm{div}(\mu(\rho)\nabla b) \\ \mathrm{div}(b) = 0 \\ \mathrm{div}(u) = 0 \end{cases}
$$

These equations are set in a two dimensional domain Ω, which is either the plane $\mathbb{R}^2$ or the torus $\mathbb{T}^2$. The vector fields u and b are the velocity and the magnetic fields, the scalar fields ρ and π represent the density and the pressure fields and $\nu(\,\cdot\,)$ and $\mu(\,\cdot\,)$ are two functions defined on $\mathbb{R}_+$. The notation $u^\perp$ refers the rotation of angle $\pi/2$ of the vector u. In other words, if $u = (u_1, u_2)$, then $u^\perp = (-u_2, u_1)$.

Magnetohydrodynamic models are used whenever describing a fluid which is subject to the magnetic field it generates through its own motion. Examples of such fluids range from the industrial scale, with plasma confinement in fusion research or some types of electrolytics, to the geophysical or astrophysical scale, with atmospheric plasmas or the solar interior. Their mathematical study thus combines the NAVIER-STOKES and the MAXWELL equations.

We focus on fluids on which the CORIOLIS force $\frac{1}{\epsilon}\rho u^\perp$ has a major influence compared to the kinematics of the said fluid, such as large-scale fluids evolving on a celestial body whose rotation speed is high when compared to the fluid velocity. The importance of this effect is measured by the ROSSBY number of the fluid $Ro = \epsilon$, the condition $\epsilon \ll 1$ defining the regime of large-scale planetary or stellar fluid dynamics.

The goal of this thesis is to study the asymptotics $\epsilon \to 0^+$. We first wish to prove that the solutions of (1) converge (in some way) to a limit $(\rho_{lim}, u_{lim}, b_{lim})$ as the rotation parameter tends to zero $\epsilon \to 0^+$, and describe the limit dynamics by finding an evolution PDE solved by the limit $(\rho_{lim}, u_{lim}, b_{lim})$. Secondly, in the case where the limit density is homogeneous $\rho_{lim} \equiv 1$, we will seek to identify the structure of the solutions for any given $\epsilon > 0$, proving that

$$
(2) \qquad u_\epsilon = u_{lim} + \delta u_\epsilon, \qquad b_\epsilon = b_{lim} + \epsilon\delta b_\epsilon \quad \text{and} \quad \rho_\epsilon = 1 + \epsilon r + \epsilon\delta r_\epsilon,
$$

where r is the limit of the quantities $\frac{\rho_\epsilon - 1}{\epsilon}$ and where functions δu_ϵ, δb_ϵ and δr_ϵ are of order depending on the initial data.

Let us give a few additional details on this precise model.

1. The fluid is assumed to be *non-relativistic*, with negligible velocities $|u| \ll c$ when compared to the speed of light. This justifies the use of an electrostatic approximation in the MAXWELL equations, which are simplified by omitting the time derivative of the electric field. Obviously, this is not a wild assumption since we intend to work on planetary or stellar fluids subject to the body's rotation.

2. This model neglects any effect due to temperature variations, which is a debatable simplification, even in the case of non-conducting fluids. However, the equations, as they are, already provide interesting challenges and are widely used by physicists for practical purposes.

[1] In an appropriate set of units, all constants irrelevant to the model can be set to unit value.

3. The electrical resistivity is described by OHM's law, which links the electrical current j to the electrical field e and the other physical quantities $j = \mu(\rho)\big(e + u \times b\big)$.

4. Both the viscosity $\nu(\rho)$ and the electrical resistivity $\mu(\rho)$ of the fluid are supposed to depend on the density, the precise nature of the functions $\nu(\,\cdot\,)$ and $\mu(\,\cdot\,)$ depend on the exact composition of the fluid. The study of a model which presents non-constant viscosity and resistivity is one of the main features of this thesis. Note that alternate models include anisotropic scaling in the viscosity to take into account the joint effects of turbulence and the asymmetry induced by the rotation, for example $\nu_h(\partial_1^2 u + \partial_2^2 u) + \epsilon\nu_v\partial_3^2 u$.

5. The CORIOLIS force reads $\frac{1}{\epsilon}\rho u^\perp$, the rotation vector being normal to the plane of the fluid. This is physically consistent with a fluid evolving at mid-latitudes, in a region small enough compared to the radius of the planetary or stellar body.

Lastly, let us say a few words concerning the two-dimensional setting. First of all, as we will see farther below, one of the common features of rotating fluids is to be, in a first approximation, planar: the fluid is devoid of vertical motion and the particules move in columns. Therefore, the 2D setting is in itself relevant for geophysical fluids.

Secondly, note that equations (1) *per se* do not describe a conducting fluid confined to a quasi-planar domain. If that were the case, the magnetic field would circulate *around* the current lines, hence being orthogonal to the plane of the fluid, assuming the form $b = b_3(t,x)e^3$ for some scalar function b_3. Our problem, which involves a 2D magnetic field $b = (b_1, b_2)$ is a projection of the full three-dimensional MHD system. Taking this step away from the physical problem brings us closer to the actual form of the physically relevant 3D problem, and we hope it will provide a step towards its understanding.

1.2 General Mathematical Remarks

As already said, the goal of this thesis is to study the asymptotics of solutions of (1) when $\epsilon \to 0^+$. Given a family of weak solutions $(\rho_\epsilon, u_\epsilon, b_\epsilon)_{\epsilon>0}$ of (1), we seek to prove some kind of convergence to solutions (ρ, u, b) of a PDE system which describes the limit dynamics.

To do this, we assume that the solutions $(\rho_\epsilon, u_\epsilon, b_\epsilon)_{\epsilon>0}$ are of finite energy: this means that, at all times $t \geq 0$, the sum of the kinetic energy, the magnetic field's energy and the energy dispersed through viscosity and resistivity is finite

$$\begin{aligned}(3)\quad \int_\Omega \left\{\frac{1}{2}|\rho_\epsilon(t)u_\epsilon(t)|^2 + \frac{1}{2}|b_\epsilon(t)|^2\right\}\mathrm{d}x + \int_0^t\int_\Omega \left\{\nu(\rho_\epsilon(s))|\nabla u_\epsilon(s)|^2 + \mu(\rho_\epsilon(s))|\nabla b_\epsilon(s)|^2\right\}\mathrm{d}t\mathrm{d}s \\ \leq \int_\Omega \left\{\frac{1}{2}|\rho_{0,\epsilon}u_{0,\epsilon}|^2 + \frac{1}{2}|b_{0,\epsilon}|^2\right\}\mathrm{d}x \leq \mathrm{Cte}.\end{aligned}$$

There are three good reasons to consider finite energy weak solutions. Firstly, existence theorems are available for such solutions. Next, physically relevant solutions of (1) should preserve energy in some way. Finally, the energy inequality (3) implies weak compactness of the family of solutions.

Since the above inequality (3) is uniform with respect to the rotation parameter ϵ, it provides uniform bounds on the solutions. Assuming that both the viscosity and the resistivity are non-degenerate, in other words $\nu(\rho) \geq \nu_* > 0$ and $\mu(\rho) \geq \mu_* > 0$ for all $\rho \geq 0$, we see that

$$(4)\qquad \forall \epsilon > 0, \quad \|\rho_\epsilon u_\epsilon\|^2_{L^\infty(L^2)} + \|\nabla u_\epsilon\|^2_{L^2(L^2)} + \|b_\epsilon\|^2_{L^\infty(L^2)} + \|\nabla b_\epsilon\|^2_{L^2(L^2)} \leq \mathrm{Cte}.$$

And likewise, the mass conservation equation $\partial_t\rho_\epsilon + \mathrm{div}(\rho_\epsilon u_\epsilon) = 0$, which is a transport equation by a divergence-free vector field, shows that we also have uniform bounds on the density

$$(5)\qquad \forall p \geq 2, \forall \epsilon > 0, \forall t \geq 0, \qquad \|\rho_\epsilon(t)\|_{L^p} = \|\rho_{\epsilon,t=0}\|_{L^p}.$$

Using the BANACH-ALAOGLU theorem with these uniform bounds, we find that, up to an extraction, we have weak convergence of the solutions $(\rho_\epsilon, u_\epsilon, b_\epsilon) \rightharpoonup (\rho, u, b)$ as $\epsilon > 0$. The difficult part of the study is to prove that the limit (ρ, u, b) solves a PDE which describes the limit dynamics. This is done by taking the limit $\epsilon \to 0^+$ in the weak form of the solutions, after taking care of the singular part of the equations.

In order to get a feel of the methods used in this text, we mention results previously obtained in the past years for different fluid models.

As a historical note, we mention that the mathematical study of rotating fluids has started in the 90s with the works of BABIN, MAHALOV and NIKOLAENKO, who studied incompressible NAVIER-STOKES equations set on the torus.

1.2.1 Homogeneous Fluids

Assume for a while that the fluid's density is constant $\rho \equiv 1$ and that there is no magnetic field $b \equiv 0$, so that the fluid is homogeneous and non-conducting. For a homogeneous (incompressible) fluid, the fast rotating NAVIER-STOKES equations read, in the more physically relevant three-dimensional setting,

$$\begin{cases} \partial_t u_\epsilon + \operatorname{div}(u_\epsilon \otimes u_\epsilon) + \frac{1}{\epsilon}\nabla \pi_\epsilon + \frac{1}{\epsilon} e^3 \times u_\epsilon = \nu \Delta u \\ \operatorname{div}(u_\epsilon) = 0. \end{cases} \tag{6}$$

Here, $e^3 = (0,0,1) \in \mathbb{R}^3$ is the axis of rotation and $e^3 \times u_\epsilon = (-u_{2,\epsilon}, u_{1,\epsilon}, 0)$ is the cross product representing the CORIOLIS force. The fluid is assumed to be confined in an infinite slab $\Omega = \mathbb{R}^2 \times]0,1[$ which represents the ocean or an atmospheric layer, and we endow system (6) with the so-called *complete-slip* boundary conditions

$$\Big(u \cdot n = 0 \qquad \text{and} \qquad [(\nabla u).n] \times n = 0 \Big) \qquad \text{on } \partial\Omega, \tag{7}$$

where $n = \pm e^3$ is the outward normal unit vector on the boundary of Ω. We will see that the choice of boundary conditions is immensely important. For example, the use of the complete-slip conditions eliminate the effect of a boundary layer (see below).

Before discussing the asymptotics of this problem, we make an elementary but important remark concerning the nature of these equations. Although system (6) is made of two equations with two unknowns u and π, the pressure can be expressed as a function of the velocity field by taking the divergence of the first equation and solving the resulting elliptic problem. Therefore problem (6) can be formulated in this way: we seek a divergence-free u such that $\partial_t u + \operatorname{div}(u \otimes u) + \frac{1}{\epsilon} e^3 \times u - \nu \Delta u$ is the gradient of *some* function π, which we can then find *a posteriori*: the momentum equation is written up to a gradient summand. Another way of saying this is that we may be content of testing the momentum equation with *divergence-free* functions.

Note that the function $\nabla \pi$ can be interpreted as a Lagrangian multiplier associated with the constraint $\operatorname{div}(u) = 0$.

Now, considering a family $(u_\epsilon, \pi_\epsilon)_{\epsilon > 0}$ of finite energy weak solutions, as above for (1), we obtain uniform bounds and weak convergence $u_\epsilon \rightharpoonup u$ in the energy space $L^\infty(L^2) \cap L^2_{loc}(H^1)$. Multiplying the weak form of system (6) by ϵ we expect to get, for all divergence-free $\phi \in \mathcal{D}(\mathbb{R}_+ \times \Omega; \mathbb{R}^3)$,

$$\int_0^{+\infty} \int_\Omega (e^3 \times u_\epsilon) \cdot \phi \, \mathrm{d}x \, \mathrm{d}t = O(\epsilon). \tag{8}$$

This means that, in the limit $\epsilon \to 0^+$, the term $e^3 \times u$ is the gradient of some function p. In other words,

$$\begin{cases} -\partial_1 p = -u_2 \\ -\partial_2 p = u_2 \\ -\partial_3 p = 0. \end{cases} \tag{9}$$

The function p is therefore independent of the x_3 variable $p(x) = p(x_1, x_2)$, and so are the components u_1 and u_2 of the velocity field. Next, computing the horizontal divergence of the horizontal velocity field $u_h = (u_1, u_2)$,

$$\mathrm{div}_h(u_h) = \partial_1 u_1 + \partial_2 u_2 = \partial_1(-\partial_2 p) + \partial_2(\partial_1 p) = 0. \tag{10}$$

Finally, the incompressibility condition gives $\mathrm{div}(u) = \partial_3 u_3 = 0$, which, associated with the boundary condition $u \cdot n = 0$ implies that $u_3 \equiv 0$ everywhere. Therefore, in the fast rotation limit $\epsilon \to 0^+$, the problem is completely planar $u = (u_1(x_1, x_2), u_2(x_1, x_2), 0)$, and the particules move in vertical columns. This result is called the Taylor-Proudman theorem and shows that the Coriolis force has a stabilizing effect on the solutions, rendering the fluid devoid of vertical motion. This is a common and important feature linked to fast rotation, and is found in different models (see below for compressible fluids).

Of course, conditions

$$\begin{cases} u(t,x) = (u_h(t, x_1, x_2), 0) \\ \mathrm{div}_h(u_h) = 0, \end{cases} \tag{11}$$

which are equivalent to the existence of a pressure function satisfying (9), do not suffice to describe the limit dynamics of the problem. Doing this requires a detailed study of the linear operator

$$L_\epsilon w = \partial_t w - \nu \Delta w + \frac{1}{\epsilon} e^3 \times w \tag{12}$$

for which we refer to [3], chapter 7.

The limit u can be shown to solve a two-dimensional Navier-Stokes system. This property of the limit is hugely important, as 2D Navier-Stokes is known to be a well-posed problem, whereas well-posedness and stability for the 3D system is notoriously difficult (and unsolved). Here, we further see the stabilizing effects of the rotation, forcing the solutions towards that of a stable PDE.

The rotation also has dispersive effects on the solutions (time decay of certain L^p norms). In the case of a homogeneous fluid, they are made plain by Strichartz estimates (see chapter 5 of [3]) or by the RAGE theorem (Ruelle, Armein, Georgescu and Enss) (see [10]). This is important for handling the convective term $\mathrm{div}(u \otimes u)$ of the equations.

Before going onwards, we mention two points of interest in this model.

The precise nature of the study of operator L_ϵ in (12) depends on the initial data $(u_{\epsilon,t=0})_{\epsilon>0}$. If these are compatible with the limit planar conditions (11) and are themselves planar functions

$$u_{\epsilon,t=0}(x) = u_{\epsilon,t=0}(x_1, x_2) \tag{13}$$

then the study is much simplified. Such initial data are called *well-prepared*. When this is not the case and no *a priori* assumption is made regarding the initial data being compatible with the conditions derived from the highest order terms (with respect to $\frac{1}{\epsilon}$), they are called *ill-prepared*.

Secondly, assume DIRICHLET boundary conditions $u = 0$ on $\partial\Omega$ rather than the complete-slip boundary conditions. Because ofTAYLOR-PROUDMAN theorem, these cannot possibly apply to the planar limit velocity field u, unless $u \equiv 0$. This is solved by the presence of two zones near the boundary $\partial\Omega$ called EKMAN layers.

For a thorough overview of the homogenous problem, we refer to [3]. The book's introduction and chapters 5 to 7 contain detailed explanations and complete proofs for the contents of this paragraph.

1.2.2 Compressible Fluids

We now turn to the case of compressible fluids. We focus on an article [10] of E. FEIREISL, I GALLAGHER and A. NOVOTNÝ and present selected features of the proof therein.

Usually, fast rotation is only one of the two properties which define geophysical fluids. The other one is stratification of the density linked gravity forces. Working on non-homogenous fluids is therefore a preliminary step. We point out that [10] is the first proof of convergence in the case of 3D compressible fluids with ill-prepared initial data.

As previously, we set the problem in the infinite slab $\Omega = \mathbb{R}^2 \times]0, 1[$. The rotating NAVIER-STOKES equations are:

$$\begin{cases} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \frac{1}{\epsilon^2}\nabla[p(\rho)] + \frac{1}{\epsilon}e^3 \times \rho u = \nu\Delta u \\ \partial_t\rho + \operatorname{div}(\rho u) = 0. \end{cases} \tag{14}$$

In these equations, the pressure is given as a function of the density, a state-equation $\pi = p(\rho)$ that the fluid satisfies. On the boundary of Ω, the same complete-slip boundary conditions (7) are satisfied by the velocity field.

The coefficient $\frac{1}{\epsilon^2}$ associated to the pressure term is related to theMACH number $Ma = \epsilon$ of the fluid. The condition $\epsilon \ll 1$ states that the velocity of the fluid is low compared to the speed of acoustic waves in the same fluid. As we will see, a typical consequence of this is that the flow is incompressible in the limit $\epsilon \to 0^+$.

First note that, provided the state equation $\pi = p(\rho)$ is non-degenerate, the constant density state $\rho(t, x) = \bar{\rho} \in \mathbb{R}$ is the only solution of the stationary equations, that is the equations in which the fluid is motionless $u \equiv 0$. As we will explain later, the density of the fluid is expected to be close the homogeneous density $\bar{\rho}$ in the regime of fast rotations: the initial density is assumed to be a perturbation of a constant density state

$$\rho_{0,\epsilon}(x) = \bar{\rho} + \epsilon r_{0,\epsilon}(x), \tag{15}$$

with the sequence $(r_{0,\epsilon})_{\epsilon>0}$ being bounded in $L^2 \cap L^\infty$. This reflects the fact that geophysical fluids are expected to be close to equilibrium. Concerning the velocity field, the initial data $(u_{0,\epsilon})_{\epsilon>0}$ is also taken bounded in $L^2 \cap L^\infty$. The appropriate definition of finite energy solution relies on the energy inequality

$$\int_\Omega \left\{ \frac{1}{2}\rho|u|^2 + \frac{1}{\epsilon^2}E(\rho) \right\} \mathrm{d}x + \nu \int_0^t \int_\Omega |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \le \int_\Omega \left\{ \frac{1}{2}\rho_{0,\epsilon}|u_{0,\epsilon}|^2 + \frac{1}{\epsilon^2}E(\rho_{0,\epsilon}) \right\} \mathrm{d}x \le \mathrm{Cte}, \tag{16}$$

where $E(\rho)$ is a function of the density depending on the form of the state equation $\pi = p(\rho)$. These uniform energy inequalities allow to find uniform bounds on the sequence $(u_\epsilon)_{\epsilon>0}$ as well as on the sequence of perturbations with respect to the constant density state $(r_\epsilon)_{\epsilon>0} = \left(\frac{\rho_\epsilon - 1}{\epsilon}\right)_{\epsilon>0}$. In the same way as before, we have weak convergence of the solutions, $(\rho_\epsilon, u_\epsilon) \rightharpoonup (1, u)$ and $r_\epsilon \rightharpoonup r$.

Note that the factor $\frac{1}{\epsilon^2}$ in front of the pressure term is not surprising, as is makes it a term of order $\frac{1}{\epsilon}$. Writing $\rho_\epsilon = \bar{\rho} + \epsilon r_\epsilon$ we get

$$\frac{1}{\epsilon^2}\nabla[p(\rho)] = \frac{1}{\epsilon^2}\nabla\{p(\bar{\rho}) + \epsilon p'(\bar{\rho})r_\epsilon + o(\epsilon)\} = \frac{1}{\epsilon}p'(\bar{\rho})\nabla r_\epsilon + o\left(\frac{1}{\epsilon}\right), \tag{17}$$

and so the pressure forces are expected to balance perfectly the CORIOLIS force $\frac{1}{\epsilon}e^3 \times \rho_\epsilon u_\epsilon$.

Exactly as in the case of a homogeneous fluid, the study of the singular part of the equations comes first. Multiplying the momentum equation by ϵ, we expect that

$$e^3 \times \rho_\epsilon u_\epsilon + p'(\bar{\rho})\nabla r_\epsilon = O(\epsilon), \tag{18}$$

and using $\rho_\epsilon = \bar{\rho} + \epsilon r_\epsilon$ in the momentum equation,

$$\mathrm{div}(u_\epsilon) = O(\epsilon). \tag{19}$$

This provides two constraints the limit points r and u must satisfy:

$$\begin{cases} \bar{\rho}e^3 \times u + p'(\bar{\rho})\nabla r = 0 \\ \mathrm{div}(u) = 0. \end{cases} \tag{20}$$

Taking advantage of these conditions as for the case of homogeneous fluids, we see they imply that the limit problem is planar: because of the cross product, the third component of the first equation is $\partial_3 r = 0$ so r must depend only on $x_h = (x_1, x_2)$, and therefore u_1 and u_2 also. Next,

$$\partial_3 u_3 = -(\partial_1 u_1 + \partial_2 u_2) = \partial_1\partial_2 r - \partial_2\partial_1 r = 0. \tag{21}$$

The complete-slip boundary conditions imply that $u_3 \equiv 0$. But there is more. Not only is the limit problem completely planar $u(x) = (u_1(x_h), u_2(x_h), 0)$, $r(x) = r(x_h)$, but the limit velocity field can be expressed by means of r, which acts as a stream function: $(u_1, u_2) = \nabla_h^\perp r$. Therefore, the limit dynamics is entirely given by an equation describing the evolution of r.

To find a PDE solved by r, two tricks are in order. First, projecting the momentum equation on the (x_1, x_2) plane and taking the 2D curl rids it of the singular terms, so that all remains is to take the limit $\epsilon \to 0^+$ in the remaining non-singular, but also non-linear, equation. In that respect, the only troublesome element is the quadratic convective term $\mathrm{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon)$ which requires strong convergence $u_\epsilon \longrightarrow u$ of the velocity field to be dealt with.

The second idea is to study the singular part of the equations a bit closer. Isolating the high order terms, we define a linear operator

$$L\begin{pmatrix} r \\ V \end{pmatrix} = \begin{pmatrix} \mathrm{div}(V) \\ e^3 \times V + p'(\bar{\rho})\nabla r \end{pmatrix}, \tag{22}$$

where $V_\epsilon = \rho_\epsilon u_\epsilon$. Using this operator, the whole system can be seen as a first order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} r_\epsilon \\ V_\epsilon \end{pmatrix} + \frac{1}{\epsilon}L\begin{pmatrix} r_\epsilon \\ V_\epsilon \end{pmatrix} = F_\epsilon. \tag{23}$$

The problem is that, since operator L is skew-symmetric and has pure imaginary spectrum, the solutions may have highly oscillating components which might come in the way of proving the strong convergence we seek. The trick is the RAGE theorem, whose object is to use the spectral properties of L to prove that the solutions collapse to the subspace $\ker(L)$ in a strong topology.

It only remains to prove strong convergence of the component QV_ϵ parallel to $\ker(L)$, where $Q : L^2 \longrightarrow \ker(L)$ be the L^2-orthogonal projector on $\ker(L)$. Applying Q to (23) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} Q \begin{pmatrix} r_\epsilon \\ V_\epsilon \end{pmatrix} = QF_\epsilon, \tag{24}$$

which can be used to obtain strong convergence by means of Ascoli's theorem. The final result this analysis is a quasi-geostrophic equation solved by r

$$\partial_t \left(\frac{1}{p'(\bar{\rho})} - \Delta \right) r - \nabla^\perp r \cdot \nabla \Delta r + \frac{\nu}{\bar{\rho}} \Delta^2 r = 0. \tag{25}$$

Once more, we see the stabilizing effects of the Coriolis force, whether it be concerning the constraints the limit flow satisfies (u is an incompressible planar flow derived from a stream function) or the fact that equation (25) is well-posed: rotation forces the solutions towards that of a stable PDE.

We end this paragraph with a few remarks concerning the density. We had asserted that the density of a compressible fluid must be (almost) constant in the regime of fast rotations. To see this, we expand both the density and the velocity in powers of ϵ,

$$\begin{aligned} \rho_\epsilon &= \rho_0 + \epsilon \rho_1 + \cdots \\ u_\epsilon &= u_0 + \epsilon u_1 + \cdots \end{aligned} \tag{26}$$

where u_0 is the limit velocity field and ρ_0 is the limit density, which is no longer assumed to be constant. Expanding the momentum equation in powers of ϵ, we get

$$\frac{1}{\epsilon^2} p'(\rho_0) \nabla \rho_0 + \frac{1}{\epsilon} \left(p'(\rho_0) \nabla \rho_1 + e^3 \times \rho_0 u_0 \right) + \cdots = 0 \tag{27}$$

Equating the term of order $\frac{1}{\epsilon^2}$ with zero, we see that $\nabla \rho_0 = 0$ so that ρ_0 must be a function of time only. Doing the same for the term of order $\frac{1}{\epsilon}$, we see that

$$e^3 \times \rho_0(t) u_0(t, x) = -p'(\rho_0(t)) \nabla \rho_1(t, x). \tag{28}$$

Exactly as in (21) above, this implies that the horizontal velocity field $u_{0,h} = (u_{0,1}, u_{0,2}, 0)$ is divergence-free: $\mathrm{div}_h(u_0) = \partial_1 u_{0,1} + \partial_2 u_{0,2} = 0$.

Now, expanding the mass equation in powers of ϵ and equating the highest order term with zero, we find that

$$\partial_t \rho_0 + \mathrm{div}(\rho_0 u_0) = \partial_t \rho_0 + \rho_0 \,\mathrm{div}(u_0) = \partial_t \rho_0 + \rho_0 \partial_3 u_{0,3} = 0. \tag{29}$$

This shows that $\partial_3 u_{0,3}$ is a function of time only. The boundary condition $u \cdot n = \pm u_3 = 0$ set on $\partial\Omega$ forces $u_{0,3} \equiv 0$, and so $\partial_t \rho_0 = 0$. We have shown that we can indeed expect the target density to be a constant.

1.2.3 Incompressible Inhomogeneous Fluids

The case of plane density-dependent incompressible fluids has been lately (2018) studied by Fanelli and Gallagher [8]. The proofs and ideas therein are especially important as the MHD system which is the focus of this thesis bears important similarities with this one.

As we saw in the previous section, even for compressible fluids, a typical consequence of fast rotation is that the limit flow is incompressible (see also [9] for the same effect in another model). This makes the incompressible approximation perfectly relevant for geophysical fluids, while the density-dependent equations (30) still can account for stratification effects, provided we take into account gravity (which we do not do in this section).

This time, the equations are set in the planar domain Ω which is either the plane $\mathbb{R}^2$ or the torus $\mathbb{T}^2$. The system studied is

$$\begin{cases} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \frac{1}{\epsilon}\rho u^\perp + \frac{1}{\epsilon}\nabla\pi = \nu\Delta u \\ \partial_t\rho + \operatorname{div}(\rho u) = 0. \end{cases} \tag{30}$$

Unlike the compressible NAVIER-STOKES system, the constant density state is no longer the only solution of the stationary equations (problem (30) for which $u \equiv 0$). The initial datum appropriate for the density is a perturbation of a non-constant equilibrium state

$$\rho_{0,\epsilon} = \rho_0(x) + \epsilon r_{0,\epsilon}, \tag{31}$$

with the sequence $(r_{0,\epsilon})_{\epsilon>0}$ being bounded in $L^2 \cap L^\infty$. Because of this, the fluid's density can be very low, having for example vacuum patches in which the velocity field is not defined. Therefore, it is more suitable to make assumptions on the initial momentum $m_{0,\epsilon} = \rho_{0,\epsilon}u_{0,\epsilon}$ rather than the initial velocity: the fluid's momentum is zero in a vacuum zone. The sequences $(m_{0,\epsilon})_{\epsilon>0}$ and $(m_{0,\epsilon}/\sqrt{\rho_{0,\epsilon}})_{\epsilon>0}$ are taken to be bounded in L^2. Finite energy weak solutions are those who satisfy the uniform inequality

$$\int_\Omega \rho|u|^2\,\mathrm{d}x + \nu\int_0^t\int_\Omega |\nabla u|^2\,\mathrm{d}t \leq \int_\Omega \frac{|m_{0,\epsilon}|^2}{\rho_{0,\epsilon}} \leq \mathrm{Cte}. \tag{32}$$

The discussion in [8] breaks down the problem into two regimes: that of a quasi-homogeneous fluid where $\rho_0(x) = 1$, and the general case of a totally non-homogeneous fluid. The case of a quasi-homogeneous fluid is much simpler, as the density can be written $\rho_\epsilon(t) = 1 + \epsilon r_\epsilon(t)$ at all times $t \geq 0$, with r_ϵ solving the linear transport equation

$$\partial_t r_\epsilon \operatorname{div}(r_\epsilon u_\epsilon) = 0. \tag{33}$$

The consequence is that the singularity disappears from the momentum equation:

$$\begin{aligned} \partial_t(\rho_\epsilon u_\epsilon) &+ \operatorname{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon) + \frac{1}{\epsilon}\nabla\pi_\epsilon + \frac{1}{\epsilon}\rho_\epsilon u_\epsilon^\perp - \nu\Delta u_\epsilon \\ &= \partial_t u_\epsilon + \operatorname{div}(u_\epsilon \otimes u_\epsilon) + r_\epsilon u_\epsilon^\perp - \Delta u_\epsilon + \frac{1}{\epsilon}\Big\{\nabla\pi_\epsilon + u_\epsilon^\perp\Big\} + O(\epsilon), \end{aligned} \tag{34}$$

the terms in the brackets are the gradient of some function, thanks to the condition $\operatorname{div}(u_\epsilon) = 0$ and do not appear in the weak form of the equations. Unfortunately, in the totally non-homogeneous case $\rho_0 \neq 1$, it is not at all obvious that $\rho_\epsilon(t,x) = \rho_0(x) + O(\epsilon)$ (see section 5.6 below), and even if such a decomposition were available (which it turns out to be), the singularity is not resolved because $\frac{1}{\epsilon}\rho_0 u_\epsilon^\perp$ is not necessarily a gradient term. To proceed to the asymptotic study, it is necessary to take the curl of the momentum equation, which we describe in section 5.1 below. Note that we need not take well-prepared initial data for this.

The other major argument in [8] is a compensated compactness argument (which we reproduce in the thesis) used to deal with the convective term $\operatorname{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon)$. This type of argument has been introduced by P.-L. LIONS and MASMOUDI for incompressible limits and applied to the context of rotating fluids by GALLAGHER and SAINT-RAYMOND [12]. Broadly speaking and omitting the density for the sake of simplicity, as the momentum equation is written up to a gradient function and

$$\operatorname{div}(u_\epsilon \otimes u_\epsilon) = \frac{1}{2}\nabla\big(|u_\epsilon|^2\big) + \operatorname{curl}(u_\epsilon)u_\epsilon^\perp, \tag{35}$$

it suffice to show strong convergence of the vorticity $\omega_\epsilon := \mathrm{curl}\,(u_\epsilon) \longrightarrow \omega := \mathrm{curl}\,(u)$ to achieve weak convergence of the convective term. This strong convergence is found by using the algebraic structure of the system and taking advantage the vorticity formulation the problem.

In the case of a quasi-homogeneous fluid, the target system is very similar to a homogeneous Navier-Stokes system,

$$\begin{cases} \partial_t u + \mathrm{div}(u \otimes u) + \nabla \pi + r u^\perp = \nu \Delta u \\ \partial_t r + \mathrm{div}(ru) = 0 \\ \mathrm{div}(u) = 0. \end{cases} \tag{36}$$

This system is well-posed (given regular enough initial data) and we, once again, see the stabilizing effects of the fast rotation.

In the case of a fully non-homogeneous fluid, the limit equations are expressed in vorticity formulation, as already remarked above, and they take the following form:

$$\begin{cases} \partial_t \big(\mathrm{curl}\,(\rho_0 u) - \sigma\big) + \mathrm{curl}\,\big(\rho_0 \nabla \Gamma - \mathrm{div}(\nu(\rho_0)\nabla u)\big) = 0 \\ \mathrm{div}(\rho_0 u) = 0 \\ \mathrm{div}(u) = 0, \end{cases} \tag{37}$$

for some function Γ. Similarly to the pressure in the incompressible Navier-Stokes system, the term $\rho_0 \nabla \Gamma$ can be seen as a Lagrangian multiplier associated with the new constraint $\mathrm{div}(\rho_0 u) = 0$.

Unlike (36), this PDE system is not well-posed: the first line is only a scalar equation for two quantities. Give the level of regularity of the velocity field, taking directly the limit of the momentum equation is impossible, whereas the curl of the equation can be shown to converge.

1.2.4 Magnetohydrodynamics in a Rotating Frame of Coordinates

Much of the work carried out for non-conducting fluids can be transported into the broader frame of MHD, uncovering interesting features. We mention for example [5] for work on the stability of boundary layers (for homogeneous fluids) or [16] for more on the stabilizing effect the rotation has on solution's lifespan.

Fast rotations asymptotics has been conducted by Kwon, Lin and Su in [14] in the case of two-dimensional compressible fluids. The MHD system studied in [14] is

$$\begin{cases} \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \frac{1}{\epsilon^2}\nabla[p(\rho)] + \frac{1}{\epsilon}\rho u^\perp \\ \qquad\qquad = \epsilon^\theta \Delta u + (\epsilon^\theta + o(1))\nabla\,\mathrm{div}(u) + \mathrm{div}(b \otimes b) - \frac{1}{2}\nabla\big(|b|^2\big) \\ \partial_t b + \mathrm{div}(u \otimes b - b \otimes u) + \mathrm{div}(u) b = \epsilon^\sigma \Delta b \\ \partial_t \rho + \mathrm{div}(\rho u) = 0, \end{cases} \tag{38}$$

for some exponents $\theta, \sigma > 0$. The limit system describes the dynamics in terms of the magnetic field and the density oscillation function $q = \lim_\epsilon \frac{1}{\epsilon}(\rho_\epsilon - 1)$ and is a quasi-geostrophic system

$$\begin{cases} \partial_t(\Delta - I)q + \nabla^\perp q \cdot \nabla \Delta q = \mathrm{div}^\perp\left((b\cdot\nabla)b\right) \\ \partial_t b + \mathrm{div}(\nabla^\perp q \otimes b - b \otimes \nabla^\perp q) = 0. \end{cases} \tag{39}$$

The method used is based on relative entropy estimates (see below for more on this method) and requires strong convergence of the initial data $(q_{0,\epsilon}, b_{0,\epsilon})$ as well as the *a priori* assumption

$$\sqrt{\rho_\epsilon}\, u_\epsilon \longrightarrow \nabla^\perp q \tag{40}$$

which make these well-prepared initial data.

In this thesis, we prove a convergence theorem for equations (1) both in the case of a quasi-homogeneous and a totally non-homogeneous fluid (we reproduce equations (1) here for the readers comfort)

$$
\begin{cases}
\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \frac{1}{\epsilon}\nabla\pi + \frac{1}{\epsilon}\rho u^\perp = \operatorname{div}(\nu(\rho)\nabla u) + \operatorname{div}(b \otimes b) - \frac{1}{2}\nabla\big(|b|^2\big) \\
\partial_t \rho + \operatorname{div}(\rho u) = 0 \\
\partial_t b + \operatorname{div}(u \otimes b - b \otimes u) = \operatorname{div}(\mu(\rho)\nabla b) \\
\operatorname{div}(b) = \operatorname{div}(u) = 0.
\end{cases}
\tag{41}
$$

All the discussion in section 1.2.3 concerning the work in [8] is relevant for the asymptotic study of (41). In particular, the proof we present is robust enough to allow the possible presence of vacuum regions in the fluid.

System (41) is different from (30) in one important way. While the presence of the magnetic field does not bear on the proof very much, as the magnetic field equations enables us to quickly prove strong convergence of the b_ϵ, the present of density-dependent viscosity and resistivity brings a new challenge. For the viscosity term $\operatorname{div}(\nu(\rho_\epsilon)\nabla u_\epsilon)$ to converge, we need strong convergence of the density $\rho_\epsilon \longrightarrow \rho$, which the methods used in [8] are insufficient to find. We do this by using well-posedness results on the linear transport equation proved by Di Perna and P.-L. Lions.

Uniform bounds on the density give weak convergence of the densities $\rho_\epsilon \rightharpoonup^* \rho$ in the space $L^\infty(L^\infty)$. To prove strong convergence in $L^2_{loc}(L^2_{loc})$, it suffice to prove that

$$
\rho_\epsilon^2 \overset{*}{\rightharpoonup} \rho^2 \quad \text{in } L^\infty(L^\infty)
\tag{42}
$$

and use characteristic functions of compact sets to test that convergence. However, the uniform bounds we have at our disposal only show the existence of a function g such that $\rho_\epsilon^2 \rightharpoonup^* g$ in the space $L^\infty(L^\infty)$. The trick is to prove that both ρ^2 and g are solutions of the same well-posed linear transport problem

$$
\begin{cases}
\partial_t w + \operatorname{div}(wu) = 0 \\
w_{t=0} = \rho_{0,t=0},
\end{cases}
\tag{43}
$$

with $w = \rho^2$ or $w = g$. If that is so, then $g = \rho^2$ almost everywhere.

In the case of a quasi-homogeneous fluid, we will prove the aforementioned quantitative approximation estimates by using relative entropy estimates. The general idea is to take the difference between system (41) and the limit system to find a PDE solved by $\delta r_\epsilon = r_\epsilon - r$, $\delta u_\epsilon = u_\epsilon - u$ and $\delta b_\epsilon = b_\epsilon - b$ (recall that $r_\epsilon = \frac{1}{\epsilon}(\rho_\epsilon - 1)$ is the density oscillation function in the quasi-homogeneous case). The next step is to find energy estimates for this PDE and deduce the approximation inequalities.

Finally, still in the case of a quasi-homogeneous fluid, we use a relative entropy method to compute quantitative approximation estimates

$$
\begin{aligned}
&\|u_\epsilon(t) - u(t)\|_{L^2} + \|b_\epsilon(t) - b(t)\|_{L^2} + \|r_\epsilon(t) - r(t)\|_{L^2} \\
&\qquad \leq C \exp\left(e^t\right) \left\{ \|u_{\epsilon,t=0}\|_{L^2} + \|b_{\epsilon,t=0}\|_{L^2} + \left\| \left(\frac{\rho_\epsilon - 1}{\epsilon} \right)_{t=0} \right\|_{L^2} \right\} + O(\epsilon),
\end{aligned}
\tag{44}
$$

which show that solutions of (41) are, for any given $\epsilon > 0$, the sum of the limit profile and a small remainder.

Both the results mentioned above are, to the best of our knowledge, new in the mathematical literature.

2 Notations and conventions

Before starting, we give a list of notations we use throughout this text.

- All derivatives are (weak) derivatives relative to the space variables, unless specified otherwise.
- For $1 \le p \le +\infty$, we will note $L^p(\Omega) = L^p$ when there is no ambiguity regarding the domain of definition of the functions. Likewise, we omit the dependency on Ω in functional spaces when no mistake can be made.
- If X is a BANACH space of functions, we note $L^p(X) = L^p(\mathbb{R}_+; X)$. For any finite $T > 0$, we note $L^p_T(X) = L^p([0, T]; X)$ and $L^p_T = L^p[0, T]$.
- Any constant will be generically noted C, and, whenever deemed useful, we will specify the dependencies by writing $C = C(a_1, a_2, a_3, ...)$.
- In all the text, $M_p(t) \in L^p(t \ge 0)$ will be a generic globally L^p function and $N_p(t) \in L^p_{loc}(t \ge 0)$ a generic locally L^p function.
- We note $\mathcal{D}(U)$ the space of compactly supported C^∞ functions on the open subset $U \subset \mathbb{R}^d$.
- If $(f_\epsilon)_{\epsilon>0}$ is a sequence of functions which is bounded in the normed space X, we note $(f_\epsilon)_{\epsilon>0} \subset X$.

3 Main assumptions and Results

The viscosity $\nu(\rho)$ and the resistivity $\mu(\rho)$ are assumed to be a continuous non-degenerate function of the density: $\nu, \mu \in C^0(\mathbb{R}_+)$ and $\nu(\rho) \ge \nu_* > 0$ as well as $\mu(\rho) \ge \mu_* > 0$.

3.1 Initial Data

We supplement system (1) with *ill-prepared* initial data: for the density, we take

$$\rho_{0,\epsilon} = \rho_0 + \epsilon r_{0,\epsilon}, \qquad \text{with} \quad \rho_0 \in C^2_b(\Omega) \text{ and } (r_{0,\epsilon})_{\epsilon>0} \subset L^2 \cap L^\infty. \tag{45}$$

The linear space C^2_b is that of C^2 bounded functions whose first and second derivatives are bounded. We also assume that there is a constant $\rho^* > 0$ such that

$$0 \le \rho_0 \le \rho^* \quad \text{and} \quad 0 \le \rho_{0,\epsilon} \le 2\rho^*. \tag{46}$$

For the velocity field, in order to avoid the trouble of defining the speed of the fluid in a vacuum zone $\{\rho = 0\}$, we work instead on the momentum $m = \rho u$. We take an initial momentum $m_{0,\epsilon}$ such that

$$(m_{0,\epsilon})_{\epsilon>0} \subset L^2, \quad \left(\frac{|m_{0,\epsilon}|^2}{\rho_{0,\epsilon}} \right)_{\epsilon>0} \subset L^1, \tag{47}$$

and we agree that $m_{0,\epsilon} = 0$ and $\frac{|m_{0,\epsilon}|^2}{\rho_{0,\epsilon}} = 0$ wherever $\rho_{0,\epsilon} = 0$. For the magnetic field, we choose initial data $(b_{0,\epsilon})_{\epsilon>0} \subset L^2$.

Because of these uniform bounds set on the initial data, we deduce that, up to an extraction,

$$\Big(m_{0,\epsilon} \rightharpoonup m_0 \quad \text{in } L^2 \Big), \qquad \Big(r_{0,\epsilon} \overset{*}{\rightharpoonup} r_0 \quad \text{in } L^2 \cap L^\infty \Big), \qquad \text{and} \qquad \Big(b_{0,\epsilon} \rightharpoonup b_0 \quad \text{in } L^2 \Big), \tag{48}$$

and we obviously have $\big(\rho_{0,\epsilon} \longrightarrow \rho_0$ in $L^\infty\big)$.

Finally, if $\Omega = \mathbb{R}^2$, we require the initial densities $\rho_{0,\epsilon}$ to fulfill an extra regularity property. We suppose that **one** of the two following (non-equivalent) conditions is satisfied:

$$\exists \delta > 0 | \quad \left(\frac{1}{\rho_{0,\epsilon}} \mathbb{1}_{\rho_{0,\epsilon}<\delta} \right)_{\epsilon>0} \subset L^1 \tag{49}$$

$$\exists p_0 \in]1, +\infty[, \exists \bar{\rho} > 0 | \quad \big((\bar{\rho} - \rho_{0,\epsilon})^+ \big)_{\epsilon>0} \subset L^{p_0}. \tag{50}$$

These two conditions make sure that the fluid velocity is uniformly L^2 (see below). We refer to [15], chapter 2, equations (2.8), (2.9) and (2.10) for more details.

3.2 Finite Energy Weak Solutions

For smooth solutions of (1) related to the initial data (ρ_0, m_0, b_0), we can multiply the momentum equation by u and the magnetic field equation by b and integrate both equations. We have, after integration by parts,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho |u|^2 \mathrm{d}x + \int_\Omega \nu(\rho) |\nabla u|^2 \mathrm{d}x = \int_\Omega (b \cdot \nabla) b \cdot u \, \mathrm{d}x, \tag{51}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |b|^2 \mathrm{d}x + \int_\Omega \mu(\rho) |\nabla b|^2 \mathrm{d}x = \int_\Omega (b \cdot \nabla) u \cdot b \, \mathrm{d}x. \tag{52}$$

One more integration by parts show that the righthand-side of both equations are opposite so that by summing the equations and integrating over $t \in [0, T]$, we get

$$\int_\Omega \Big(\rho(T) |u(T)|^2 + |b(T)|^2 \Big) \mathrm{d}x + \int_0^T \int_\Omega \Big(\nu_* |\nabla u|^2 + \mu_* |\nabla b|^2 \Big) \mathrm{d}x\mathrm{d}t \leq \int_\Omega \left(\frac{|m_0|^2}{\rho_0} + |b_0|^2 \right) \mathrm{d}x. \tag{53}$$

We have used the fact that both the viscosity and the resistivity are non-degenerate: $\nu(\rho) \geq \nu_*$ and $\mu(\rho) \geq \mu_*$. On the other hand, ρ is simply transported by the divergence-free velocity field u, so that all the L^p norms of ρ must be independent of time (for $2 \leq p \leq +\infty$). This gives us grounds to define the notion of finite energy weak solution.

Definition 3.1. Let $T > 0$ and let (ρ_0, m_0, b_0) be initial data fulfilling the conditions described in section 3.1 above. We say that (ρ, u, b) is a finite energy weak solution to (1) in $[0, T[\times \Omega$ if

1. $\rho \in L^\infty([0, T[\times \Omega)$ and $\rho \in C_T^0(L_{loc}^q)$ for all $1 \leq q < +\infty$.
2. $\rho |u|^2 \in L_T^\infty(L^1)$ with $u \in L_T^2(H^1)$.
3. $b \in L_T^2(H^1)$.
4. The mass equation is satisfied in the weak sense: for any test function $\phi \in \mathcal{D}([0, T[\times \Omega)$
$$\int_0^T \int_\Omega \Big\{ \rho \partial_t \phi + \rho u \cdot \nabla \phi \Big\} \mathrm{d}x\mathrm{d}t + \int_\Omega \rho_0 \phi_{t=0} \mathrm{d}x = 0. \tag{54}$$
5. The divergence-free condition $\mathrm{div}(u) = 0$ is satisfied in $\mathcal{D}'(]0, T[\times \Omega)$.
6. The momentum equation is satisfied in the weak sense: for any divergence-free test function $\phi \in \mathcal{D}([0, T[\times \Omega; \mathbb{R}^2)$ (with $\mathrm{div}(\phi) = 0$),
$$\int_0^T \int_\Omega \Big\{ \rho u \cdot \partial_t \phi + (\rho u \otimes u - b \otimes b) : \nabla \phi - \frac{1}{\epsilon} \rho u^\perp \cdot \phi - \nu(\rho) \nabla u : \nabla \phi \Big\} \mathrm{d}x\mathrm{d}t + \int_\Omega m_0 \phi_{t=0} \mathrm{d}x = 0. \tag{55}$$

7. The equation for the magnetic field is satisfied in the weak sense : for all $\phi \in \mathcal{D}([0,T[\times\Omega)$,

$$\int_0^T \int_\Omega \Big\{ b\cdot\partial_t\phi + (u\otimes b - b\otimes u) : \nabla\phi + \mu\nabla b : \nabla\phi \Big\}\,\mathrm{d}x\mathrm{d}t + \int_\Omega b_0\cdot\phi_{t=0}\,\mathrm{d}x = 0. \tag{56}$$

8. For almost every $t\in[0,T[$, the energy inequality (53) is satisfied.

The solution (ρ, u) is said to be *global* if the above holds for all $T>0$.

Existence of such weak finite energy solutions has been shown for fluids with density dependent viscosity in the case where there is no magnetic field $b\equiv 0$ by P.-L. LIONS (see the second chapter of [15]). This theorem even allows the initial density to vanish under conditions (49) and (50).

Concerning conductive fluids, more limited results are available. GERBEAU and LE BRIS prove in [13] existence of weak (finite energy) solutions in a bounded domain of $\mathbb{R}^3$ (the proof can be extended to $\mathbb{R}^2$ or $\mathbb{T}^2$ with slight modifications), but only for fluids with non-vanishing initial densities. DESJARDINS and LE BRIS do so in [6] for cylindrical or toroidal domains based on bounded subsets of $\mathbb{R}^2$ and for flows with translation invariance.

Even if we do not have a full existence result for flows having possible vacuum patches, as described above, we nonetheless work with these to accommodate possible future existence results.

3.3 Statement of the Results

We consider a sequence of initial data $(\rho_{0,\epsilon}, m_{0,\epsilon}, b_{0,\epsilon})_{\epsilon>0}$ satisfying all the assumptions and uniform bounds described in section 3.1 above. We further consider a sequence $(\rho_\epsilon, u_\epsilon, b_\epsilon)_{\epsilon>0}$ of finite energy weak solutions related to those initial data as in definition 3.1 above.

We aim to prove some kind of convergence of the solutions $(\rho_\epsilon, u_\epsilon, b_\epsilon)_{\epsilon>0}$ and identify the limit dynamics for $\epsilon\to 0^+$ in the form of an evolution PDE solved by the limit points of the sequence. We consider two cases:

1. The case of a quasi-homogeneous density, where the initial density $\rho_{0,\epsilon}$ is supposed to be a perturbation of a constant density state $\rho_{0,\epsilon} = 1+\epsilon r_{0,\epsilon}$. This assumption simplifies the equations very much: On the one hand we can write $\rho_\epsilon = 1+\epsilon r_\epsilon$ with r_ϵ solving a linear transport equation

$$\begin{cases} \partial_t r_\epsilon + \operatorname{div}(r_\epsilon u_\epsilon) = 0 \\ r_{\epsilon,t=0} = r_{0,\epsilon}, \end{cases} \tag{57}$$

thanks to the divergence-free condition $\operatorname{div}(u_\epsilon)=0$. On the other hand, the uniform bounds thus obtained simplify the momentum equation

$$\begin{aligned} \partial_t(\rho_\epsilon u_\epsilon) + \operatorname{div}(\rho_\epsilon u_\epsilon\otimes u_\epsilon - b_\epsilon\otimes b_\epsilon) + \frac{1}{\epsilon}\nabla\pi_\epsilon + \frac{1}{2}\nabla\big(|b_\epsilon|^2\big) + \frac{1}{\epsilon}\rho_\epsilon u_\epsilon^\perp - \nu\Delta u_\epsilon \\ = \partial_t u_\epsilon + \operatorname{div}(u_\epsilon\otimes u_\epsilon - b_\epsilon\otimes b_\epsilon) - \nu\Delta u_\epsilon + r_\epsilon u_\epsilon^\perp \\ + \Big\{ \frac{1}{\epsilon}\nabla\pi_\epsilon + \frac{1}{2}\nabla\big(|b_\epsilon|^2\big) + \frac{1}{\epsilon}u_\epsilon^\perp \Big\} + O(\epsilon), \end{aligned} \tag{58}$$

where the terms in the brackets are gradient terms and do not appear in the weak form of the equations.

2. The case of fully non-homogeneous fluids is understandably more difficult. None of the two previous simplifications can obviously be made. However, we will see that by using the structure of system (1) we can find an analogous decomposition $\rho_\epsilon = \rho_0(x) + \epsilon\sigma_\epsilon$, where σ_ϵ is bounded in a low-regularity space. Unfortunately, this does not simplify much the singular term, as $\frac{1}{\epsilon}\rho_0 u_\epsilon^\perp$ is not a gradient term.

Another problem is that the bounds we will find on σ are in such a low regularity space ($H^{-3-\delta}$ in fact) that taking the limit $\epsilon \to 0^+$ directly in the momentum equation is impossible. We will need to, instead, work on the vorticity and take the curl of the momentum equation.

Theorem 3.2 (Quasi-homogeneous case)**.** *Suppose that $\rho_0 = 1$ and consider a sequence $(\rho_{0,\epsilon}, m_{0,\epsilon}, b_{0,\epsilon})_{\epsilon>0}$ of initial data satisfying the assumptions fixed in section 3.1. Let $(\rho_\epsilon, u_\epsilon, b_\epsilon)_{\epsilon>0}$ be a sequence of corresponding weak solutions to* (1) *as in definition 3.1 above. Finally, let m_0, b_0 and r_0 as in* (48) *and $r_\epsilon = \frac{1}{\epsilon}(\rho_\epsilon - 1)$.*

There exists $(r, u, b) \in L^\infty(L^2 \cap L^\infty) \times L^\infty(L^2) \cap L^2_{loc}(H^1) \times L^\infty(L^2) \cap L^2_{loc}(H^1)$ such that $\mathrm{div}(u) = 0$ *and, up to the extraction of a subsequence and for any $T > 0$,*

1. $r_\epsilon \rightharpoonup^* r$ *in* $L^\infty_T(L^2 \cap L^\infty)$,
2. $u_\epsilon \rightharpoonup^* u$ *in* $L^\infty(L^2) \cap L^2_T(H^1)$,
3. $\big(b_\epsilon \longrightarrow b$ *in* $L^2_T(H^s)\big)$, *for any* $0 < s < 1$, *and* $\big(b_\epsilon \rightharpoonup b$ *in* $L^2_T(H^1)\big)$.

The limit dynamics for the triplet (r, u, b) is described by a homogeneous MHD system which it solves in the weak sense, namely

$$\begin{cases} \partial_t u + \mathrm{div}(u \otimes u) + \nabla \pi + \frac{1}{2}\nabla\big(|b|^2\big) + r u^\perp = \nu(1)\Delta u + \mathrm{div}(b \otimes b) \\ \partial_t b + \mathrm{div}(u \otimes b - b \otimes u) = \mu(1)\Delta b \\ \partial_t r + \mathrm{div}(ru) = 0 \\ \mathrm{div}(u) = \mathrm{div}(b) = 0, \end{cases} \tag{59}$$

for some pressure function π, with initial data

$$r_{t=0} = r_0, \quad u_{t=0} = m_0, \quad \text{and} \quad b_{t=0} = b_0. \tag{60}$$

For the fully non-homogeneous case, we need an extra technical assumption on the reference density ρ_0, namely that it's critical points are non-degenerate. More precisely, we suppose that

$$\mathrm{meas}\big\{|\nabla \rho_0| \le \delta\big\} \underset{\delta \to 0^+}{\longrightarrow} 0. \tag{61}$$

Theorem 3.3 (Fully non-homogeneous case)**.** *Consider a sequence $(\rho_{0,\epsilon}, m_{0,\epsilon}, b_{0,\epsilon})_{\epsilon>0}$ of initial data satisfying the assumptions fixed in section 3.1. Let $(\rho_\epsilon, u_\epsilon, b_\epsilon)_{\epsilon>0}$ be a sequence of corresponding weak solutions to* (1) *as in definition 3.1 above. Finally, let m_0, b_0 and r_0 as in* (48) *and $\sigma_\epsilon = \frac{1}{\epsilon}(\rho_\epsilon - 1)$.*

Then, for any $\delta > 0$ small enough, there exists $\sigma \in L^\infty_{loc}(H^{-3-\delta})$, $u \in L^\infty(L^2) \cap L^2_{loc}(H^1)$ and $b \in L^\infty(L^2) \cap L^2_{loc}(H^1)$ such that $\mathrm{div}(\rho_0 u) = 0$ *and, up to the extraction of a subsequence and for any $T > 0$,*

1. $\rho_\epsilon \longrightarrow \rho_0$ *in* $L^2_T(L^2_{loc})$),
2. $\sigma_\epsilon \rightharpoonup^* \sigma$ *in* $L^\infty_T(H^{-3-\delta})$,
3. $u_\epsilon \rightharpoonup^* u$ *in* $L^\infty(L^2) \cap L^2_T(H^1)$,
4. $\big(b_\epsilon \longrightarrow b$ *in* $L^2_T(H^s)\big)$, *for any* $0 < s < 1$, *and* $\big(b_\epsilon \rightharpoonup b$ *in* $L^2_T(H^1)\big)$.

There exists a distribution Γ *of order at most* 1 *such that*

$$\begin{cases} \partial_t\big(\mathrm{curl}\,(\rho_0 u) - \sigma\big) + \mathrm{curl}\left(\rho_0\nabla\Gamma - \mathrm{div}(\nu(\rho_0)\nabla u) - \mathrm{div}(b\otimes b)\right) = 0 \\ \partial_t b + \mathrm{div}(u\otimes b - b\otimes u) = \mathrm{div}(\mu(\rho_0)\nabla b) \\ \mathrm{div}(\rho_0 u) = 0 \\ \mathrm{div}(u) = \mathrm{div}(b) = 0, \end{cases} \tag{62}$$

with initial data

$$r_{t=0} = r_0, \quad \mathrm{curl}\,\big(\rho_0 u_{t=0}\big) - \sigma_{t=0} = \mathrm{curl}\,(m_0) - r_0, \quad \text{and } b_{t=0} = b_0. \tag{63}$$

In the case of a quasi-homogeneous fluid, we can find quantitative approximation estimates, provided that the solutions of the limit system be regular enough.

Theorem 3.4. *Given* $0 < \beta < 1$ *and* $(r_0, u_0, b_0) \in H^{1+\beta} \times H^1 \times H^1$ *such that* $\mathrm{div}(u_0) = 0$, *there exists a unique weak solution* (r, u, b) *to system* (59) *with those initial data such that, for all* $T > 0$,

1. $r \in C^0_T(H^{1+\gamma})$ *for all* $0 \le \gamma < \beta$,
2. $u \in C^0_T(H^1) \cap L^2_T(H^2)$,
3. $b \in C^0_T(H^1) \cap L^2_T(H^2)$,

Assume moreover that $\nu, \mu \in C^1(\mathbb{R}_+)$. *In that case, the whole sequence* $(\rho_\epsilon, u_\epsilon, b_\epsilon)_{\epsilon>0}$ *of theorem 3.2 above converges (without needing to extract a subsequence) and we have the following quantitative estimates: for all* $T > 0$ *and for almost every* $0 \le t \le T$,

$$\begin{aligned} \|r_\epsilon(t) - r(t)\|^2_{L^2} &+ \|u_\epsilon(t) - u(t)\|^2_{L^2} + \|b_\epsilon(t) - b(t)\|^2_{L^2} \\ &+ \frac{1}{2}\int_0^t \Big\{\nu_*|\nabla(u_\epsilon - u)|^2 + \mu_*|\nabla(b_\epsilon - b)|^2\Big\}\,\mathrm{d}x \\ &\le C\Big\{\|r_{0,\epsilon} - r_0\|^2_{L^2} + \|u_{0,\epsilon} - u_0\|^2_{L^2} + \|b_{0,\epsilon} - b_0\|^2_{L^2} + \epsilon^2\Big\}, \end{aligned} \tag{64}$$

where the constant $C > 0$ *depends on* $(T, \nu_*, \mu_*, \|u_0\|_{H^1}, \|b_0\|_{H^1}, \|r_0\|_{H^{1+\beta}}, \|\nu'\|_{L^\infty}, \|\mu'\|_{L^\infty}, M)$ *where* M *is such that*

$$\|r_{0,\epsilon}\|_{L^\infty} + \|u_{0,\epsilon}\|_{L^2} + \|b_{0,\epsilon}\|_{L^2} \le M. \tag{65}$$

Corollary 3.5. *With the same assumptions as in theorem 3.4 above, assume moreover that*

$$\|r_{0,\epsilon} - r_0\|^2_{L^2} + \|u_{0,\epsilon} - u_0\|^2_{L^2} + \|b_{0,\epsilon} - b_0\|^2_{L^2} \underset{\epsilon\to 0^+}{\longrightarrow} 0. \tag{66}$$

Then, we have strong convergence of the solutions

$$\Big((u_\epsilon, b_\epsilon) \longrightarrow (u, b) \quad \text{in } L^\infty_T(L^2) \cap L^2_T(H^1)\Big) \quad \text{and} \quad \Big(r_\epsilon \longrightarrow r \quad \text{in } L^\infty_T(L^2)\Big). \tag{67}$$

Remark 3.6. Note that if the initial data $(r_{0,\epsilon}, u_{0,\epsilon}, b_{0,\epsilon})$ and (r_0, u_0, b_0) coincide for every $\epsilon > 0$ (or are a perturbation of order $O(\epsilon)$ of (r_0, u_0, b_0)), then the order of convergence is in fact $O(\epsilon)$.

4 Uniform Bounds and Further Convergence Properties

The next two sections are devoted to the proofs of theorems 3.2 and 3.3. In all that follows, $(\rho_{0,\epsilon}, m_{0,\epsilon}, b_{0,\epsilon})_{\epsilon>0}$ is a sequence of initial data satisfying all the assumptions and uniform bounds described in section 3.1 above and $(\rho_\epsilon, u_\epsilon, b_\epsilon)_{\epsilon>0}$ is an associated sequence of finite energy weak solutions related to those initial data as in definition 3.1 above.

In this section, we first use uniform bounds on the solutions to prove their weak convergence. Then, we focus on a strong convergence result for the density, which we will need later to handle the non-constant viscosity and resistivity.

4.1 Uniform Bounds

In this section, we establish uniform bounds (independent of ϵ) on the sequence of solutions $(\rho_\epsilon, u_\epsilon, b_\epsilon)_{\epsilon>0}$, thus enabling to extract (weakly) converging subsequences.

First of all, because the solutions satisfy the energy inequality (53) (see assumption 8 of definition 3.1)

$$\int_\Omega \Big(\rho_\epsilon(t)|u_\epsilon(t)|^2 + |b_\epsilon(T)|^2 \Big) \,\mathrm{d}x + \int_0^t \int_\Omega \Big(\nu_*|\nabla u_\epsilon|^2 + \mu|\nabla b_\epsilon|^2 \Big) \,\mathrm{d}x\mathrm{d}s \leq \int_\Omega \left(\frac{|m_{0,\epsilon}|^2}{\rho_{0,\epsilon}} + |b_{0,\epsilon}|^2 \right) \mathrm{d}x \leq C, \tag{68}$$

the conditions of section 3.1 show that the righthand side of this inequality is uniformly bounded and hence

$$\big(\sqrt{\rho_\epsilon} u_\epsilon\big)_{\epsilon>0}, (b_\epsilon)_{\epsilon>0} \subset L^\infty(L^2), \qquad \text{and} \qquad (\nabla u_\epsilon)_{\epsilon>0}, (\nabla b_\epsilon)_{\epsilon>0} \subset L^2(L^2). \tag{69}$$

Secondly, in the quasi-homogeneous case, because both ρ_ϵ and $r_\epsilon = \frac{1}{\epsilon}(\rho_\epsilon - 1)$ solve a pure transport equation by the divergence-free vector field u_ϵ, we see that (this is the same property as in theorem 2.1, chapter 2 p. 23 of [15])

$$\forall\, 0 \leq \alpha \leq \beta < +\infty, \qquad \mathrm{meas}\big\{\alpha \leq \rho_\epsilon \leq \beta\big\} = \mathrm{meas}\big\{\alpha \leq \rho_{0,\epsilon} \leq \beta\big\}, \tag{70}$$

$$\forall\, 0 \leq \alpha \leq \beta < +\infty, \qquad \mathrm{meas}\big\{\alpha \leq r_\epsilon \leq \beta\big\} = \mathrm{meas}\big\{\alpha \leq r_{0,\epsilon} \leq \beta\big\}. \tag{71}$$

This of course implies that $(\rho_\epsilon)_{\epsilon>0} \subset L^\infty(L^\infty)$ and $(r_\epsilon)_{\epsilon>0} \subset L^\infty(L^2 \cap L^\infty)$, so that (up to extracting a subsequence)

$$\Big(\rho_\epsilon \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty(L^\infty)\Big), \qquad \Big(r_\epsilon \overset{*}{\rightharpoonup} r \quad \text{in } L^\infty(L^2 \cap L^\infty)\Big), \tag{72}$$

but also shows that $\rho_\epsilon(t)$ satisfies the extra regularity properties, (49) or (50), that we had required if $\Omega = \mathbb{R}^2$, and it does so independently of ϵ.

$$\left((49) \qquad \Rightarrow \qquad \exists \delta > 0 | \forall \epsilon > 0, \quad \left(\frac{1}{\rho_\epsilon(t)} \mathbb{1}_{\rho_{0,\epsilon}<\delta} \right)_{\epsilon>0} \subset L^\infty(L^1) \right) \tag{73}$$

$$\left((50) \qquad \Rightarrow \qquad \exists p_0 \in]1, +\infty[, \exists \bar\rho > 0 | \forall \epsilon > 0, \quad \Big((\bar\rho - \rho_\epsilon(t))^+ \Big)_{\epsilon>0} \subset L^\infty(L^{p_0}) \right). \tag{74}$$

Finally, $(u_\epsilon)_{\epsilon>0}$ is in fact bounded in $L^2_T(L^2)$ for any finite time $T > 0$. If $\Omega = \mathbb{R}^2$, this is a consequence of either one of the two previous conditions (73) and (74) ([15] point 8 in remark 2.1

pp. 24-25). If $\Omega = \mathbb{T}^2$, the same can be shown without the extra assumptions, by means of the Poincaré-Wirtinger inequality ([15] paragraph 2.3 p. 37). Therefore, up to an extraction, for all $T > 0$,

$$\left(u_\epsilon \rightharpoonup u \quad \text{in } L^2_T(H^1)\right), \qquad \left(b_\epsilon \rightharpoonup b \quad \text{in } L^2_T(H^1)\right). \tag{75}$$

4.2 Strong Convergence of the Densities

This section is dedicated to the quest of pointwise convergence for the ρ_ϵ. So far, we only have obtained mere weak convergence

$$\rho_\epsilon \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty(L^\infty). \tag{76}$$

If somehow we proved that

$$\rho_\epsilon^2 \overset{*}{\rightharpoonup} \rho^2 \quad \text{in } L^\infty(L^\infty), \tag{77}$$

then by using the characteristic function $\mathbb{1}_K$ of a compact subset $K \subset \mathbb{R}^2$ as a test function, we would recover the convergence of the L^2 norms

$$\langle \rho_\epsilon^2, \mathbb{1}_K \rangle = \|\rho_\epsilon\|^2_{L^2_T(L^2(K))} \longrightarrow \langle \rho^2, \mathbb{1}_K \rangle = \|\rho\|^2_{L^2_T(L^2(K))}. \tag{78}$$

Using the euclidean structure of $L^2_T(L^2(K))$, we deduce local strong convergence (and hence pointwise convergence, after extraction). Therefore, the argument boils down to proving (77). The quadratic non-linearity is the main challenge as, by the uniform bounds $(\rho_\epsilon)_{\epsilon>0} \subset L^\infty(L^\infty)$, we only know the existence of some function $g \in L^\infty(L^\infty)$ such that

$$\rho_\epsilon^2 \overset{*}{\rightharpoonup} g \quad \text{in } L^\infty(L^\infty), \tag{79}$$

and this function g need not be ρ^2. The trick is that both g and ρ^2 are (weak) solutions of a well-posed transport PDE with the same initial data, so that they are equal. We use some of the well-posedness results for linear transport equations proven by Di Perna and P. -L. Lions in [7].

Proposition 4.1. *Let $T > 0$. We have strong convergence of the densities*

$$\rho_\epsilon \underset{\epsilon \to 0^+}{\longrightarrow} \rho = \rho_0 \quad \text{in } L^2_T(L^2_{loc}). \tag{80}$$

Proof. **STEP 1.** We show that the ρ_ϵ^2 are solutions of the continuity equation. Since this fact is to be shown for all $\epsilon > 0$ independently, we drop the ϵ indices for more clarity.

Lemma 4.2. *As above, we assume $d = 2$. Let $u \in L^2_T(H^1)$ be a divergence-free velocity field $\operatorname{div}(u) = 0$ and $\rho \in L^\infty(L^\infty)$ be a weak solution of*

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 \\ \rho_{t=0} = \rho_0 \in L^\infty \end{cases}. \tag{81}$$

Then ρ^2 is also a weak solution of the same equation, with according initial datum:

$$\begin{cases} \partial_t \left(\rho^2\right) + u \cdot \nabla \left(\rho^2\right) = 0 \\ \rho_{t=0}^2 = \rho_0^2 \end{cases}. \tag{82}$$

Proof of the lemma. We wish to prove that ρ^2 is a weak solution of (82), which means that for all $\phi \in \mathcal{D}([0,T[\times\mathbb{R}^2)$,

$$\int_0^T \int_\Omega \rho^2 \left(\partial_t \phi + u \cdot \nabla \phi\right) \mathrm{d}x\mathrm{d}t + \int_\Omega \rho_0^2 \phi_{t=0} \mathrm{d}x = 0. \tag{83}$$

We consider a regularization sequence $(\mu_\alpha)_{\alpha>0}$ such that, if $\Omega = \mathbb{R}^2$, we have $\mu_\alpha(x) = \frac{1}{\alpha^d}\mu\left(\frac{x}{\alpha}\right)$ with

$$\mu \in C^\infty(\mathbb{R}^2), \quad \mathrm{supp}(\mu) \subset B(0,1) \quad \text{and} \quad \mu(x) = \mu(-x), \tag{84}$$

and we set $\rho_\alpha = \mu_\alpha * \rho$ for all $\alpha > 0$. For $\Omega = \mathbb{T}^2$, we take essentially the same functions but take into account the periodicity, that is, for α small enough, we set $\mu'_\alpha(x) = \sum_{k\in\mathbb{Z}^2} \mu_\alpha(x+k)$. Anyhow, ρ_α solves an approximate equation:

$$\partial_t \rho_\alpha = -u \cdot \nabla \rho_\alpha + u \cdot \nabla \left(\mu_\alpha * \rho\right) - \mu_\alpha * \left(u \cdot \nabla \rho\right). \tag{85}$$

This equation is also a transport equation

$$\partial_t \rho_\alpha + u \cdot \nabla \rho_\alpha = \left[u \cdot \nabla, \mu_\alpha *\right] \rho \tag{86}$$

and holds in the weak sense. We have noted $[u\cdot\nabla, \mu_\alpha *]$ the commutator of $u \cdot \nabla$ and the convolution by μ_α linear operator. Multiplying this equation by $2\rho_\alpha$ shows that ρ_α^2 is the solution of an approximate transport equation:

$$\partial_t \left(\rho_\alpha^2\right) + u \cdot \nabla \left(\rho_\alpha^2\right) = 2\rho_\alpha \left[u \cdot \nabla, \mu_\alpha *\right] \rho. \tag{87}$$

The space differentiation $2\rho_\alpha \nabla \rho_\alpha = \nabla(\rho_\alpha^2)$ is justified because $\rho_\alpha(t) \in C^\infty$ for almost all times $0 \le t \le T$, and the time differentiation $2\rho_\alpha \partial_t \rho_\alpha = \partial_t(\rho_\alpha^2)$ is justified because $\rho_\alpha \in W_T^{1,2}(H^s)$ for every $\alpha > 0$ and $s > 0$. This comes from $\rho u \in L_T^2(L^2)$ and

$$\partial_t \rho_\alpha = -\mu_\alpha * \left(u \cdot \nabla \rho\right) = -\operatorname{div}\left(\mu_\alpha * (\rho u)\right) \in L_T^2(H^s), \tag{88}$$

where we have used the fact that u is divergence-free.

Using the fact that ρ is a weak solution of equation (81) and taking $\varphi = \mu_\alpha * \phi$ as a test function, with $\phi \in \mathcal{D}([0,T[\times\Omega)$, gives the following: (remember that we have chosen μ so that $\mu(x) = \mu(-x)$)

$$\int_0^T \int_\Omega \rho_\alpha \Big\{ \partial_t \phi + u \cdot \nabla \phi \Big\} \mathrm{d}x\mathrm{d}t + \int_\Omega (\mu_\alpha * \rho_0)\phi \mathrm{d}x = 0. \tag{89}$$

Next, we wish to take $\phi = 2\rho_\alpha \psi$, with $\psi \in \mathcal{D}([0,T[\times\Omega)$, in the previous equation. This is justified since $\rho_\alpha \in W_T^{1,2}(H^s)$ by (88), so an integration by parts gives, at first formally, noting $\langle\, .\, ,\, .\, \rangle$ the duality brackets with respect to both time and space,

$$\langle \partial_t \rho_\alpha, 2\psi\rho_\alpha \rangle = \langle \partial_t(\rho_\alpha^2), \psi \rangle - \int_\Omega (\mu_\alpha * \rho_0)^2 \psi_{t=0} \mathrm{d}x, \tag{90}$$

and this last line is made rigorous, thanks to a topological density argument, by noticing that both sides of the equation are continuous functions of ρ_α with respect to the $W_T^{1,2}(H^s)$ topology.

Putting everything together, we have shown that ρ_α solves the integral equation

$$\forall \psi \in \mathcal{D}([0,T[\times\mathbb{R}^2), \quad \int_0^T \int_\Omega \rho_\alpha^2 \Big\{ \partial_t \psi + u \cdot \nabla \psi \Big\} \mathrm{d}x\mathrm{d}t + \int_\Omega (\mu_\alpha * \rho_0)^2 \psi_{t=0} \mathrm{d}x \\ + \int_0^T \int_\Omega 2\psi \rho_\alpha \left[u \cdot \nabla, \mu_\alpha *\right] \rho \,\mathrm{d}x\mathrm{d}t = 0. \tag{91}$$

Finally, we wish to let $\alpha \to 0^+$ in this weak form to recover (82).

1. On the one hand, the commutator converges strongly to 0. Since we have $u \in L^2_T(H^1)$ and $\rho \in L^\infty(L^\infty)$, we also have $u \in L^1_T(W^{1,1}_{loc})$ and $\rho \in L^\infty_T(L^\infty_{loc})$. We can apply lemma II.1 in [7] pp. 516-517 to get

$$[u \cdot \nabla, \mu_\alpha *]w \underset{\alpha\to 0^+}{\longrightarrow} 0 \quad \text{in } L^1_T(L^2_{loc}). \tag{92}$$

Using this, we find that the commutator term in (87) cancels in the limit $\alpha \to 0^+$, as, for every compact $K \subset \mathbb{R}^2$,

$$\left\|2\rho_\alpha \left[u \cdot \nabla, \mu_\alpha *\right] \rho\right\|_{L^1_T(L^2(K))} \leq 2\|\rho\|_{L^\infty(t,x)} \left\| \left[u \cdot \nabla, \mu_\alpha *\right] \rho\right\|_{L^1_T(L^2(K))} \underset{\alpha\to 0^+}{\longrightarrow} 0. \tag{93}$$

2. On the other hand, we show that $\left(\rho_\alpha \longrightarrow \rho \text{ in } L^2_T(L^2_{loc})\right)$.

Let $U \subset \mathbb{R}^d$ be an open bounded subset and $x \in U$. Then,

$$|\mu_\alpha * (\mathbb{1}_U \rho)(x) - \mu_\alpha * \rho(x)| = \int_{^cU} \rho(y)\mu_\alpha(x-y)\mathrm{d}y. \tag{94}$$

This last integral vanishes for α small enough. Indeed, x is a point of the open subset U and, because of our choice (84) of μ,

$$\mu_\alpha(x-y) \neq 0 \Rightarrow |x-y| \leq \alpha. \tag{95}$$

As a consequence,

$$\left|\mu_\alpha * (\mathbb{1}_U \rho)(t,x) - \mu_\alpha * \rho(t,x)\right|^2 \underset{\alpha\to 0^+}{\longrightarrow} 0 \quad \text{a.e. in } [0,T[\times U \tag{96}$$

and thanks to the uniform bound $|\rho_\alpha(t,x)| \leq \|\rho\|_{L^\infty(t,x)}\|\mu_\alpha\|_{L^1} = \|\rho\|_{L^\infty(t,x)}$, the dominated convergence theorem gives

$$\int_0^T \int_U \left|\mu_\alpha * (\mathbb{1}_U \rho) - \mu_\alpha * \rho\right|^2 \mathrm{d}x\mathrm{d}t \longrightarrow 0. \tag{97}$$

Finally, we have the desired convergence:

$$\mu_\alpha * \rho - \rho = \Big(\mu_\alpha * \rho - \mu_\alpha * (\mathbb{1}_U \rho)\Big) + \Big(\mu_\alpha * (\mathbb{1}_U \rho) - \mathbb{1}_U \rho\Big) \longrightarrow 0 \quad \text{in } L^2_T(L^2(U)). \tag{98}$$

In the exact same way, we see that $\left((\mu_\alpha * \rho_0)^2 \longrightarrow \rho_0^2 \text{ in } L^2_{loc}\right)$.

Using both (93) and (98) in (91) finishes the proof of lemma.

□

STEP 2. The uniform bound $(\rho_\epsilon)_{\epsilon>0} \subset L^\infty(L^\infty)$ shows that there exists a function $g \in L^\infty(L^\infty)$ such that, up to an extraction,

$$\rho_\epsilon^2 \overset{*}{\rightharpoonup} g \quad \text{in } L^\infty(L^\infty). \tag{99}$$

We now show that g is a solution of the linear transport equation

$$\begin{cases} \partial_t g + u \cdot \nabla g = 0 \\ g_{t=0} = \rho_0^2 \end{cases}. \tag{100}$$

The main issue is showing that $\rho_\epsilon^2 u_\epsilon$ converges, in some sense, to gu. In doing so, we proceed as in section 3.1.2. of [8], where it is shown that $\rho_\epsilon u_\epsilon$ converges to ρu in some suitable space.

The main idea is to use the transport equation solved by the ρ_ϵ^2 to trade space regularity for time compactness.

Because of the first step, the ρ_ϵ solve the linear transport equation

$$\begin{cases} \partial_t \left(\rho_\epsilon^2\right) + u \cdot \nabla \left(\rho_\epsilon^2\right) = 0 \\ \rho_{t=0}^2 = \rho_{0,\epsilon}^2 \end{cases} \tag{101}$$

so that we have $\partial_t(\rho_\epsilon^2) = -\operatorname{div}(\rho_\epsilon^2 u_\epsilon)$. Since $|\rho_\epsilon^2 u_\epsilon| \leq (\rho^*)^{3/2} |\sqrt{\rho_\epsilon} u_\epsilon|$, we see that $\left(\partial_t(\rho_\epsilon^2)\right)_{\epsilon>0} \subset L^\infty(H^{-1})$, and

$$(\rho_\epsilon^2)_{\epsilon>0} \subset W_T^{1,\infty}(H_{loc}^{-1}), \tag{102}$$

where the *loc* comes from the fact that the initial data ρ_0^2 is L^∞ and so H_{loc}^{-1}. Let $0 < \eta < 1$ and consider $\theta \in]0,1[$ such that $-1+\eta = -\theta + 0 \times (1-\theta)$. Interpolation lemma A.3 gives, for (almost) all $0 \leq s, t \leq T$,

$$\|(\rho_\epsilon(t) - \rho_\epsilon(s))\chi\|_{H^{-1+\eta}} \leq \|(\rho_\epsilon(t) - \rho_\epsilon(s))\chi\|_{H^{-1}}^\theta \|(\rho_\epsilon(t) - \rho_\epsilon(s)\chi\|_{L^2}^{1-\theta}, \tag{103}$$

where $\chi \in \mathcal{D}(\Omega)$ is an arbitrary compactly supported function. This shows that $(\rho_\epsilon^2)_{\epsilon>0}$ is bounded in every space $C_T^{0,1-\eta}(H_{loc}^{-1+\eta})$ and so, by Ascoli's theorem,

$$\rho_\epsilon^2 \longrightarrow g \quad \text{in } C_T^{0,1-\eta}(H_{loc}^{-1+\eta}) \tag{104}$$

for all $0 < \eta < 1$. Combining this with the paraproduct lemma B.5, which provides continuity of the function product $(a, b) \mapsto ab$ in the $H^{-\eta} \times H^1 \to H^{-\eta+\delta}$ topology (for $\delta > 0$ small enough), we get

$$\rho_\epsilon^2 u_\epsilon \rightharpoonup gu \quad \text{in } \mathcal{D}'(]0,T[\times\Omega). \tag{105}$$

It is now possible to take the limit $\epsilon \to 0^+$ in (101). We recover equation (100).

STEP 3. It follows from lemma 4.2 that ρ^2 solves the transport equation, so we have two weak solutions $w = \rho^2, g \in L^\infty(L^\infty)$ of the initial value problem

$$\begin{cases} \partial_t w + u \cdot \nabla w = 0 \\ w_{t=0} = \rho_0^2. \end{cases} \tag{106}$$

We will see that problem (106) is well-posed in $L^\infty(L^\infty)$ so that we do in fact have $\rho^2 = g$. This is a consequence of a theorem proved by Di Perna and P. -L. Lions in their article [7] revolving around transport theory and differential equations. We reproduce this theorem here.

Theorem 4.3 ([7], theorem II.2. p. 517)**.** *Assume that $u \in L_T^1(W_{loc}^{1,1})$ is such that $\operatorname{div}(u) \in L_T^1(L^\infty)$ and*

$$\frac{u(t,x)}{1+|x|} \in L_T^1(L^1) + L_T^1(L^\infty). \tag{107}$$

Then, for $w_0 \in L^\infty$ there is a unique weak solution $w \in L^\infty(L^\infty)$ of the linear transport equation (106)

$$\begin{cases} \partial_t w + u \cdot \nabla w = 0 \\ w_{t=0} = w_0. \end{cases} \tag{108}$$

It is not entirely obvious that condition (107) is fulfilled for $u \in L^2_T(H^1)$. In order to fall in the assumptions of the theorem, we set an arbitrary $R > 0$ and decompose according to whether $|u| < R$ or not.

$$\frac{|u(t,x)|}{1+|x|} = \mathbb{1}_{|u|<R}\frac{|u(t,x)|}{1+|x|} + \mathbb{1}_{|u|\geq R}\frac{|u(t,x)|}{1+|x|}. \tag{109}$$

On the one hand, the measure of the set $A_R := \{|u| \geq R\}$ is bounded by the BIENAYMÉ-TCHEBYCHEV inequality, as

$$\mathrm{mes}\{|u(t)| \geq R\}\,\mathrm{d}t \leq \frac{1}{R^2}\int_\Omega |u(t,x)|^2 \mathrm{d}x \in L^1_T. \tag{110}$$

Therefore, HÖLDER's inequality allows us to assert that $\mathbb{1}_{|u|\geq R}u(t,x)/(1+|x|) \in L^1_T(L^1)$ since

$$\int_0^T \int_{A_R} \frac{|u(t,x)|}{1+|x|}\mathrm{d}x\,\mathrm{d}t \leq \int_0^T \|u(t)\|_{L^1(A_R)}\,\mathrm{d}t \leq \int_0^T \|u\|_{L^2}|A_R|^{1/2}\,\mathrm{d}t < +\infty. \tag{111}$$

On the other hand, we obviously have

$$\mathbb{1}_{|u|<R}\frac{|u(t,x)|}{1+|x|} \leq R \in L^1_T(L^\infty), \tag{112}$$

so that (107) is indeed satisfied by u.

We have proven that $\big(g = \rho^2 \text{ a.e.}\big)$, thus proving the weak convergence

$$\rho_\epsilon^2 \overset{*}{\rightharpoonup} \rho^2 \quad \text{in } L^\infty(L^\infty). \tag{113}$$

STEP 4. We now prove strong convergence of the ρ_ϵ to ρ using (113).

Let $K \subset \mathbb{R}^d$ be a compact subset. Then, using $\mathbb{1}_K$ as a test function in (113), we get

$$\langle \rho_\epsilon^2, \mathbb{1}_K\rangle = \|\rho_\epsilon\|^2_{L^2_T L^2(K)} \longrightarrow \langle \rho^2, \mathbb{1}_K\rangle = \|\rho\|^2_{L^2_T L^2(K)}, \tag{114}$$

and using the fact that $L^2_T(L^2(K))$ has a Euclidean structure gives strong convergence: as $\rho \in L^1_T(L^1)$, because of the weak(∗) convergence (76) of the ρ_ϵ, we have ${}_{L^\infty_{t,x}}\langle \rho_\epsilon, \rho\rangle_{L^1_{t,x}} \longrightarrow \|\rho\|_{L^2_T(L^2)}$ and

$$\|\rho_\epsilon - \rho\|^2_{L^2_T(L^2)} = \underbrace{\|\rho_\epsilon\|^2_{L^2_T(L^2(K))} + \|\rho\|^2_{L^2_T(L^2(K))}}_{\to 2\|\rho\|^2_{L^2_T(L^2(K))}} - \underbrace{2\int_0^T \int_K \rho_\epsilon \rho\,\mathrm{d}x\mathrm{d}t}_{\to 2\|\rho\|^2_{L^2_T(L^2(K))}} \underset{\epsilon\to 0^+}{\longrightarrow} 0. \tag{115}$$

Hence, after extracting one more time, we deduce the pointwise convergence $\big(\rho_\epsilon \longrightarrow \rho \text{ a. e.}\big)$.
□

5 Convergence

This section is dedicated to the proof of theorems 3.2 and 3.3. After looking at the singular terms and inferring constraints the limit points must satisfy, we study convergence of each of the summands in (1), leaving the more difficult convective term for last.

5.1 The Singular part of the Equations

In this part, we focus our attention on the singular part of the equations, namely $\frac{1}{\epsilon}\big(\nabla\pi_\epsilon + \rho_\epsilon u_\epsilon^\perp\big)$. The method we use here is identical to the one used in [8] section 3.1.2.

Proposition 5.1. *Let $T > 0$. The following results hold:*

1. *In the case of a quasi-homogeneous density, we have, for all divergence-free test function $\phi \in \mathcal{D}([0,T[\times\Omega;\mathbb{R}^2)$,*

$$\Big\langle \frac{1}{\epsilon}\big(\nabla\pi_\epsilon + \rho_\epsilon u_\epsilon^\perp\big), \phi \Big\rangle \underset{\epsilon\to 0^+}{\longrightarrow} \int_0^T \int_\Omega r u^\perp \cdot \phi \,\mathrm{d}x dt. \tag{116}$$

2. *In the case of a fully non-homogeneous density, the limit density satisfies $\rho(t,x) = \rho_0(x)$ almost everywhere in $]0,T[\times\Omega$ and we have the relation $\mathrm{div}(\rho_0 u) = \mathrm{div}(u) = 0$ in $\mathcal{D}'$. In particular, $\nabla\rho_0 \cdot u = 0$ almost everywhere in $]0,T[\times\Omega$.*

Proof. We start by attending to the quasi-homogeneous setting, where the singularity *de facto* disappears as

$$\frac{1}{\epsilon}\big(\nabla\pi_\epsilon + \rho_\epsilon u_\epsilon^\perp\big) = \frac{1}{\epsilon}\big\{\nabla\pi_\epsilon + u_\epsilon^\perp\big\} + r_\epsilon u_\epsilon, \tag{117}$$

the terms in the brackets being gradient terms, which do not appear in the weak form of the equations since the relevant test functions are divergence-free. Therefore, if $\phi \in \mathcal{D}([0,T[\times\Omega;\mathbb{R}^2)$ is a divergence-free test function, what remains to study is

$$\Big\langle \frac{1}{\epsilon}\big(\nabla\pi_\epsilon + \rho_\epsilon u_\epsilon^\perp\big), \phi \Big\rangle = \int_0^T \int_\Omega r_\epsilon u_\epsilon^\perp \cdot \phi \,\mathrm{d}x\mathrm{d}t. \tag{118}$$

To take the limit $\epsilon \to 0^+$, we seek some form of strong convergence on r_ϵ. Since the r_ϵ solve the linear transport equation

$$\partial_t r_\epsilon + \mathrm{div}(r_\epsilon u_\epsilon) = 0 \tag{119}$$

we can simply replace ρ_ϵ by r_ϵ in the proofs of section 4.2. Proposition 4.1 therefore applies to $(r_\epsilon)_{\epsilon>0}$ and

$$r_\epsilon \underset{\epsilon\to 0^+}{\longrightarrow} r \quad \text{in } L^2_T(L^2_{loc}). \tag{120}$$

Using the weak convergence $u_\epsilon \rightharpoonup u$ we had in $L^2_T(H^1)$, we get convergence for the product

$$r_\epsilon u_\epsilon \longrightarrow ru \quad \text{in } \mathcal{D}'(]0,T[\times\Omega). \tag{121}$$

This gives the first property of the proposition:

$$\Big\langle \frac{1}{\epsilon}\big(\nabla\pi_\epsilon + \rho_\epsilon u_\epsilon^\perp\big), \phi \Big\rangle \underset{\epsilon\to 0^+}{\longrightarrow} \int_0^T \int_\Omega r u^\perp \cdot \phi \,\mathrm{d}x\mathrm{d}t. \tag{122}$$

The study of the fully non-homogeneous case is very much similar, with the exception that the singularity does not disappear. We already know that $\rho_\epsilon \longrightarrow \rho$ in the space $L^2_T(L^2_{loc})$. We infer weak convergence

$$\rho_\epsilon u_\epsilon \rightharpoonup \rho u \quad \text{in } \mathcal{D}'(]0,T[\times\Omega). \tag{123}$$

Multiplying the momentum equation in its weak form (55) by ϵ, we see that, for any divergence-free $\phi \in \mathcal{D}([0,T[\times\Omega;\mathbb{R}^2)$,

$$\int_0^T \int_\Omega \rho_\epsilon u_\epsilon^\perp \cdot \phi \, \mathrm{d}x \, \mathrm{d}t = O(\epsilon). \tag{124}$$

Indeed, the uniform bounds of section 4.1 show that the sequences $(\rho_\epsilon u_\epsilon)_{\epsilon>0}$, $(\rho_\epsilon u_\epsilon \otimes u_\epsilon)_{\epsilon>0}$, $(b_\epsilon \otimes b_\epsilon)_{\epsilon>0}$, $\big(\nu(\rho_\epsilon)\nabla u_\epsilon\big)_{\epsilon>0}$ and $(m_{0,\epsilon})_{\epsilon>0}$ are bounded in respectively $L^2_T(L^2)$, $L^\infty_T(L^1)$, $L^\infty(L^1)$, $L^2(L^2)$ and L^2. Therefore

$$\forall \phi \in \mathcal{D}([0,T[\times\Omega;\mathbb{R}^2), \quad \mathrm{div}(\phi) = 0 \Rightarrow \int_0^T \int_\Omega \rho u^\perp \cdot \phi \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{125}$$

This means that $\rho u^\perp = \nabla p$ for some suitable function p. Taking the curl of this relation gives $\mathrm{div}(\rho u) = 0$.

Finally, we look at the mass equation. Since we already know that $\big(\rho_\epsilon u_\epsilon \rightharpoonup \rho u \quad \text{in } L^2_T(H^{-1-\eta}_{loc})\big)$, we have no trouble taking the limit $\epsilon \to 0^+$ and we get

$$\partial_t \rho + \mathrm{div}(\rho u) = \partial_t \rho = 0. \tag{126}$$

The consequence of this is that $\rho(t,x) = \rho_0(x) \in C^2_b$ and that the relation $\mathrm{div}(\rho_0 u) = \nabla\rho_0 \cdot u = 0$ is satisfied almost everywhere in $]0,T[\times\Omega$.

□

5.2 The Magnetic Field

In this section, We take care of all the terms containing the magnetic field b. As in the previous section, we use the magnetic field equation to trade space regularity against time compactness.

Proposition 5.2. *Let $T > 0$. We have the following strong convergence for the magnetic field: for any $0 < s < 1$,*

$$b_\epsilon \underset{\epsilon\to 0^+}{\longrightarrow} b \quad \text{in } L^2_T(H^s_{loc}). \tag{127}$$

We deduce the convergence of all bilinear terms involving the magnetic field:

$$\mathrm{div}(b_\epsilon \otimes b_\epsilon) \longrightarrow \mathrm{div}(b \otimes b) \quad \text{in } \mathcal{D}'(]0,T[\times\Omega), \tag{128}$$

and we have identical results for $u_\epsilon \otimes b_\epsilon$ and $b_\epsilon \otimes u\epsilon$.

Note that this proposition does not complete the study of the magnetic field equation: the resistivity term $\mathrm{div}(\mu(\rho_\epsilon)\nabla b_\epsilon)$ still remains.

Proof. Recall that we have $(u_\epsilon)_{\epsilon>0}, (b_\epsilon)_{\epsilon>0} \subset L^2_T(H^1)$ and $(b_\epsilon)_{\epsilon>0} \subset L^\infty(L^2)$. The magnetic field equation reads

$$\partial_t b_\epsilon = \mathrm{div}(b_\epsilon \otimes u_\epsilon - u_\epsilon \otimes b_\epsilon) + \mathrm{div}(\mu(\rho_\epsilon)\nabla b). \tag{129}$$

Interpolation between Lebesgue spaces and Sobolev embeddings (see lemma A.2) gives $(b_\epsilon)_{\epsilon>0} \subset L^4_T(L^4)$ and $(u_\epsilon)_{\epsilon>0} \subset L^2_T(L^4)$. Hence $(u_\epsilon \otimes b_\epsilon)_{\epsilon>0} \subset L^{4/3}_T(L^2)$ and $(\partial_t b_\epsilon)_{\epsilon>0} \subset L^{4/3}_T(H^{-1})$. As a result, we get the Hölder bound $(b_\epsilon)_{\epsilon>0} \subset C^{0,3/4}_T(H^{-1})$ and the Ascoli theorem gives compactness of the sequence $(b_\epsilon)_{\epsilon>0}$ in $C^0_T(H^{-1-\delta}_{loc})$ for any small $\delta > 0$. Using interpolation lemma

A.3 one more time gives strong convergence of the magnetic fields b_ϵ. Indeed, let θ be such that $\theta \times 1 - (1+\delta)(1-\theta) = s \in [0,1[$. Then, for every compactly supported $\chi \in \mathcal{D}(\Omega)$,

$$\|(b_\epsilon - b)\chi\|_{H^s} \leq \|(b_\epsilon - b)\chi\|_{H^{-1-\delta}}^{1-\theta} \|(b_\epsilon - b)\chi\|_{H^1}^{\theta} \longrightarrow 0 \quad \text{in } L^{2/\theta}(0 \leq t \leq T), \tag{130}$$

so that we have secured strong convergence in some positive regularity space

$$b_\epsilon \underset{\epsilon \to 0^+}{\longrightarrow} b \quad \text{in } L_T^{2/\theta}(H^s_{loc}). \tag{131}$$

The precise values of θ and s are not important. All that matters is that $2/\theta \geq 2$ and $s \geq 0$ so that we do indeed have a strong convergence of the tensor products $b_\epsilon \otimes b_\epsilon$ in, say, $L^1_T(L^1_{loc})$. We deduce

$$\mathrm{div}(b_\epsilon \otimes b_\epsilon) \longrightarrow \mathrm{div}(b \otimes b) \quad \text{in } \mathcal{D}'(]0,T[\times\Omega). \tag{132}$$

On the other hand, this also means that we can achieve weak convergence of the mixed tensor products $b_\epsilon \otimes u_\epsilon$ and $u_\epsilon \otimes b_\epsilon$. □

5.3 The Viscosity and Resistivity Terms

In this section, we take care of the viscosity term $\mathrm{div}(\nu(\rho_\epsilon)\nabla u_\epsilon)$. Remember that we have taken $\nu(\,\cdot\,)$ to be continuous on $\mathbb{R}_+$. Convergence for the resistivity term $\mathrm{div}(\mu(\rho_\epsilon)\nabla b_\epsilon)$ is shown exactly in the same way.

Proposition 5.3. *We have weak convergence of the viscosity and resistivity terms*

$$\mathrm{div}(\nu(\rho_\epsilon)\nabla u_\epsilon) \longrightarrow \mathrm{div}(\nu(\rho_0)\nabla u) \quad \textit{in } \mathcal{D}'(]0,T[\times\mathbb{R}^2), \tag{133}$$

$$\mathrm{div}(\mu(\rho_\epsilon)\nabla b_\epsilon) \longrightarrow \mathrm{div}(\mu(\rho_0)\nabla b) \quad \textit{in } \mathcal{D}'(]0,T[\times\mathbb{R}^2). \tag{134}$$

Proof. We only prove the first convergence property, the second one being, in all that matters, identical.

In the case where the density is quasi-homogeneous, we already have strong convergence $\big(\rho_\epsilon \longrightarrow 1 \quad \text{in } L^\infty(L^\infty)\big)$ because $(r_\epsilon)_{\epsilon>0}$ is bounded in $L^\infty(L^2 \cap L^\infty)$. This makes the viscosity term easy to handle: let $\phi \in \mathcal{D}([0,T[\times\Omega;\mathbb{R}^2)$ be a divergence-free test function. Then

$$\begin{aligned}\int_0^T \int_\Omega \nu(\rho_\epsilon)\nabla u_\epsilon : \nabla\phi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega \nu(1)\nabla u : \nabla\phi \,\mathrm{d}x\,\mathrm{d}t = \\ \int_0^T \int_\Omega \Big\{\nu(\rho_\epsilon) - \nu(1)\Big\}\nabla u_\epsilon : \nabla\phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \nu(1)\Big\{\nabla u_\epsilon - \nabla u\Big\} : \nabla\phi \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{135}$$

The second integral has limit zero, because of the weak convergence of the u_ϵ in $L^2_T(H^1)$. As for the first integral, uniform convergence of the ρ_ϵ and continuity of $\nu(\,\cdot\,)$ gives

$$\left|\int_0^T \int_\Omega \Big\{\nu(\rho_\epsilon) - \nu(1)\Big\}\nabla u_\epsilon : \nabla\phi \,\mathrm{d}x\,\mathrm{d}t\right| \leq \|\nu(\rho_\epsilon) - \nu(1)\|_{L^\infty(L^\infty)} \|\nabla u_\epsilon\|_{L^2_T(L^2)} \|\nabla\phi\|_{L^2_T(L^2)} \tag{136}$$

$$\underset{\epsilon \to 0^+}{\longrightarrow} 0. \tag{137}$$

Obviously, this does not work as well in the fully non-homogeneous case, so we use the strong convergence theorem 4.1.

Recall from proposition 4.1 that we have strong convergence of the densities

$$\rho_\epsilon \underset{\epsilon\to 0^+}{\longrightarrow} \rho = \rho_0 \quad \text{in } L^2_T(L^2_{loc}). \tag{138}$$

The uniform bounds $(\rho_\epsilon)_{\epsilon>0} \subset L^\infty(L^\infty)$ and the continuity of ν show that the $\nu(\rho_\epsilon)$ are also uniformly bounded in $L^\infty(L^\infty)$. There is a constant $\nu^* > 0$ such that $0 < \nu_* \leq \nu(\rho_\epsilon) \leq \nu^*$ for all $\epsilon > 0$. The dominated convergence theorem then gives strong convergence of the viscosities:

$$\nu(\rho_\epsilon) \underset{\epsilon\to 0^+}{\longrightarrow} \nu(\rho_0) \quad \text{in } L^2_T(L^2_{loc}). \tag{139}$$

Let $\phi \in \mathcal{D}([0,T[\times\mathbb{R}^2)$ and $K \subset \mathbb{R}^2$ be a compact subset with $\operatorname{supp}(\phi) \subset K$. Then, recalling that $\big(\nabla u_\epsilon \rightharpoonup \nabla u \text{ in } L^2(H^1)\big)$,

$$\begin{aligned} &\left| \int_0^T \int_\Omega \nu(\rho_\epsilon)\nabla u_\epsilon \cdot \nabla\phi \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega \nu(\rho_0)\nabla u \cdot \nabla\phi \mathrm{d}x\mathrm{d}t \right| \\ &\leq \left| \int_0^T \int_\Omega \nu(\rho_0)(\nabla u_\epsilon - \nabla u)\cdot\nabla\phi \,\mathrm{d}x\mathrm{d}t \right| + \|\nabla\phi\|_{L^\infty(L^\infty)} \|\nu(\rho_\epsilon) - \nu(\rho_0)\|_{L^2_T(L^2(K))} \|\nabla u\|_{L^2(L^2)} \\ &\underset{\epsilon\to 0^+}{\longrightarrow} 0. \end{aligned} \tag{140}$$

and so

$$\operatorname{div}(\nu(\rho_\epsilon)\nabla u_\epsilon) \longrightarrow \operatorname{div}(\nu(\rho_0)\nabla u) \quad \text{in } \mathcal{D}'(]0,T[\times\mathbb{R}^2). \tag{141}$$

□

5.4 The Coriolis Force Term

This paragraph centers on the density function in the fully non-homogeneous case. As we have seen, there is no obvious way to write $\rho_\epsilon = \rho_0 + \epsilon\sigma_\epsilon$ with the σ_ϵ solving some PDE or being bounded in some Banach space. Doing this is the goal of this section. We refer to [8] sections 3.3.1 and 4.1 for the original proofs (the presence of the magnetic field introduces only minor modifications).

Proposition 5.4. *Let $T > 0$. The function $\sigma_\epsilon = \frac{1}{\epsilon}(\rho_\epsilon - \rho_0)$ is bounded in $L^\infty_T(H^{-3-\delta})$ for all $\delta > 0$, and so $(*)$-weakly converges in that same space to some σ. Moreover, the* Coriolis *force term satisfies, for any divergence-free $\phi \in \mathcal{D}([0,T[\times\Omega;\mathbb{R}^2)$,*

$$\frac{1}{\epsilon}\int_0^T \int_\Omega \rho_\epsilon u_\epsilon^\perp \cdot \phi \,\mathrm{d}x\,\mathrm{d}t \underset{\epsilon\to 0^+}{\longrightarrow} -\int_0^T \int_\Omega \sigma\partial_t\psi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega r_0\psi_{t=0}\,\mathrm{d}x, \tag{142}$$

where $\psi \in \mathcal{D}([0,T[\times\Omega)$ is a function such that $\phi = \nabla^\perp\psi$ and exists owing to the fact that $\operatorname{div}(\phi) = 0$.

Proof. For more convenience, set $V_\epsilon = \rho_\epsilon u_\epsilon$ and

$$f_\epsilon = \operatorname{div}(\nu(\rho_\epsilon)\nabla u_\epsilon) + \operatorname{div}(b_\epsilon \otimes b_\epsilon - \rho_\epsilon u_\epsilon \otimes u_\epsilon). \tag{143}$$

Because of the Sobolev embedding $H^{1+\delta} \subset L^\infty$ (see lemma A.2), which holds for any $\delta > 0$, we see that $L^1 \subset H^{-1-\delta}$. Hence

$$(f_\epsilon)_{\epsilon>0} \subset L^2_T(H^{-1}) + L^\infty_T(H^{-2-\delta}) \subset L^2_T(H^{-2-\delta}). \tag{144}$$

Now, because ρ_0 is time-independent, we can write the mass and momentum equations as

$$\epsilon\partial_t\sigma_\epsilon + \mathrm{div}(V_\epsilon) = 0, \tag{145}$$

$$\epsilon\partial_t V_\epsilon + \nabla\pi_\epsilon + V_\epsilon^\perp = \epsilon f_\epsilon. \tag{146}$$

Taking the curl of the second equation and computing the difference with the first leads to

$$\big(\partial_t(\eta_\epsilon - \sigma_\epsilon)\big)_{\epsilon>0} = \big(\mathrm{curl}\,(f_\epsilon)\big)_{\epsilon>0} \subset L^2_T(H^{-3-\delta}), \tag{147}$$

where we have set $\eta_\epsilon = \mathrm{curl}\,(V_\epsilon)$, which is bounded in $L^\infty_T(H^{-1})$. Hence the uniform bound

$$(\sigma_\epsilon)_{\epsilon>0} \subset L^\infty_T(H^{-3-\delta}) \tag{148}$$

which holds for every $\delta > 0$. In particular, there exists $\sigma \in L^\infty_T(H^{-3-\delta})$ such that $\sigma_\epsilon \rightharpoonup^* \sigma$ in this space.

We use this to rewrite the CORIOLIS force term. Let $\phi = \nabla^\perp\psi$ be a divergence-free test function, with $\psi \in \mathcal{D}([0,T[\times\Omega)$.

$$\frac{1}{\epsilon}\int_0^T\int_\Omega \rho_\epsilon u_\epsilon^\perp\cdot\phi\,\mathrm{d}x\,\mathrm{d}t = -\frac{1}{\epsilon}\big\langle\,\mathrm{div}(\rho_\epsilon u_\epsilon),\psi\big\rangle = \frac{1}{\epsilon}\big\langle\partial_t\rho_\epsilon,\psi\big\rangle \tag{149}$$

$$= \big\langle\partial_t\sigma_\epsilon,\psi\big\rangle = -\int_0^T\int_\Omega \sigma_\epsilon\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega r_{0,\epsilon}\psi_{t=0}\,\mathrm{d}x. \tag{150}$$

This last trick of using ψ as a test function rather than the divergence-free ϕ allows us to get rid of the singularity in the case of non-homogeneous densities. However, it forces us to take the curl of the whole equation.

$\square$

5.5 The Convective Term: the Quasi-Homogeneous Case

The convective term $\mathrm{div}(\rho_\epsilon u_\epsilon\otimes u_\epsilon)$ is the last we have to study. In the slightly non-homogeneous case, the argument is in three steps. First of all, we reduce the problem to the study of $\mathrm{div}(u_\epsilon\otimes u_\epsilon)$, taking advantage of the approximation $\rho_\epsilon \approx 1$. Then, we use uniform H^1 regularity to find a uniform approximation of u_ϵ by smooth functions which we will need for the last step, a compensated compactness argument. All of this section is an account of what can be found in [8], sections 4.2 and 4.3.

Proposition 5.5. *Let $T > 0$. For all divergence-free $\phi \in \mathcal{D}([0,T[\times\Omega;\mathbb{R}^2)$,*

$$\int_0^T\int_\Omega \rho_\epsilon u_\epsilon\otimes u_\epsilon : \nabla\phi\,\mathrm{d}x\,\mathrm{d}t \underset{\epsilon\to0^+}{\longrightarrow} \int_0^T\int_\Omega u\otimes u : \nabla\phi\,\mathrm{d}x\,\mathrm{d}t. \tag{151}$$

Proof. To start the proof, we notice that because we can write $\rho_\epsilon = 1 + \epsilon r_\epsilon$ with $(r_\epsilon)_{\epsilon>0} \subset L^\infty(L^2\cap L^\infty)$, we have, for all divergence-free $\phi \in \mathcal{D}([0,T[\times\Omega;\mathbb{R}^2)$,

$$\left|\int_0^T\int_\Omega\Big\{\rho_\epsilon u_\epsilon\otimes u_\epsilon - u_\epsilon\otimes u_\epsilon\Big\} : \nabla\phi\,\mathrm{d}x\,\mathrm{d}t\right| \underset{\epsilon\to0^+}{\longrightarrow} 0. \tag{152}$$

Next, we seek a uniform approximation of u_ϵ by a smooth function (that is, smooth in the space variable). Let S_j be the low-frequency cut-off operator from the LITTLEWOOD-PALEY decomposition given by (338) in the appendix and assume first that $\Omega = \mathbb{R}^2$. Then, using the uniform bound $(u_\epsilon)_{\epsilon>0} \subset L^2_T(H^1)$ and the definition of S_j as a low-frequency cut-off operator,

$$\big\|(I-S_j)u_\epsilon\big\|^2_{L^2} \leq \int_{|\xi|\geq 2^j} |\hat{u}_\epsilon(\xi)|^2\mathrm{d}\xi \leq 2^{-j}\int_{\mathbb{R}^2}(1+|\xi|^2)|\hat{u}_\epsilon(\xi)|^2\,\mathrm{d}x, \tag{153}$$

and for some constant $C > 0$ independent of both j and ϵ,

$$\big\|(I - S_j)u_\epsilon\big\|_{L^2_T(L^2)} \leq C 2^{-j/2}. \tag{154}$$

On the other hand, if $\Omega = \mathbb{T}^2$, the proof is exactly the same except that the frequency integral has to be replaced by a sum on the discreet FOURIER coefficients. Anyhow,

$$\begin{aligned} \left|\int_0^T \int_\Omega \left\{ u_\epsilon \otimes u_\epsilon - S_j u_\epsilon \otimes S_j u_\epsilon \right\} : \nabla\phi \,\mathrm{d}x\,\mathrm{d}t\right| \leq \|\nabla\phi\|_{L^\infty(L^\infty)} \\ \times \left(\big\|(I - S_j)u_\epsilon \otimes u_\epsilon\big\|_{L^1_T(L^1)} + \big\|S_j u_\epsilon \otimes (I - S_j)u_\epsilon\big\|_{L^1_T(L^1)} \right) \leq C2^{-j/2}. \end{aligned} \tag{155}$$

Now consider a given $j \geq 1$ and note $u_{\epsilon,j} = S_j u_\epsilon$ for convenience purposes. Because the operator S_j is a FOURIER-multiplier, it commutes with all the partial derivatives. In particular $\mathrm{div}(u_{\epsilon,j}) = 0$ and

$$\mathrm{div}(u_{\epsilon,j} \otimes u_{\epsilon,j}) = \frac{1}{2}\nabla\big|u_{\epsilon,j}\big|^2 + \mathrm{curl}\,(u_{\epsilon,j})u^\perp_{\epsilon,j}. \tag{156}$$

The first summand in the righthand side disappears when tested against a divergence-free function

$$-\int_0^T \int_\Omega u_{\epsilon,j} \otimes u_{\epsilon,j} : \nabla\phi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \mathrm{curl}\,(u_{\epsilon,j})u^\perp_{\epsilon,j} \cdot \phi \,\mathrm{d}x\,\mathrm{d}t. \tag{157}$$

As for the vorticity term, we reformulate the momentum equation in the following way (recall that we had set $V_\epsilon = \rho_\epsilon u_\epsilon$): then,

$$\epsilon\partial_t V_\epsilon + \nabla\pi_\epsilon + \frac{1}{2}\epsilon\nabla|b|^2 + u^\perp_\epsilon = \epsilon(f_\epsilon - r_\epsilon u^\perp_\epsilon) \tag{158}$$

with $f_\epsilon = \mathrm{div}(\nu(\rho_\epsilon)u_\epsilon + b_\epsilon \otimes b_\epsilon - \rho_\epsilon u_\epsilon \otimes u_\epsilon)$. Then, applying the S_j operator to the momentum equation and taking the curl gives

$$\partial_t \eta_{\epsilon,j} = \mathrm{curl}\,\big(f_{\epsilon,j} - S_j(r_\epsilon u^\perp_\epsilon)\big), \tag{159}$$

with $\eta_{\epsilon,j} = S_j \mathrm{curl}\,(V_\epsilon)$ and $f_{\epsilon,j} = S_j f_\epsilon$. Now, we had seen that the f_ϵ are bounded in $L^2(H^{-2-\delta})$ (see section 5.4). Therefore, we see that, for every fixed j, the sequence $(\eta_{\epsilon,j})_{\epsilon>0}$ is compact in any $L^\infty_T(H^m_{loc})$ for all $m \in \mathbb{R}$. We deduce strong convergence (up to the extraction of a subsequence) to some $\eta_j \in L^\infty_T(H^m_{loc})$,

$$\forall j \geq 1, \quad \eta_{\epsilon,j} \underset{\epsilon\to 0^+}{\longrightarrow} \eta_j \quad \text{in } L^\infty_T(H^m_{loc}). \tag{160}$$

But since we already know that $\big(V_\epsilon \rightharpoonup u \text{ in } L^2_T(L^2)\big)$, it follows that $\eta_j = \omega_j = S_j\mathrm{curl}\,(u)$ and so, for fixed $j \geq 1$ and for every $m \in \mathbb{R}$,

$$\omega_{\epsilon,j} = \mathrm{curl}\,(u_{\epsilon,j}) = \eta_{\epsilon,j} - \epsilon S_j \mathrm{curl}\,(r_\epsilon u_\epsilon) \underset{\epsilon\to 0^+}{\longrightarrow} \omega_j \quad \text{in } L^\infty(H^m_{loc}). \tag{161}$$

Finally, the weak convergence $\big(u_\epsilon \rightharpoonup u \text{ in } L^2_T(H^1)\big)$ gives weak convergence of the product

$$\omega_{\epsilon,j} u^\perp_{\epsilon,j} \underset{\epsilon\to 0^+}{\longrightarrow} \omega_j u^\perp_j \quad \text{in } \mathcal{D}'(]0,T[\times\Omega). \tag{162}$$

We are close to ending the proof. First, we have

$$\big\langle\, \mathrm{div}(u_{\epsilon,j} \otimes u_{\epsilon,j}, \phi)\big\rangle = \langle \omega_{\epsilon,j} u^\perp_{\epsilon,j}, \phi\rangle \underset{\epsilon\to 0^+}{\longrightarrow} \langle \omega_j u^\perp_j, \phi\rangle = \big\langle\, \mathrm{div}(u_j \otimes u_j, \phi)\big\rangle, \tag{163}$$

and then using (152) and the uniform approximation property (155), we conclude:

$$\big\langle\, \mathrm{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon, \phi)\big\rangle \underset{\epsilon\to 0^+}{\longrightarrow} \big\langle\, \mathrm{div}(u \otimes u), \phi\big\rangle. \tag{164}$$

□

5.6 Further Properties on the Density

We show a quantitive convergence estimate for the density which we will need in the study of the fully non-homogeneous case, to which we restrict ourselves in this paragraph. The arguments are those of [8] section 3.3.

Note that in subsection 4.2 we had proved strong convergence of the densities. However, this strong convergence is neither quantitative (there is no rate of convergence) nor uniform with respect to time.

We set $s_\epsilon = \rho_\epsilon - \rho_0$.

Proposition 5.6. *Let $T > 0$. Given $0 < \gamma < 1$, there exists $0 < \theta < 1$ and β, k such that*

$$0 < \beta < \gamma < k < 1 \tag{165}$$

and that the uniform embeddings

$$\left(\epsilon^{-\theta} s_\epsilon\right)_{\epsilon>0} \subset C_T^{0,\beta}(H^{-k}) \quad \text{and} \quad \left(\epsilon^{-\theta} s_\epsilon u_\epsilon\right)_{\epsilon>0} \subset L_T^2(H^{-k-\delta}) \tag{166}$$

hold true for any $\delta > 0$. Moreover,

$$\left(\epsilon^{-\theta} s_\epsilon \longrightarrow 0 \quad \text{in } L_T^\infty(H_{loc}^{-k-\delta})\right) \quad \text{and} \quad \left(\epsilon^{-\theta} s_\epsilon u_\epsilon \rightharpoonup 0 \quad \text{in } L_T^2(H_{loc}^{-k-\delta})\right), \tag{167}$$

for any $\delta > 0$.

Proof. The function s_ϵ solves a transport equation with a second member:

$$\partial_t s_\epsilon + \operatorname{div}(s_\epsilon u_\epsilon) = -u_\epsilon \cdot \nabla \rho_0. \tag{168}$$

with initial data $s_{\epsilon,t=0} = \epsilon r_{0,\epsilon}$. Because we have assumed that $\rho_0 \in C_b^2$, Sobolev embeddings show that the $(u_\epsilon \cdot \nabla \rho_0)$ are bounded in every space $L_T^2(L^q)$ for $2 \leq q < +\infty$. And because $(s_\epsilon)_{\epsilon>0} = (\rho_\epsilon - \rho_0)_{\epsilon>0} \subset L^\infty(L^\infty)$, we infer the uniform bounds

$$(s_\epsilon)_{\epsilon>0} \subset L_T^\infty(L^2 \cap L^\infty). \tag{169}$$

Furthermore, writing $\partial_t s_\epsilon = -\operatorname{div}(\rho_\epsilon u_\epsilon)$ and reasoning as in section 5.1, we see that $(s_\epsilon)_{\epsilon>0} \subset W_T^{1,\infty}(H^{-1})$ and after interpolation between Sobolev spaces (lemma A.3),

$$(s_\epsilon)_{\epsilon>0} \subset C_T^{0,\gamma}(H^{-\gamma}) \tag{170}$$

for every $0 \leq \gamma \leq 1$. On the other hand, recall from the previous section (proposition 5.4) that $(\sigma_\epsilon) = (s_\epsilon/\epsilon) \subset L_T^\infty(H^{-3-\delta})$ for $\delta > 0$ arbitrarily small. Therefore, for $0 \leq t_1, t_2 \leq T$, and $0 < \theta < 1$ such that $-k = -\gamma(1-\theta) + (-3-\delta)\theta$,

$$\|s_\epsilon(t_2) - s_\epsilon(t_1)\|_{H^{-k}} \leq \|s_\epsilon(t_2) - s_\epsilon(t_1)\|_{H^{-\gamma}}^{1-\theta} \|s_\epsilon(t_2) - s_\epsilon(t_1)\|_{H^{-3-\delta}}^{\theta} \tag{171}$$

$$\leq 2\|s_\epsilon\|_{C_T^{0,\gamma}(H^{-\gamma})}^{1-\theta} |t_2 - t_1|^{\gamma(1-\theta)} \epsilon^\theta \|\sigma_\epsilon\|_{L_T^\infty(H^{-3-\delta})}^{\theta}. \tag{172}$$

By setting $\beta = (1-\theta)\gamma$, we get the expected result. We deduce from Ascoli's theorem that the sequence $(\epsilon^{-\theta} s_\epsilon)_{\epsilon>0}$ is compact in $L_T^\infty(H_{loc}^{-k-\delta})$ so that it converges strongly to some s in that space. We must have $s = 0$ because $\epsilon^{1-\theta}\sigma_\epsilon = \epsilon^{-\theta} s_\epsilon \longrightarrow 0$ (in $\mathcal{D}'$).

Finally, the function product is a continuous map in the $H^{-k} \times H^1 \longrightarrow H^{-k-\delta}$ topology (lemma B.5) for arbitrarily small $\delta > 0$, hence the uniform bound

$$\left(\epsilon^{-\theta} s_\epsilon u_\epsilon\right)_{\epsilon>0} \subset L_T^2(H^{-k-\delta}). \tag{173}$$

And because of the strong convergence property $\left(\epsilon^{-\theta} s_\epsilon \longrightarrow 0 \quad \text{in } L^\infty_T(H^{-k-\delta}_{loc})\right)$, we get the weak convergence

$$\epsilon^{-\theta} s_\epsilon u_\epsilon \rightharpoonup 0 \quad \text{in } L^2_T(H^{-k-\delta}_{loc}). \tag{174}$$

□

5.7 The Convective Term: the Fully Non-Homogeneous Case

The main ideas for handling the convective term in the non-homogeneous case are very similar to those used for the quasi-homogeneous case, although many complications occur. Because $\rho_0(x)$ is not constant, decomposition (156) will instead be (omitting for the time being the regularization argument)

$$\operatorname{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon) \approx \operatorname{div}(\rho_0 u_\epsilon \otimes u_\epsilon) = \frac{1}{2}\rho_0 \nabla |u_\epsilon|^2 + \rho_0 \omega_\epsilon u_\epsilon^\perp + (u_\epsilon \cdot \nabla \rho_0) u_\epsilon. \tag{175}$$

To simplify those terms, we can no longer rely on the fact that we use divergence-free test functions: $\langle \rho_0 \nabla |u_\epsilon|^2, \phi \rangle \neq 0$ even when $\operatorname{div}(\phi) = 0$. However, any term of the form $\rho_0 \nabla \Lambda_\epsilon$ or $\Lambda_\epsilon \nabla \rho_0$ will give rise to a term of the form $\rho_0 \nabla \Gamma$ in the limit (see below). All that follows is nearly identical to section 5.2 of [8].

Proposition 5.7. *Let $T > 0$. There is a distribution Γ (of order at most one) such that, for all $\phi \in \mathcal{D}([0, T[\times \Omega; \mathbb{R}^2)$ with $\operatorname{div}(\phi) = 0$,*

$$\int_0^T \int_\Omega \rho_\epsilon u_\epsilon \otimes u_\epsilon : \nabla \phi \, \mathrm{d}x \, \mathrm{d}t \underset{\epsilon \to 0^+}{\longrightarrow} \langle \rho_0 \nabla \Gamma, \phi \rangle. \tag{176}$$

Proof. **STEP 1: Approximation of the densities.** We justify the approximation $\rho_\epsilon u_\epsilon \otimes u_\epsilon \approx \rho_0 u_\epsilon \otimes u_\epsilon$.

Note that because we have accounted for the presence of vacuum, the best uniform bound we have for the velocity field is $(u_\epsilon)_{\epsilon>0} \subset L^2_T(H^1)$ (instead of $L^\infty(L^2) \cap L^2_T(H^1)$ if we assume that $\rho_0 \geq c > 0$). This means that when estimating the difference $(\rho_\epsilon - \rho_0) u_\epsilon \otimes u_\epsilon$ we must use strong convergence of the densities *uniformly with respect to time*. In particular, we cannot benefit of the convergence properties proved in section 5.3 (proposition 5.3) which only provides strong convergence $\rho_\epsilon \longrightarrow \rho_0$ in the spaces $L^p_T(L^q_{loc})$ for $1 \leq p, q < +\infty$ thanks to the uniform bound $0 \leq \rho_\epsilon \leq \rho^*$ and dominated convergence.

Instead, we know that $\rho_\epsilon = \rho_0 + s_\epsilon$ with $\left(s_\epsilon \longrightarrow 0 \quad \text{in } L^\infty_T(H^{-\gamma})\right)$ for $0 < \gamma < 1$. Using the paraproduct lemma B.5 twice, we see that the product is continuous in the $H^1 \times H^1 \longrightarrow H^{1-\delta}$ and $H^{-\gamma} \times H^{1-\delta} \longrightarrow H^{-\gamma-\delta}$ topologies provided that $\delta > 0$ be small enough (*i.e.* small enough for $1 - \gamma - \delta$ to be positive). We therefore gather that

$$s_\epsilon u_\epsilon \otimes u_\epsilon \rightharpoonup 0 \quad \text{in } L^1_T(H^{-\gamma-\delta}). \tag{177}$$

STEP 2: Regularization. We use the same regularization procedure than in the quasi-homogeneous case: here S_j still is the LITTLEWOOD-PALEY operator defined by (338) in the appendix and has the same properties of uniform approximation. We continue to note $S_j g_\epsilon = g_{\epsilon,j}$ for any sequence of functions $(g_\epsilon)_{\epsilon>0}$ whenever we feel it make things more legible.

Lemma 5.8. *The following uniform properties hold: (recall that $\eta_\epsilon = \operatorname{curl}(V_\epsilon) = \operatorname{curl}(\rho_\epsilon u_\epsilon)$ is bounded in $L^\infty_T(H^{-1})$, and that σ_ϵ is bounded in every $L^\infty_T(H^{-3-\delta})$ with $\delta > 0$)*

1. *For all $s > 3$, we have* $\sup_{\epsilon>0} \|\sigma_\epsilon - S_j \sigma_\epsilon\|_{L^\infty_T(H^{-s})} \longrightarrow_{j \to +\infty} 0$,

2. For all $s > 1$, we have $\sup_{\epsilon>0} \|\eta_\epsilon - S_j\eta_\epsilon\|_{L^\infty_T(H^{-s})} \longrightarrow_{j\to+\infty} 0$.

The first new problem we face, after introducing S_j, is that $S_j[\rho_\epsilon u_\epsilon] \neq \rho_0 S_j u_\epsilon$ because ρ_0 is no longer constant. We write, recalling the notations of section 5.6,

$$S_j[\rho_\epsilon u_\epsilon] = S_j[\rho_0 u_\epsilon] + \epsilon^\theta S_j[\epsilon^{-\theta} s_\epsilon u_\epsilon] = \rho_0 u_{\epsilon,j} + \big[S_j, \rho_0\big] u_\epsilon + \epsilon^\theta S_j[\epsilon^{-\theta} s_\epsilon u_\epsilon]. \tag{178}$$

In the above, $\big[S_j, \rho_0\big]$ is the commutator of S_j and the multiplication by ρ_0 operator. We already know that $\big(S_j[\epsilon^{-\theta} s_\epsilon u_\epsilon]\big)_{\epsilon>0} \subset L^2_T(H^s)$ for any $s \geq 0$. We next use the following commutator estimate (see [1] lemma 2.97 for the proof):

Lemma 5.9. *Let $\chi \in C^1(\mathbb{R}^d)$ be such that $H(\xi) := (1+|\xi|)\hat{\chi} \in L^1$. There exists a constant $C > 0$ depending only on $\|H\|_{L^1}$ such that*[2]*,*

$$\forall f \in W^{1,\infty}, \forall g \in L^2, \forall \lambda > 0, \qquad \left\| \left[\chi\left(\frac{1}{\lambda}D\right), f\right] g \right\|_{L^2} \leq C\frac{1}{\lambda} \|\nabla f\|_{L^\infty} \|g\|_{L^2}. \tag{179}$$

We deduce from the lemma two estimates on the commutator term $\big[S_j, \rho_0\big] u_\epsilon$. On the one hand

$$\big\| \big[S_j, \rho_0\big] u_\epsilon \big\|_{L^2_T(L^2)} \leq \frac{C}{2^j} \|\nabla \rho_0\|_{L^\infty} \|u_\epsilon\|_{L^2_T(L^2)}, \tag{180}$$

and on the other hand, by differentiating the commutator we get, for $i \in \{1, 2\}$,

$$\partial_i \big[S_j, \rho_0\big] u_\epsilon = \big[S_j, \partial_i \rho_0\big] u_\epsilon + \big[S_j, \rho_0\big] \partial_i u_\epsilon \tag{181}$$

so that

$$\big\| \partial_i \big[S_j, \rho_0\big] u_\epsilon \big\|_{L^2_T(L^2)} \leq \frac{C}{2^j} \Big\{ \|\nabla^2 \rho_0\|_{L^\infty} \|u_\epsilon\|_{L^2_T(L^2)} + \|\nabla \rho_0\|_{L^\infty} \|\nabla u_\epsilon\|_{L^2_T(L^2)} \Big\}. \tag{182}$$

We have obtained a decomposition of $S_j(\rho_\epsilon u_\epsilon)$,

$$S_j[\rho_\epsilon u_\epsilon] = \rho_0 u_{\epsilon,j} + \epsilon^\theta \zeta_{\epsilon,j} + h_{\epsilon,j}, \tag{183}$$

with

$$\zeta_{\epsilon,j} = S_j[\epsilon^{-\theta} s_\epsilon u_\epsilon] \subset L^2_T(H^s), \tag{184}$$

$$h_{\epsilon,j} = \big[S_j, \rho_0\big] u_\epsilon \subset L^2_T(H^1) \quad \text{and} \quad \left(\overline{\lim_{\epsilon>0}} \|h_{\epsilon,j}\|_{L^2_T(H^1)} \underset{j\to+\infty}{\longrightarrow} 0 \right). \tag{185}$$

We make one last remark before going onwards. We had seen in section 5.6 that $\big(\epsilon^{-\theta} s_\epsilon u_\epsilon\big)$ is bounded in $L^2_T(H^{-k-\delta})$. Therefore, we can write

$$\eta_{\epsilon,j} = S_j \mathrm{curl}\, \big[\rho_0 u_\epsilon + \epsilon^\theta (\epsilon^{-\theta} s_\epsilon u_\epsilon)\big] = \eta^{(1)}_{\epsilon,j} + \epsilon^\theta \eta^{(2)}_{\epsilon,j} \tag{186}$$

with the uniform bounds (with respect to ϵ)

$$\|\eta^{(1)}_{\epsilon,j}\|_{L^2_T(L^2)} \leq C_1 \quad \text{and} \quad \|\eta^{(2)}_{\epsilon,j}\|_{L^2_T(H^s)} \leq C(s,j) \tag{187}$$

for any given $s \geq 0$. Note that the constant C_1 does not depend on either ϵ or j.

[2] For any function F, we note $F(D)$ the pseudo-differential operator $F(D)u := \mathcal{F}^{-1}\big[F(\xi)\hat{u}(\xi)\big]$.

Finally, exactly as in the quasi-homogeneous case, the uniform approximation properties of S_j yield, for any divergence-free $\phi \in \mathcal{D}([0,T[\times\Omega;\mathbb{R}^2)$,

$$\overline{\lim_{\epsilon>0}}\left|\int_0^T\int_\Omega \rho_0\nabla\phi : \Big\{u_\epsilon\otimes u_\epsilon - u_{\epsilon,j}\otimes u_{\epsilon,j}\Big\}\,\mathrm{d}x\,\mathrm{d}t\right| \underset{j\to+\infty}{\longrightarrow} 0. \tag{188}$$

STEP 3: Reformulation. We are left with $\mathrm{div}(\rho_0 u_{\epsilon,j}\otimes u_{\epsilon,j})$. Since all functions are smooth, we write

$$\mathrm{div}(\rho_0 u_{\epsilon,j}\otimes u_{\epsilon,j}) = (u_{\epsilon,j}\cdot\nabla\rho_0)u_{\epsilon,j} + \rho_0\omega_{\epsilon,j}u_{\epsilon,j}^\perp + \frac{1}{2}\rho_0\nabla|u_{\epsilon,j}|^2 \tag{189}$$

with $\omega_{\epsilon,j} = S_j\mathrm{curl}\,(u_\epsilon)$. We remark that the last term in the righthand side of this equation contributes $\rho_0\nabla\Gamma$ in the limit for some distribution Γ of order at most one and up to some extraction. In the same way, any term of the form $\langle\Lambda_{\epsilon,j}\nabla\rho_0,\phi\rangle$ has a limit of the same form $\langle\rho_0\nabla\Gamma,\phi\rangle$, which we see by integrating by parts

$$\int_0^T\int_\Omega \Lambda\epsilon,j\nabla\rho_0\cdot\phi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \Lambda_{\epsilon,j}\,\mathrm{div}(\rho_0\phi)\,\mathrm{d}t\,\mathrm{d}t = -\int_0^T\int_\Omega \rho_0\nabla\Lambda_{\epsilon,j}\cdot\phi\,\mathrm{d}x\,\mathrm{d}t. \tag{190}$$

Since all terms of the form $\rho_0\nabla\Lambda_{\epsilon,j}^{(1)} + \Lambda_{\epsilon,j}^{(2)}\nabla\rho_0$ can be treated in this way, we will generically note any of them $\Gamma_{\epsilon,j}$. Likewise, we note $R_{\epsilon,j}$ any remainder term, that is any term such that

$$\overline{\lim_{\epsilon>0}}\left|\int_0^T\int_\Omega R_{\epsilon,j}\cdot\phi\,\mathrm{d}x\,\mathrm{d}t\right| \underset{j\to+\infty}{\longrightarrow} 0. \tag{191}$$

With those notations, we have

$$\mathrm{div}(\rho_0 u_{\epsilon,j}\otimes u_{\epsilon,j}) = (u_{\epsilon,j}\cdot\nabla\rho_0)u_{\epsilon,j} + \rho_0\omega_{\epsilon,j}u_{\epsilon,j}^\perp + \Gamma_{\epsilon,j}. \tag{192}$$

STEP 4: The vorticity term. By use of (183), we get

$$\eta_{\epsilon,j} = \mathrm{curl}\,(\rho_0 u_{\epsilon,j}) + \epsilon^\theta\mathrm{curl}\,(\zeta_{\epsilon,j}) + \mathrm{curl}\,(h_{\epsilon,j}) \tag{193}$$

$$= \rho_0\omega_{\epsilon,j} + \nabla^\perp\rho_0\cdot u_{\epsilon,j} + \epsilon^\theta\mathrm{curl}\,(\zeta_{\epsilon,j}) + \mathrm{curl}\,(h_{\epsilon,j}). \tag{194}$$

and so

$$\rho_0\omega_{\epsilon,j}u_{\epsilon,j}^\perp = \eta_{\epsilon,j}u_{\epsilon,j}^\perp - \big(u_{\epsilon,j}\cdot\nabla^\perp\rho_0\big)u_{\epsilon,j}^\perp + R_{\epsilon,j}. \tag{195}$$

$$\mathrm{div}(\rho_0 u_{\epsilon,j}\otimes u_{\epsilon,j}) = \eta_{\epsilon,j}u_{\epsilon,j}^\perp + (u_{\epsilon,j}\cdot\nabla\rho_0)u_{\epsilon,j} - \big(u_{\epsilon,j}\cdot\nabla^\perp\rho_0\big)u_{\epsilon,j}^\perp + \Gamma_{\epsilon,j} + R_{\epsilon,j}. \tag{196}$$

STEP 5. We focus on the term $X_{\epsilon,j} := (u_{\epsilon,j}\cdot\nabla\rho_0)u_{\epsilon,j} - \big(u_{\epsilon,j}\cdot\nabla^\perp\rho_0\big)u_{\epsilon,j}^\perp$. The main idea in these few lines is to decompose $u_{\epsilon,j}(x)$ in the orthonormal basis of $\mathbb{R}^2$ given by $\Big(\frac{\nabla\rho_0}{|\nabla\rho_0|},\frac{\nabla^\perp\rho_0}{|\nabla\rho_0|}\Big)$. However, to avoid complications, we will discriminate between those $x\in\mathbb{R}^2$ for which $|\nabla\rho_0|$ is large enough and the others.

More precisely, let $B\in\mathcal{D}(\mathbb{R}^2)$ be such that

$$0\le B\le 1 \qquad\text{and}\qquad \begin{cases} B(y)=1 & \text{for } |y|\le 1,\\ B(y)=0 & \text{for } |y|\ge 2,\end{cases} \tag{197}$$

and let $B_j(x) = B\big(2^{j/2}\nabla\rho_0(x)\big)$. The function B_j is so chosen that $|\nabla\rho_0|\ge 2^{-j/2}$ on $\mathrm{supp}(1-B_j)$. Then, for any $2<q<+\infty$, HÖLDER's inequality with $1=\frac{2}{q}+\frac{q-2}{q}$ yields

$$\|B_jX_{\epsilon,j}\|_{L^1_T(L^1)} \le C\|X_{\epsilon,j}\|_{L^1_T(L^{q/2})}\mathrm{meas}\Big\{\|\nabla\rho_0|\le 2^{1-j/2}\Big\}^{(q-2)/q}, \tag{198}$$

and using the SOBOLEVembedding $H^1 \subset L^q$,

$$\|X_{\epsilon,j}\|_{L^1_T(L^{q/2})} \leq C\|\nabla\rho_0\|_{L^\infty}\|u_{\epsilon,j}\|^2_{L^2_T(H^1)}. \tag{199}$$

We see that $B_j X_{\epsilon,j} = R_{\epsilon,j}$ is a remainder term thanks to the assumption (61) we had made on ρ_0 to have non-degenerate critical points. Now we look at $(1-B_j)X_{\epsilon,j}$. We decompose $u_{\epsilon,j}$ and $u^\perp_{\epsilon,j}$ on the orthonormal basis $\left(\frac{\nabla\rho_0}{|\nabla\rho_0|}, \frac{\nabla^\perp\rho_0}{|\nabla\rho_0|}\right)$.

$$(1-B_j)u_{\epsilon,j} = \frac{1-B_j}{|\nabla\rho_0|^2}\Big\{(u_{\epsilon,j}\cdot\nabla\rho_0)\nabla\rho_0 + (u_{\epsilon,j}\cdot\nabla^\perp\rho_0)\nabla^\perp\rho_0\Big\}, \tag{200}$$

$$(1-B_j)u^\perp_{\epsilon,j} = \frac{1-B_j}{|\nabla\rho_0|^2}\Big\{-(u_{\epsilon,j}\cdot\nabla^\perp\rho_0)\nabla\rho_0 + (u_{\epsilon,j}\cdot\nabla\rho_0)\nabla^\perp\rho_0\Big\}. \tag{201}$$

Putting this in $(1-B_j)X_{\epsilon,j}$, we see that the two terms parallel to $\nabla^\perp\rho_0$ cancel out. All that remains is

$$(1-B_j)X_{\epsilon,j} = \frac{1-B_j}{|\nabla\rho_0|^2}\Big\{(u_{\epsilon,j}\cdot\nabla\rho_0)^2 - (u_{\epsilon,j}\cdot\nabla^\perp\rho_0)^2\Big\}\nabla\rho_0 = \Gamma_{\epsilon,j}. \tag{202}$$

We have proven that $X_{\epsilon,j} = \Gamma_{\epsilon,j} + R_{\epsilon,j}$ and

$$\operatorname{div}(\rho_0 u_{\epsilon,j}\otimes u_{\epsilon,j}) = \eta_{\epsilon,j}u^\perp_{\epsilon,j} + \Gamma_{\epsilon,j} + R_{\epsilon,j}. \tag{203}$$

STEP 6. We use one last time the decomposition of $u_{\epsilon,j}$ on the basis $\left(\frac{\nabla\rho_0}{|\nabla\rho_0|}, \frac{\nabla^\perp\rho_0}{|\nabla\rho_0|}\right)$. First we prove that $B_j\eta_{\epsilon,j}u^\perp_{\epsilon,j}$ is a remainder term $R_{\epsilon,j}$. Writing $\eta_{\epsilon,j} = \eta^{(1)}_{\epsilon,j} + \epsilon^\theta\eta^{(2)}_{\epsilon,j}$ as in (186), we see that, for $2 < q < +\infty$,

$$\begin{aligned}\big\|B_j\eta_{\epsilon,j}u^\perp_{\epsilon,j}\big\|_{L^1_T(L^1)} \leq C\|\eta^{(1)}_{\epsilon,j}\|_{L^2_T(L^2)}\|u_{\epsilon,j}\|_{L^2_T(L^q)}\operatorname{meas}\Big\{|\nabla\rho_0| \leq 2^{1-j/2}\Big\}^{(q-2)/2q}\\ + C\epsilon^\theta\|\eta^{(2)}_{\epsilon,j}\|_{L^2_T(L^2)}\|u_{\epsilon,j}\|_{L^2_T(L^2)}.\end{aligned} \tag{204}$$

In other words,

$$\big\|B_j\eta_{\epsilon,j}u^\perp_{\epsilon,j}\big\|_{L^1_T(L^1)} \leq C\operatorname{meas}\Big\{|\nabla\rho_0| \leq 2^{1-j/2}\Big\}^{(q-2)/2q} + C(j)\epsilon^\theta \tag{205}$$

so that we do in fact see that $B_j\eta_{\epsilon,j}u^\perp_{\epsilon,j} = R_{\epsilon,j}$. Next,

$$(1-B_j)\eta_{\epsilon,j}u^\perp_{\epsilon,j} = \frac{1-B_j}{|\nabla\rho_0|^2}\eta_{\epsilon,j}\Big\{(u^\perp_{\epsilon,j}\cdot\nabla\rho_0)\nabla\rho_0 + (u_{\epsilon,j}\cdot\nabla\rho_0)\nabla^\perp\rho_0\Big\} \tag{206}$$

$$= \frac{1-B_j}{|\nabla\rho_0|^2}\eta_{\epsilon,j}(u_{\epsilon,j}\cdot\nabla\rho_0)\nabla^\perp\rho_0 + \Gamma_{\epsilon,j}. \tag{207}$$

Taking the divergence of (183) to rewrite the scalar product, we see that $(u_{\epsilon,j}\cdot\nabla\rho_0) = \operatorname{div}(\rho_0 u_{\epsilon,j}) = \operatorname{div}(V_{\epsilon,j}) - \operatorname{div}(\epsilon^\theta\zeta_{\epsilon,j} + h_{\epsilon,j})$ and so

$$(1-B_j)\eta_{\epsilon,j}u^\perp_{\epsilon,j} = \frac{1-B_j}{|\nabla\rho_0|^2}\eta_{\epsilon,j}\operatorname{div}(V_{\epsilon,j})\nabla^\perp\rho_0 + \Gamma_{\epsilon,j} + R_{\epsilon,j}, \tag{208}$$

as on the one hand $\epsilon^\theta\operatorname{div}(\zeta_{\epsilon,j})\eta_{\epsilon,j} = R_{\epsilon,j}$ and on the other we use the uniform H^1 bound (185) on $(h_{\epsilon,j})_{\epsilon>0}$ to get

$$(209) \quad \left\| \frac{1-B_j}{|\nabla\rho_0|^2} \eta_{\epsilon,j} \operatorname{div}(h_{\epsilon,j}) \right\|_{L^1_T(L^1)}$$
$$\leq C 2^{-j} \left(\|\eta^{(1)}_{\epsilon,j}\|_{L^2_T(L^2)} \|h_{\epsilon,j}\|_{L^2_T(H^1)} + \epsilon^\theta \|\eta^{(2)}_{\epsilon,j}\|_{L^2_T(L^2)} \|h_{\epsilon,j}\|_{L^2_T(H^1)} \right)$$
$$\underset{j\to+\infty}{\longrightarrow} 0 \quad \text{uniformly in } \epsilon.$$

So far, we have obtained

$$(210) \qquad \operatorname{div}(\rho_0 u_{\epsilon,j} \otimes u_{\epsilon,j}) = \frac{1-B_j}{|\nabla\rho_0|^2} \eta_{\epsilon,j} \operatorname{div}(V_{\epsilon,j}) \nabla^\perp \rho_0 + \Gamma_{\epsilon,j} + R_{\epsilon,j}.$$

STEP 7: end of the proof. Recalling equations (145) and (146) from section 5.4, we compute

$$(211) \qquad \eta_{\epsilon,j} \operatorname{div}(V_{\epsilon,j}) = -\epsilon \eta_{\epsilon,j} \partial_t \sigma_{\epsilon,j} = -\frac{1}{2} \epsilon \frac{\partial}{\partial t} |\sigma_{\epsilon,j}|^2 - \epsilon(\eta_{\epsilon,j} - \sigma_{\epsilon,j}) \partial_t \sigma_{\epsilon,j}.$$

Using equation (147) we get

$$(212) \qquad \eta_{\epsilon,j} \operatorname{div}(V_{\epsilon,j}) = -\frac{1}{2} \epsilon \partial_t \big(|\sigma_{\epsilon,j}|^2\big) - \epsilon \partial_t \big\{ (\eta_{\epsilon,j} - \sigma_{\epsilon,j}) \sigma_{\epsilon,j} \big\} + \epsilon \operatorname{curl}(f_{\epsilon,j}) \sigma_{\epsilon,j}$$

so that

$$(213) \qquad \frac{1-B_j}{|\nabla\rho_0|^2} \eta_{\epsilon,j} \operatorname{div}(V_{\epsilon,j}) \nabla^\perp \rho_0 = R_{\epsilon,j}$$

because of the uniform bounds we already have on $(\sigma_{\epsilon,j})_{\epsilon>0}$, $(\eta_{\epsilon,j})_{\epsilon>0}$ and $(f_{\epsilon,j})_{\epsilon>0}$. Finally, we have proved that

$$(214) \qquad \operatorname{div}(\rho_0 u_{\epsilon,j} \otimes u_{\epsilon,j}) = \Gamma_{\epsilon,j} + R_{\epsilon,j}.$$

□

5.8 Conclusion: Taking the Limit

In previous paragraphs, we have accounted for all the terms of the equations. For the quasi-homogeneous case, we already have obtained exactly what we expected. In the fully non-homogeneous case, it remains to take the curl of the whole momentum equation. All propositions from 5.1 to 5.7 show that for any divergence-free $\phi \in \mathcal{D}([0,T[\times\Omega;\mathbb{R}^2)$, which we can write $\phi = \nabla^\perp \psi$ with $\psi \in \mathcal{D}([0,T[\times\Omega)$,

$$(215) \quad \int_0^T \int_\Omega \sigma \partial_t \psi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega \Big(\rho_0 u \partial_t \nabla^\perp \psi - b \otimes b : \nabla\nabla^\perp \psi - \nu(\rho_0) \nabla u : \nabla\nabla^\perp \psi \Big) \mathrm{d}x \, \mathrm{d}t$$
$$+ \big\langle \Gamma, \operatorname{div}(\rho_0 \nabla^\perp \psi) \big\rangle = \int_\Omega \Big(r_0 \psi_{t=0} - m_0 \nabla^\perp \psi_{t=0} \Big) \mathrm{d}x \, \mathrm{d}t.$$

Integration by parts show that we have indeed the weak form of the sought equation

6 Quantitative Convergence Estimates

In this section, we focus our attention on the limit system for the quasi-homogeneous case, which we recall for the reader's convenience:

$$(216) \qquad \begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla\pi + \frac{1}{2}\nabla\big(|b|^2\big) + r u^\perp = \nu(1)\Delta u + \operatorname{div}(b \otimes b) \\ \partial_t b + \operatorname{div}(u \otimes b - b \otimes u) = \mu(1)\Delta b \\ \partial_t r + \operatorname{div}(r u) = 0 \\ \operatorname{div}(u) = \operatorname{div}(b) = 0, \end{cases}$$

for some pressure function π. We shall prove that, under certain regularity assumptions, the solutions to system (216) are unique given regular enough initial data. Under these hypotheses, the whole sequence of solutions $(r_\epsilon, u_\epsilon, b_\epsilon)_{\epsilon>0}$ will (weakly) converge to the limit point (r, u, b), without the need to extract a subsequence. After that, we will show that, with the same regularity assumptions, provided that we have strong L^2 convergence of the initial data $(r_{0,\epsilon}, u_{0,\epsilon}, b_{0,\epsilon})_{\epsilon>0}$, the whole sequence of solutions also converge in $L^2_{loc}(L^2)$ with a quantitative convergence inequality.

We proceed in four steps. First, we find energy estimates for (216) at the order of regularity suited to prove uniqueness with stability estimates. Then, we show rigorously the existence of solutions at this level of regularity. The third step is to prove uniqueness for system (216). Finally, we focus on the relative entropy estimates and the proof of theorem 3.4.

In what follows, we will make extensive use of the Gagliardo-Nirenberg inequality (GN inequality for short, see lemma A.1) as well as the Young inequality in the following form: if $\frac{1}{p} + \frac{1}{q} = 1$ then, for any $\eta > 0$ and $a, b \geq 0$, we have $2ab \leq \eta a^2 + \frac{1}{\eta} b^2$. From now on, $\eta > 0$ will always note a small positive constant to be fixed in the later parts of the proofs.

In addition, except for the uniqueness theorem, we try to find inequalities that are as precise as (reasonably) possible in order to highlight which terms have the most impact on the final estimates.

6.1 Order 2 Energy Estimates

In this section, we focus on finding order 2 *a priori* estimates for the limit system. One way to do this is to use Δu and Δb as test functions in (216), as done in [8] section 4.4.1. Even though this is not useful for the computation of quantitative estimates later on, we attempt to optimize the *a priori* estimates as far as growth in time is concerned and show that the L^2 norms $\|\Delta u(t)\|_{L^2_T(L^2)}$, $\|\Delta b(t)\|_{L^2_T(L^2)}$ and $\|\partial_t u(t)\|_{L^2_T(L^2)}$ grow slower than any polynomial function of $T \geq 0$ with positive degree $h > 0$.

Proposition 6.1. *Let* (r, u, b) *be a regular enough solution of* (216) *related to the (regular) initial data* (r_0, u_0, b_0)*. Then for all finite* $T > 0$*, we have the following properties:*

1. *we have* $u, b \in L^\infty(L^2)$ *and* $\nabla u, \nabla b \in L^2(L^2)$*, with the standard energy estimate* (219) *below,*
2. *we have* $\nabla u, \nabla b \in L^\infty_T(L^2)$ *and* $\Delta u, \Delta b \in L^2_T(L^2)$ *with explicit bounds: for all* $h > 0$*, there is a constant* $C = C(\nu(1), \mu(1), \|r_0\|_{L^2}, \|r_0\|_{L^\infty}, \|u_0\|_{H^1}, \|b_0\|_{H^1}, h) > 0$ *such that*
$$\|\nabla u\|_{L^\infty_T(L^2)} + \|\Delta u\|_{L^2_T(L^2)} + \|\nabla b\|_{L^\infty_T(L^2)} + \|\Delta b\|_{L^2_T(L^2)} \leq C(1 + T^h), \tag{217}$$
3. *we have* $\partial_t u, \partial_t b \in L^2_T(L^2)$ *with, similarly,* $\|\partial_t u\|_{L^2_T(L^2)} + \|\partial_t b\|_{L^2_T(L^2)} \leq C(1 + T^h)$ *for all* $h > 0$*, the constant* C *having the same dependencies as before;*
4. *for all* $2 \leq p \leq +\infty$*,* $\|r(t)\|_{L^p} = \|r_0\|_{L^p}$ *and for all* $0 < \gamma < \beta < 1$*, we also have*
$$\|r\|_{L^\infty_T(H^{1+\gamma})} \leq C(\beta, \gamma) \exp\left\{ C(\beta, \gamma) \left(\int_0^T \|\nabla u\|_{H^1} \, \mathrm{d}t \right)^2 \right\} \|r_0\|_{H^{1+\beta}}. \tag{218}$$

Proof. First, testing the momentum equation with u and the magnetic field equation with b gives a basic energy estimate similar to (53)

$$\frac{1}{2} \int_\Omega \left\{ |u(t)|^2 + |b(t)|^2 \right\} \mathrm{d}x + \int_0^t \int_\Omega \left\{ \nu(1) |\nabla u|^2 + \mu(1) |\nabla b|^2 \right\} \mathrm{d}x \mathrm{d}s \tag{219}$$

$$\leq \frac{1}{2}\int_\Omega \left\{|u_0|^2+|b_0|^2\right\}\mathrm{d}x.$$

Next, we use the fact that r solves a pure transport equation with a divergence free flow u to see that the L^p norms of $r(t)$ are preserved $\|r(t)\|_{L^p}=\|r_0\|_L$.

We test the momentum equation with Δu and the magnetic field equation with Δb. Summing and integrating by parts gives

(220)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left\{|\nabla u|^2+|\nabla b|^2\right\}\mathrm{d}x+\int_\Omega\left\{\nu(1)|\Delta u|^2+\mu|\Delta b|^2\right\}\mathrm{d}x+\int_\Omega(b\cdot\nabla)b\cdot\Delta u\,\mathrm{d}x+\int_\Omega(b\cdot\nabla)u\cdot\Delta b\,\mathrm{d}x$$
$$=\int_\Omega(u\cdot\nabla)u\cdot\Delta u\,\mathrm{d}x+\int_\Omega ru^\perp\Delta u\,\mathrm{d}x+\int_\Omega(u\cdot\nabla)b\cdot\Delta b\,\mathrm{d}x.$$

We start by handling the two inegrals which do not involve the magnetic field. On the one hand, using HÖLDER's inequality with exponents $\frac{1}{2}+\frac{1}{2}+\frac{1}{\infty}=1$ and then proposition B.6,

$$\left|\int_\Omega(u\cdot\nabla)u\cdot\Delta u\,\mathrm{d}x\right|\leq\|\Delta u\|_{L^2}\|u\|_{L^\infty}\|\nabla u\|_{L^2} \tag{221}$$

$$\leq C\|\Delta u\|_{L^2}^{3/2}\|u\|_{L^2}1/2\|\nabla u\|_{L^2}. \tag{222}$$

YOUNG's inequality with exponents $\frac{3}{4}+\frac{1}{4}=1$ gives in turn

$$\left|\int_\Omega(u\cdot\nabla)u\cdot\Delta u\,\mathrm{d}x\right|\leq\eta\|\Delta\|_{L^2}^2+C(\eta)\|u\|_{L^2}^2\|\nabla u\|_{L^2}^4=\eta\|\Delta\|_{L^2}^2+M_1(t)\|\nabla u\|_{L^2}^2, \tag{223}$$

where $M_1(t)=C(\eta)\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2}^2\in L^1(t\geq 0)$ is a globally integrable function, with $\|M_1\|_{L^1(\mathbb{R}_+)}$ depending only on η, $\|u_0\|_{L^2}$ and $\|b_0\|_{L^2}$, thanks to the basic energy estimates.

On the other hand, we use HÖLDER's inequality with exponents $\frac{1}{p}+\frac{1}{q}+\frac{1}{2}=1$, followed by the GN inequality: for all $\eta>0$,

$$\left|\int_\Omega ru^\perp\cdot\Delta u\,\mathrm{d}x\right|\leq\|r\|_{L^p}\|u\|_{L^q}\|\Delta u\|_{L^2} \tag{224}$$

$$\leq C(q)\|r\|_{L^2}^{2/p}\|r\|_{L^\infty}^{1-2/p}\|u\|_{L^2}^{2/q}\|\nabla u\|_{L^2}^{1-2/q}\|\Delta u\|_{L^2} \tag{225}$$

$$\leq\eta\|\Delta u\|_{L^2}^2+C(\eta,\|r_0\|_{L^2},\|r_0\|_{L^\infty},q)\|u\|_{L^2}^{4/q}\|\nabla u\|_{L^2}^{2\left(1-\frac{2}{q}\right)}. \tag{226}$$

Since $\|\nabla u\|_{L^2}^2\in L^1(t\geq 0)$, we see that, for any arbitrary $h>0$, we can chose q so large that $C\|u\|_{L^2}^{4/q}\|\nabla u\|_{L^2}^{2\left(1-\frac{2}{q}\right)}=M_{1+h}(t)\in L^{1+h}(t\geq 0)$. Therefore:

$$\left|\int_\Omega ru^\perp\cdot\Delta u\,\mathrm{d}x\right|\leq\eta\|\Delta u\|_{L^2}^2+M_{1+h}(t). \tag{227}$$

Moreover, the norm $\|M_{1+h}\|_{L^{1+h}(\mathbb{R}_+)}$ only depends on the quantities $(\eta,\|r_0\|_{L^2},\|r_0\|_{L^\infty},\|u_0\|_{L^2},\|b_0\|_{L^2},h)$.

Now we take care of the remaining integrals. Similarly to the first integral, proposition B.6 yields

$$\left|\int_\Omega(b\cdot\nabla)b\cdot u\,\mathrm{d}x\right|\leq\|\Delta u\|_{L^2}\|b\|_{L^\infty}\|\nabla b\|_{L^2} \tag{228}$$

$$\leq C\|\Delta u\|_{L^2}\|\Delta b\|_{L^2}^{1/2}\|b\|_{L^2}^{1/2}\|\nabla b\|_{L^2} \tag{229}$$

$$\leq \eta\|\Delta u\|_{L^2}^2 + C(\eta)\|\Delta b\|_{L^2}\|b\|_{L^2}\|\nabla b\|_{L^2}^2 \tag{230}$$

$$\leq \eta\Big\{\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2\Big\} + \Big(C(\eta)\|b\|_{L^2}^2\|\nabla b\|_{L^2}^2\Big)\|\nabla b\|_{L^2}^2 \tag{231}$$

$$= \eta\Big\{\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2\Big\} + M_1(t)\|\nabla b\|_{L^2}^2. \tag{232}$$

In the previous inequality, $\|M_1\|_{L^1(\mathbb{R}_+)}$ depends only on η, $\|u_0\|_{L^2}$ and $\|b_0\|_{L^2}$.

Next, using HÖLDER's inequality with exponents $\frac{1}{4}+\frac{1}{4}+\frac{1}{2}=1$, the GN inequality, and then YOUNG's inequality twice we get

$$\left|\int_\Omega (b\cdot\nabla)u\cdot\Delta b\,\mathrm{d}x\right| \leq \|b\|_{L^4}\|\nabla u\|_{L^4}\|\Delta b\|_{L^2} \leq C\|b\|_{L^2}^{1/2}\|\nabla b\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2}\|\Delta b\|_{L^2} \tag{233}$$

$$\leq \eta\|\Delta b\|_{L^2}^2 + C(\eta)\|b\|_{L^2}\|\nabla b\|_{L^2}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2} \tag{234}$$

$$\leq \eta\Big\{\|\Delta b\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\Big\} + C(\eta, \|u_0\|_{L^2}, \|b_0\|_{L^2})\|\nabla u\|_{L^2}^2\|\nabla b\|_{L^2}^2 \tag{235}$$

$$= \eta\Big\{\|\Delta b\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\Big\} + M_1(t)\|\nabla b\|_{L^2}^2, \tag{236}$$

where $M_1(t) = C\|\nabla u\|_{L^2}^2 \in L^1(t\geq 0)$ and $\|M_1\|_{L^1(\mathbb{R}_+)} = C(\eta, \|u_0\|_{L^2}, \|b_0\|_{L^2})$. Exactly the same computations *mutatis mutandi* yield

$$\left|\int_\Omega (u\cdot\nabla)b\cdot\Delta b\,\mathrm{d}x\right| \leq \eta\Big\{\|\Delta b\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\Big\} + M_1(t)\|\nabla u\|_{L^2}^2 \tag{237}$$

with this time $M_1(t) = C\|\nabla b\|_{L^2}^2 \in L^1(t\geq 0)$ and $\|M_1\|_{L^1(\mathbb{R}_+)} = C(\eta, \|u_0\|_{L^2}, \|b_0\|_{L^2})$.

Putting all these estimates together, we get, by choosing η small enough (*e.g.* $\eta = \frac{1}{100}\max\{\nu(1),\mu(1)\}$), a differential inequality to which we apply GRONWALL's lemma:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big\{|\nabla u|^2+|\nabla b|^2\Big\}\,\mathrm{d}x + \frac{1}{2}\int_\Omega\Big\{\nu(1)|\Delta u|^2+\mu(1)|\Delta b|^2\Big\}\,\mathrm{d}x \leq M_{1+h}(t)+M_1(t)\int_\Omega\Big\{|\nabla u|^2+|\nabla b|^2\Big\}\,\mathrm{d}x, \tag{238}$$

$$\begin{aligned}\frac{1}{2}\int_\Omega \Big\{|\nabla u|^2+|\nabla b|^2\Big\}\,\mathrm{d}x + \frac{1}{2}\int_0^T\int_\Omega\Big\{\nu(1)|\Delta u|^2+\mu(1)|\Delta b|^2\Big\}\,\mathrm{d}x \\ \leq \left(\frac{1}{2}(\|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2) + \int_0^T M_{1+h}\right)\exp\left\{\int_0^{+\infty} M_1\right\}.\end{aligned} \tag{239}$$

Now that we have shown $L^2_T(H^2)$ bounds on the velocity field, we can apply proposition 5.2. of [4] (which is also expressed in a much more thorough form in [1], theorem 3.33). We see that because r solves the pure transport equation with a velocity field u of $L^2_T(H^2)$ regularity, then for all $0<\gamma<\beta<1$ we have

$$\|r\|_{L^\infty_T(H^{1+\gamma})} \leq C(\beta,\gamma)\exp\left\{C(\beta,\gamma)\left(\int_0^T \|\nabla u\|_{H^1}\,\mathrm{d}t\right)^2\right\}\|r_0\|_{H^{1+\beta}}. \tag{240}$$

We look at $\partial_t u$. In order to get rid of the pressure term, we apply the LERAY projector $\mathbb{P}$, which is the L^2-orthogonal projector on the subspace of divergence-free functions and can be defined as a FOURIER multiplier

$$\forall f\in L^2,\quad \widehat{\mathbb{P}f}(\xi) = \hat{f}(\xi) - \left(\frac{\xi}{|\xi|^2}\cdot\hat{f}(\xi)\right)\xi = \left(\frac{\xi^\perp}{|\xi|^2}\cdot\hat{f}(\xi)\right)\xi^\perp. \tag{241}$$

Applying the LERAY projector to the momentum equation in (216) gives,

$$\partial_t u + \mathbb{P}\big[\operatorname{div}(u \otimes u - b \otimes b) + ru^\perp\big] = \nu(1)\Delta u, \tag{242}$$

as divergence-free functions remain unchanged by $\mathbb{P}$, and as $\mathbb{P}$ commutes with differential operators because it is a FOURIER multiplier. In fact, since $\mathbb{P}$ is a FOURIER multiplier associated to a homogeneous function of degree zero, is is a continuous function $L^2 \longrightarrow L^2$, so we need only estimates on Δu, ru and $\operatorname{div}(u \otimes u - b \otimes b)$ to conclude.

First of all, for $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, HÖLDER's and the GN inequalities give

$$\|ru\|_{L^2} \leq \|r\|_{L^p}\|u\|_{L^q} \leq C(q, \|r_0\|_{L^2}, \|r_0\|_{L^\infty}, \|u_0\|_{L^2}, \|b_0\|_{L^2})\|\nabla u\|_{L^2}^{1-2/q}. \tag{243}$$

so that, by taking q large enough, $ru \in L^{2+h}(L^2)$. Secondly, because of proposition B.6,

$$\|\operatorname{div}(u \otimes u)\|_{L^2} = \|(u \cdot \nabla)u\|_{L^2} \leq C\|u\|_{L^\infty}\|\nabla u\|_{L^2} = C\Big(\|u\|_{L^2} + \|\Delta u\|_{L^2}\Big)\|\nabla u\|_{L^2} \tag{244}$$

$$\leq 2\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \|\nabla u\|_{L^2}\|u\|_{L^2}. \tag{245}$$

Squaring this inequality and integrating over $t \in [0, T]$ gives, with the previous results on Δu and ∇u, for all $h > 0$,

$$\int_0^T \|\operatorname{div}(u \otimes u)(t)\|_{L^2}^2 \mathrm{d}t \leq C(\|r_0\|_{L^2}, \|r_0\|_{L^\infty}, \|u_0\|_{H^1}, \|b_0\|_{H^1}, h)\,\big(1 + T^h\big), \tag{246}$$

The same computations with the magnetic field yield

$$\int_0^T \|\operatorname{div}(b \otimes b)(t)\|_{L^2}^2 \mathrm{d}t \leq C(\|r_0\|_{L^2}, \|r_0\|_{L^\infty}, \|u_0\|_{H^1}, \|b_0\|_{H^1}, h)\,\big(1 + T^h\big), \tag{247}$$

The combination of all this in ends the argument: for all $h > 0$, there is a constant C (with the dependencies specified above) such that

$$\forall T > 0, \quad \|\partial_t u\|_{L^2_T(L^2)} \leq C(1 + T^h). \tag{248}$$

It only remains to find the estimate on $\partial_t b$. We use the magnetic field equation

$$\partial_t b = \mu(1)\Delta b + \operatorname{div}(b \otimes u - u \otimes b). \tag{249}$$

The quadratic terms can be estimated exactly as in (244) and (245), and we already have an $L^2_T(L^2)$ bound for Δb. All this gives

$$\|\partial_t b\|_{L^2_T(L^2)} \leq C(1 + T^h). \tag{250}$$

□

6.2 Existence Result

In this section, we explain quickly how solutions of (216) with the level of regularity described in proposition 6.1 can be constructed.

Proposition 6.2. *Assume that $r_0 \in H^{1+\beta}$ for some $\beta > 0$ and that $u_0, b_0 \in H^1$ are divergence-free. Then there exists a solution (r, u, b) of system (216) related to those initial data such that, for all $T > 0$,*

1. *For all $0 < \gamma < \beta$, we have $r \in C^0_T(H^{1+\gamma})$,*

2. We have $u, b \in C^0_T(H^1) \cap L^2_T(H^2)$.

Moreover, the solution (r, u, b) satisfies the inequalities described in proposition 6.1.

Proof. We give the approximate system, whose solutions tend (weakly) to solutions of (216) and show how the solutions to the approximate system satisfy the *a priori* bounds we have obtained in the previous section. This is an implementation of the so-called FRIEDRICHS scheme.

STEP 1. Approximate system. Let $j \geq 2$ and A_j be the spectral projection operator defined in the following way:

$$\forall f \in L^2, \quad \mathcal{F}\big[A_j f\big](\xi) = \mathbb{1}_{|\xi| \leq j} \hat{f}(\xi). \tag{251}$$

Recall the LERAY projector $\mathbb{P}$ from (242). Set $r_1(t, x) = S_1 r_0(x)$. We consider the sequences of approximate systems

$$\begin{cases} \partial_t u_j + \mathbb{P} A_j \operatorname{div}(u_j \otimes u_j - b_j \otimes b_j) + \mathbb{P} A_j \big[r_{j-1} u_j^\perp\big] = \nu(1) \Delta u_j \\ \partial_t b_j + A_j \operatorname{div}(u_j \otimes b_j - b_j \otimes u_j) = \mu(1) \Delta b_j \\ \operatorname{div}(u_j) = \operatorname{div}(b_j) = 0, \end{cases} \tag{252}$$

which we equip with the initial data

$$\begin{cases} u_j(0) = A_j u_0 \\ b_j(0) = A_j b_0 \end{cases} \tag{253}$$

and where the function r_{j-1} solves the linear transport equation (remember that S_j is the low frequency cut-off LITTLEWOOD-PALEY operator defined by (338) in the appendix)

$$\begin{cases} \partial_t r_{j-1} + \operatorname{div}(r_{j-1} u_{j-1}) = 0 \\ r_{j-1, t=0} = S_j r_0. \end{cases} \tag{254}$$

Applying the CAUCHY-LIPSCHITZ theorem in the BANACH space

$$X_j = \big\{f, \quad \hat{f} \in L^2 \text{ and } \operatorname{supp}(\hat{f}) \subset B(0, j)\big\} \tag{255}$$

gives the existence of a (unique) $C^\infty(]T_-(j), T_+(j)[; X_j)$ maximal solution which satisfies both the energy estimate and the order 2 estimate in the previous section. Indeed, testing the momentum equation in (252) with $\Delta A_j u$ (for example), which is both in X_j and divergence free, gives for all $0 \leq t < T_+(j)$,

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla u_j|^2 \,\mathrm{d}x + \nu(1) \int_\Omega |\Delta u_j| \,\mathrm{d}x = \int_\Omega \operatorname{div}(u_j \otimes u_j - b_j \otimes b_j) \cdot \Delta u_j \,\mathrm{d}x + \int_\Omega r_{j-1} u_j^\perp \cdot \Delta u_j \,\mathrm{d}x. \tag{256}$$

In the same way, all the estimates derived from the momentum and the magnetic field equations hold for the approximate velocity and magnetic fields. For the density perturbation r_{j-1}, note that the operator S_j is a convolution operator with a function $K_j(x) = 2^{dj} K_1(2^j x) = \mathcal{F}^{-1}[\chi(2^{-j}\xi)](x)$ of constant L^1 norm, and hence defines a continuous operator for the $L^p \longrightarrow L^p$ topologies with $1 \leq p \leq +\infty$. Therefore, using the fact that r_{j-1} solves a pure transport equation by a divergence-free velocity field u_j,

$$\|r_{j-1}\|_{L^2} + \|r_{j-1}\|_{L^\infty} = \|S_j r_0\|_{L^2} + \|S_j r_0\|_{L^\infty} \leq C\big(\|r_0\|_{L^2} + \|r_0\|_{L^\infty}\big). \tag{257}$$

Next we show that the approximate solutions do not blow-up in finite time, that is that $T_+(j) = +\infty$. Fix $j \geq 2$. The basic energy estimates state that $\|u_j(t)\|_{L^2}$ and $\|b_j(t)\|_{L^2}$ are bounded for $0 \leq t < T_+(j)$, and therefore so are the norms of the time derivatives $\|\partial_t u_j(t)\|_{L^2}$

and $\|\partial_t b_j(t)\|_{L^2}$. Hence, the solution of ODE system (252) satisfies the CAUCHY criterion for $t < T_+(j)$ and necessarily $T_+(j) = +\infty$.

STEP 2. Convergence of u and b. In order to achieve convergence of the approximate solutions, we prove that $(\partial_t u_j)_{j\geq 2} \subset L^2_T(H^{-1})$. We will be using only the following basic energy estimates:

$$(u_j)_{j\geq 2}\,,\,(b_j)_{j\geq 2} \subset L^\infty(L^2) \cap L^2_{loc}(H^1) \quad \text{and} \quad (r_j)_{j\geq 2} \subset L^\infty(L^2 \cap L^\infty). \tag{258}$$

which yield the following weak convergences (up to an extraction): for some $(r, u, b) \in L^\infty(L^2 \cap L^\infty) \times L^2_T(H^1) \times L^2_T(H^1)$,

$$\Big((u_j, b_j) \rightharpoonup (u, b) \quad \text{in } L^2_T(H^1)\Big) \qquad \text{and} \qquad \Big(r_j \overset{*}{\rightharpoonup} r \quad \text{in } L^\infty(L^2 \cap L^\infty)\Big). \tag{259}$$

Since both $\mathbb{P}$ and A_j are L^2-orthogonal projectors, they are continuous for all the H^s topologies (with $s \in \mathbb{R}$). Therefore,

$$\begin{aligned} &\|\partial_t u_j\|_{L^2_T(H^{-1})} + \|\partial_t b_j\|_{L^2_T(H^{-1})} \\ &\quad \leq \nu(1)\|u_j\|_{L^2_T(H^1)} + \mu(1)\|b_j(t)\|_{L^2_T(H^{-1})} + \Big\| \operatorname{div}\big(u \otimes u - b \otimes b + b \otimes u - u \otimes b\big) \Big\|_{L^2_T(H^{-1})} \\ &\qquad + \|r_{j-1}u_j\|_{L^2_T(H^{-1})}. \end{aligned} \tag{260}$$

The last term is bounded by $\|r_{j-1}u_j\|_{L^2_T(H^{-1})} \leq \|r_{j-1}\|_{L^\infty_T(L^\infty)}\|u_j\|_{L^\infty_T(L^2)}$, so we only have to worry about the quadratic terms. If $f, g \in L^2_T(H^1)$, then, for all $0 \leq t \leq T$, using the SOBOLEV embedding $H^1 \subset L^4$ followed by the CAUCHY-SCHWARZ inequality in L^2_T,

$$\| \operatorname{div}(f(t) \otimes g(t))\|_{L^2_T(H^{-1})} \leq \int_0^T \|f(t)\|_{L^4}^{1/2}\|g(t)\|_{L^4}^{1/2} \leq \|f(t)\|_{L^2_T(H^1)}\|g(t)\|_{L^2_T(H^1)} < +\infty. \tag{261}$$

These computation show that $(\partial_t u_j)_{j\geq 2}$ is indeed bounded in $L^2_T(H^{-1})$. Therefore, we have the uniform bound $(u_j)_{j\geq 2} \subset C^{0,1/2}_T(H^{-1})$ and we wish to make use of ASCOLI's theorem. This is possible because the sequences $(u_j(t))_{j\geq 2}$ are relatively compact in H^{-1}_{loc}, thanks to the compact embedding $L^2(K) \subset H^{-1}(K)$ for all compact $K \subset \mathbb{R}^2$. We have proven

$$(u_j, b_j) \longrightarrow (u, b) \quad \text{in } L^\infty_T(H^{-1}_{loc}). \tag{262}$$

STEP 3. Convergence of r_j and $r_j u_{j-1}$. Using the fact that the r_j solve the linear transport equation and arguing exactly as in section 5.1, we get the weak convergence

$$r_j \overset{*}{\rightharpoonup} r \quad \text{in } L^\infty_T(H^{-\eta}_{loc}), \tag{263}$$

which, in turn, gives convergence of the product $r_j u_j$ thanks to the paraproduct lemma B.5

$$r_j u_{j-1} \rightharpoonup ru \quad \text{in } L^2_T(H^{-\eta-\delta}_{loc}) \tag{264}$$

for any $\delta > 0$ small enough.

STEP 4. Weak solutions. We aim to prove that (r, u, b) is a weak solution of (216). The only terms whose convergence is non-obvious at this point are the quadratic terms in u_j and b_j. Let $\phi \in \mathcal{D}([0, T[\times\Omega; 2^2)$ be a divergence-free test function. We will prove the convergence of

$$\int_0^T \int_\Omega A_j\mathbb{P}(u_j \otimes b_j) : \nabla\phi \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \int_\Omega (u_j \otimes b_j) : A_j\nabla\phi \,\mathrm{d}x \,\mathrm{d}t, \tag{265}$$

all other quadratic terms being similar. Taking the difference between the previous integral and the one we desire,

$$\tag{266} \begin{aligned} &\left|\int_0^T\int_\Omega \Big\{(u_j\otimes b_j) : A_j\nabla\phi - (u\otimes b):\nabla\phi\Big\}\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\qquad \le \left|\int_0^T\int_\Omega (u_j\otimes b_j):(A_j-I)\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\right| + \left|\int_0^T\int_\Omega (u_j\otimes b_j - u\otimes b):\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\right|. \end{aligned}$$

Using the SOBOLEV embedding $H^1\subset L^4$, we see that the first integral on the righthand side is bounded by

$$\tag{267} \left|\int_0^T\int_\Omega (u_j\otimes b_j):(A_j-I)\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\right| \le \|u_j\|_{L^2_T(L^4)}\|b_j\|_{L^2_T(L^4)}\|(A_j-I)\nabla\phi\|_{L^\infty_T(L^2)} \underset{j\to+\infty}{\longrightarrow} 0.$$

For the other integral, we decompose the quadratic term as

$$\tag{268} \begin{aligned} &\int_0^T\int_\Omega (u_j\otimes b_j - u\otimes b):\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad = \int_0^T\int_\Omega ((u_j-u)\otimes b_j):\nabla\phi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega (u\otimes(b_j-b)):\nabla\phi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Recall that we had strong convergence $\Big((u_j,b_j)\longrightarrow(u,b) \quad \text{in } L^\infty_T(H^{-1}_{loc})\Big)$. Using this, we deal with the first integral in the righthand side of (268),

$$\tag{269} \begin{aligned} \left|\int_0^T\int_\Omega ((u_j-u)\otimes b_j):\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\right| &\le \int_0^T \|b_j\|_{H^1}\|(u_j-u).\nabla\phi\|_{H^{-1}}\,\mathrm{d}t \\ &\le \|b_j\|_{L^2_T(H^1)}\|(u_j-u).\nabla\phi\|_{L^\infty_T(H^{-1})} \underset{j\to+\infty}{\longrightarrow} 0. \end{aligned}$$

The last integral in (268) tends to zero exactly in the same way, and we have proven convergence for the quadratic term:

$$\tag{270} \int_0^T\int_\Omega A_j\mathbb{P}(u_j\otimes b_j):\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \underset{j\to+\infty}{\longrightarrow} \int_0^T\int_\Omega (u\otimes b):\nabla\phi\,\mathrm{d}x\,\mathrm{d}t.$$

We have proven that (r,u,b) is indeed a weak solution.

STEP 5. Bounds for the solutions. Finally, the BANACH-STEINHAUS theorem makes sure that the inequalities of proposition 6.1 are carried from the approximate solutions to (r,u,b). □

6.3 Uniqueness for the Limit System

The proof of the uniqueness for the limit system runs very much in the same lines as the uniqueness theorem in section 4.4 of [8]. Here we also require enough regularity to perform simple energy estimates, which is appropriate considering that we aim at proving quantitative results based on relative entropy estimates later on.

Proposition 6.3. *Let (r_0,u_0,b_0) be initial data satisfying $r_0\in H^{1+\beta}$ and $u_0,b_0\in H^1$ for some $0<\beta<1$. There is at most one solution (r,u,b) of system (216) associated to these initial data such that, for all $T>0$,*

1. *$r\in C^0_T(H^{1+\gamma})$ for all $0<\gamma<\beta$,*

2. $u, b \in L^\infty_T(H^1) \cap L^2_T(H^2)$.

Remark 6.4. In addition to the previous proposition, we will see from the proof that if (r_1, u_1, b_1) and (r_2, u_2, b_2) are two such solutions, then setting $\delta r = r_2 - r_1$, $\delta u = u_2 - u_1$ and $\delta b = b_2 - b_1$ and δr_0, δu_0, δb_0 being these quantities computed on the initial data,

$$\|\delta u(t)\|^2_{L^2} + \|\delta b(t)\|^2_{L^2} + \|\delta r(t)\|^2_{L^2} + \frac{1}{2}\int_0^t \Big\{\nu(1)\|\nabla\delta u\|^2_{L^2} + \mu(1)\|\nabla\delta b\|^2_{L^2}\Big\}\mathrm{d}s \\ \leq C\Big(\|\delta u_0\|^2_{L^2} + \|\delta b_0\|^2_{L^2} + \|\delta r_0\|^2_{L^2}\Big), \tag{271}$$

where $C > 0$ depends on T, $\mu(1)$, $\nu(1)$ and $(\|r_{i,0}\|_{H^{1+\beta}}, \|u_{i,0}\|_{H^1}, \|b_{i,0}\|_{H^1})$ for $i \in \{1,2\}$.

Proof. We consider (r_1, u_1, b_1) and (r_2, u_2, b_2) two solutions as described in the proposition above. Let $\delta r = r_2 - r_1$, $\delta u = u_2 - u_1$ and $\delta b = b_2 - b_1$. Then, taking the difference between the equation solved by (r_2, u_2, b_2) and the one solved by (r_1, u_1, b_1) gives

$$\begin{cases} \partial_t(\delta u) + (u_2\cdot\nabla)\delta u + (\delta u\cdot\nabla)u_1 + \nabla\big(\pi_2 - \pi_1 + \frac12|b_2|^2 - \frac12|b_1|^2\big) + r_2\delta u^\perp + \delta r u_1^\perp \\ \qquad = \nu(1)\Delta(\delta u) + (\delta b\cdot\nabla)b_2 + (b_1\cdot\nabla)\delta b \\ \partial_t(\delta b) + (\delta u\cdot\nabla)b_2 + (u_1\cdot\nabla)\delta b = (\delta b\cdot\nabla)u_2 + (b_1\cdot\nabla)\delta u + \mu(1)\Delta(\delta b) \\ \partial_t(\delta r) + (u_2\cdot\nabla)\delta r = -\delta u\cdot\nabla r_1 \\ \mathrm{div}(\delta u) = \mathrm{div}(\delta b) = 0. \end{cases} \tag{272}$$

Testing the first equation with δu, the second one with δb and the third one with δr gives

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\delta u|^2\,\mathrm{d}x + \int_\Omega(u_2\cdot\nabla)\delta u\cdot\delta u\,\mathrm{d}x + \int_\Omega(\delta u\cdot\nabla)u_1\cdot\delta u\,\mathrm{d}x + \int_\Omega\delta r u_1^\perp\cdot\delta u\,\mathrm{d}x + \nu(1)\int_\Omega|\nabla\delta u|^2\,\mathrm{d}x \\ = \int_\Omega(\delta b\cdot\nabla)b_2\cdot\delta u\,\mathrm{d}x + \int_\Omega(b_1\cdot\nabla)\delta b\cdot\delta u\,\mathrm{d}x, \tag{273}$$

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\delta b|^2\,\mathrm{d}x + \int_\Omega(\delta u\cdot\nabla)b_2\cdot\delta b\,\mathrm{d}x + \int_\Omega(u_1\cdot\nabla)\delta b\cdot\delta b\,\mathrm{d}x + \mu\int_\Omega|\nabla\delta b|^2\,\mathrm{d}x \\ = \int_\Omega(\delta b\cdot\nabla)u_2\cdot\delta b\,\mathrm{d}x + \int_\Omega(b_1\cdot\nabla)\delta u\cdot\delta b\,\mathrm{d}x, \tag{274}$$

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\delta r|^2\,\mathrm{d}x + \int_\Omega(u_2\cdot\nabla\delta r)\delta r\,\mathrm{d}x = -\int_\Omega(\delta u\cdot\nabla r_1)\delta r\,\mathrm{d}x. \tag{275}$$

Now note that the second integral in (273), the third in (274) and the second in (275) are equal to zero, since integration by parts show that they are equal to their opposite. Next, note that the last integrals in (273) and (274) are opposite, which can again be seen by integration by parts. Therefore, adding the three equations together gives:

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big\{|\delta u|^2 + |\delta b|^2 + |\delta r|^2\Big\}\,\mathrm{d}x + \int_\Omega\Big\{\nu(1)|\nabla\delta u|^2 + \mu|\nabla b|^2\Big\}\,\mathrm{d}x \leq \\ \left|\int_\Omega(\delta u\cdot\nabla)u_1\cdot\delta u\,\mathrm{d}x\right| + \left|\int_\Omega\delta r u_1^\perp\cdot\delta u\,\mathrm{d}x\right| + \left|\int_\Omega(\delta u\cdot\nabla r_1)\delta r\,\mathrm{d}x\right| \\ + \left|\int_\Omega(\delta u\cdot\nabla)b_2\cdot\delta b\,\mathrm{d}x\right| + \left|\int_\Omega(\delta b\cdot\nabla)u_2\cdot\delta b\,\mathrm{d}x\right| + \left|\int_\Omega(\delta b\cdot\nabla)b_2\cdot\delta u\,\mathrm{d}x\right|. \tag{276}$$

The first three integrals, which do not involve the magnetic field, can be dealt with exactly as in [8]. We briefly summarize the computations. Firstly, using in turn the HÖLDER, the GN and YOUNG's inequalities with exponents $\frac{1}{4}+\frac{3}{4}=1$ yields

$$\left|\int_\Omega (\delta u\cdot\nabla)u_1\cdot\delta u\,\mathrm{d}x\right| \leq \|\delta u\|_{L^4}\|u_1\|_{L^4}\|\nabla\delta u\|_{L^2} \tag{277}$$

$$\leq C\|\delta u\|_{L^2}^{1/2}\|\nabla\delta u\|_{L^2}^{3/2}\|u_1\|_{L^4} \tag{278}$$

$$\leq \eta\|\nabla\delta u\|_{L^2}^2 + C(\eta)\|u_1\|_{L^4}^4\|\delta u\|_{L^2}^2. \tag{279}$$

Because of the SOBOLEV embedding $H^1\subset L^4$, we see that $N_1(t)=\|u_1(t)\|_{L^4}^4\leq C\|u_1(t)\|_{L^2}^2\|\nabla u_1(t)\|_{L^2}^2\in L^1(\mathbb{R}_+)$ in an integrable function thanks to the GN inequality.

Next, making use of proposition B.6, we see that

$$\left|\int_\Omega \delta r u_1^\perp\cdot\delta u\,\mathrm{d}x\right| \leq \|u_1\|_{L^\infty}\big(\|\delta u\|_{L^2}^2+\|\delta r\|_{L^2}^2\big) \tag{280}$$

with $N_2(t)=\|u_1\|_{L^\infty}\leq C\|u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2}\in L^2_T$.

As for the third integral, we use the fact that $\nabla r_1\in L^\infty_T(H^\gamma)$ for some $\gamma>0$. By fractional SOBOLEV embedding (see lemma A.2), we know that $\nabla r_1\in L^\infty_T(L^p)$ for some $p>2$. Let q be the associated exponent $\frac{1}{p}+\frac{1}{q}=1$. Then

$$\begin{aligned}\left|\int_\Omega \delta u\cdot\nabla r_1\delta r\,\mathrm{d}x\,\mathrm{d}x\right| &\leq \eta\|\nabla r_1\cdot\delta u\|_{L^2}^2+C(\eta)\|\delta r\|_{L^2}^2\leq\eta\|\nabla r_1\|_{L^p}^2\|\delta u\|_{H^1}^2+C(\eta)\|\delta r\|_{L^2}^2\\ &\leq C(\eta,\|r_{1,0}\|_{H^{1+\beta}},\|u_{1,0}\|_{H^1},\|b_{1,0}\|_{H^1})\Big(\|\delta u\|_{L^2}^2+\|\delta r\|_{L^2}^2\Big)+\eta\|\nabla\delta u\|_{L^2}^2.\end{aligned} \tag{281}$$

We have three remaining integrals which involve the magnetic field. Firstly, integration by parts gives

$$\left|\int_\Omega (\delta u\cdot\nabla)b_2\cdot\delta b\,\mathrm{d}x\right| = \left|\int_\Omega (\delta u\cdot\nabla)\delta b\cdot b_2\,\mathrm{d}x\right| \leq \|\delta u\|_{L^4}\|\nabla\delta b\|_{L^2}\|b_2\|_{L^4} \tag{282}$$

$$\leq \eta\|\nabla\delta b\|_{L^2}^2+C(\eta)\|\delta u\|_{L^4}^2\|b_2\|_{L^4}^2 \tag{283}$$

The GN inequality with exponents $\frac{1}{4}+\frac{3}{4}=1$ gives in turn

$$\left|\int_\Omega (\delta u\cdot\nabla)b_2\cdot\delta b\,\mathrm{d}x\right| \leq \eta\|\nabla\delta b\|_{L^2}^2+\|\delta u\|_{L^2}\|\nabla\delta u\|_{L^2}\|b_2\|_{L^4}^2 \tag{284}$$

$$\leq \eta\big(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\big)+N_\infty(t)\|\delta b\|_{L^2}^2, \tag{285}$$

where $N_\infty(t)=C(\eta)\|b_2\|_{L^4}^4\leq C(\eta)\|b_2\|_{L^\infty_T(H^1)}^4\in L^\infty_T$. The last integral in (276) can be treated in the same way: integration by parts gives

$$\left|\int_\Omega (\delta b\cdot\nabla)b_2\cdot\delta u\,\mathrm{d}x\right| = \left|\int_\Omega (\delta b\cdot\nabla)\delta u\cdot b_2\,\mathrm{d}x\right| \leq \|\delta u\|_{L^4}\|\nabla\delta b\|_{L^2}\|b_2\|_{L^4}\leq N_\infty(t)\|\delta b\|_{L^2}^2. \tag{286}$$

Finally, applying one last time the GN inequality with the same exponents $\frac{1}{4}+\frac{3}{4}=1$ yields

$$\left|\int_\Omega (\delta b\cdot\nabla)u_2\cdot\delta b\,\mathrm{d}x\right| \leq \|\nabla u_2\|_{L^2}^2\|\delta b\|_{L^4}^2\leq C\|\nabla u_2\|_{L^2}\|\delta b\|_{L^2}\|\nabla\delta b\|_{L^2} \tag{287}$$

$$\leq \eta\|\nabla\delta b\|_{L^2}^2+C(\eta)\|\nabla u_2\|_{L^2}^2\|\delta b\|_{L^2}^2\leq\eta\|\nabla\delta b\|_{L^2}^2+M_1(t)\|\delta b\|_{L^2}^2, \tag{288}$$

where $M_1(t) = C(\eta)\|\nabla u_2\|_{L^2}^2 \in L^1(t \geq 0)$.

Putting everything together and choosing η small enough that the gradient terms can be absorbed in the lefthand side, we have the differential inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left\{\|\delta u\|_{L^2}^2 + \|\delta b\|_{L^2}^2 + \|\delta r\|_{L^2}^2\right\}\mathrm{d}x + \frac{1}{2}\int_\Omega \left\{\nu(1)|\nabla \delta u|^2 + \nu(1)|\nabla \delta b|^2\right\}\mathrm{d}x \\ \leq N_1(t)\int_\Omega \left\{\|\delta u\|_{L^2}^2 + \|\delta b\|_{L^2}^2 + \|\delta r\|_{L^2}^2\right\}\mathrm{d}x \tag{289}$$

where $N_1(t) \in L^1_T$ is the sum of all the functions $M_p(t), N_p(t) \in L^p_T$ in the previous inequalities. We set

$$E(t) = \|\delta u\|_{L^2}^2 + \|\delta b\|_{L^2}^2 + \|\delta r\|_{L^2}^2. \tag{290}$$

Then we have $E'(t) \leq N_1(t)E(t)$ and GRONWALL's lemma gives

$$\int_\Omega \left\{\|\delta u\|_{L^2}^2 + \|\delta b\|_{L^2}^2 + \|\delta r\|_{L^2}^2\right\}\mathrm{d}x + \ \leq\ C(T)\int_\Omega \left\{\|\delta u_0\|_{L^2}^2 + \|\delta b_0\|_{L^2}^2 + \|\delta r_0\|_{L^2}^2\right\}\mathrm{d}x. \tag{291}$$

Using this last inequality in the righthand side of (289) gives the full order one inequality of remark 6.4.

□

6.4 Quantitative Estimates

In this section, we seek quantitative estimates for the functions $\delta r_\epsilon = r_\epsilon - r$, $\delta u_\epsilon = u_\epsilon - u$ and $\delta b_\epsilon = b_\epsilon - b$. The proof is based on relative entropy estimates. We will write differential inequalities which can be justified by working as in [11], but complete justification is left for a later work.

Proposition 6.5. *Let $T > 0$. Assume that $\mu, \nu \in C^1(\mathbb{R}_+)$. For almost every $0 \leq t \leq T$,*

$$\|\delta r_\epsilon(t)\|_{L^2}^2 + \|\delta u_\epsilon(t)\|_{L^2}^2 + \|\delta b_\epsilon(t)\|_{L^2}^2 + \frac{1}{2}\int_0^t \left\{\nu_*|\nabla \delta u_\epsilon|^2 + \mu_*|\nabla \delta b_\epsilon|^2\right\}\mathrm{d}x \\ \leq C\left\{\|r_{0,\epsilon} - r_0\|_{L^2}^2 + \|u_{0,\epsilon} - u_0\|_{L^2}^2 + \|b_{0,\epsilon} - b_0\|_{L^2}^2 + \epsilon^2\right\}, \tag{292}$$

where the constant $C > 0$ depends where the constant $C > 0$ depends on $(T, \nu_*, \mu_*, \|u_0\|_{H^1}, \|b_0\|_{H^1}, \|r_0\|_{H^{1+\beta}}, \|r_0\|_{L^\infty}, \|\nu'\|_{L^\infty}, \|\mu'\|_{L^\infty}, M)$, with M such that

$$\|r_{0,\epsilon}\|_{L^\infty} + \|u_{0,\epsilon}\|_{L^2} + \|b_{0,\epsilon}\|_{L^2} \leq M. \tag{293}$$

Proof. We seek a PDE system which is solved by the error functions δr_ϵ, δu_ϵ and δb_ϵ by taking the difference between the MHD system system (1) and the limit system (216). In order to do this, we write $\partial_t u = (\rho_\epsilon - \epsilon r_\epsilon)\partial_t u$ and we get

(294)

$$\begin{cases} \rho_\epsilon \partial_t(\delta u_\epsilon) + \rho_\epsilon(u_\epsilon\cdot\nabla)\delta u_\epsilon + (\delta u_\epsilon\cdot\nabla)u + r_\epsilon u_\epsilon^\perp - ru^\perp + \epsilon\left\{r_\epsilon(u_\epsilon\cdot\nabla)u + r_\epsilon\partial_t u\right\} \\ \qquad + \frac{1}{\epsilon}\nabla\pi_\epsilon - \nabla\pi + \frac{1}{\epsilon}u_\epsilon^\perp = \mathrm{div}\left\{\big(\nu(\rho_\epsilon) - \nu(1)\big)\nabla u + \nu(\rho_\epsilon)\nabla(\delta u_\epsilon)\right\} + (b_\epsilon\cdot\nabla)\delta u_\epsilon + (\delta b_\epsilon\cdot\nabla)\delta u_\epsilon \\ \partial_t(\delta b_\epsilon) + (u_\epsilon\cdot\nabla)\delta b_\epsilon + (\delta u_\epsilon\cdot\nabla)b = (b_\epsilon\cdot\nabla)\delta u_\epsilon + (\delta b_\epsilon\cdot\nabla)u + \mathrm{div}\left\{\big(\mu(\rho_\epsilon) - \mu(1)\big)\nabla b + \mu(\rho_\epsilon)\nabla(\delta b_\epsilon)\right\} \\ \partial_t(\delta r_\epsilon) + \mathrm{div}(\delta r_\epsilon u_\epsilon) = -\nabla r\cdot\delta u_\epsilon \\ \mathrm{div}(\delta u_\epsilon) = \mathrm{div}(\delta b_\epsilon) = 0 \end{cases}$$

By multiplying the first equation of (294) by δu_ϵ and integrating over all of Ω, we get:

(295)
$$\int_\Omega \rho_\epsilon \partial_t(\delta u_\epsilon)\cdot\delta u_\epsilon \mathrm{d}x + \int_\Omega \rho_\epsilon (u_\epsilon\cdot\nabla)\delta u_\epsilon\cdot\delta u_\epsilon \mathrm{d}x + \int_\Omega \big(\nu(\rho_\epsilon)-\nu(1)\big)\nabla u\cdot\nabla\delta u_\epsilon \,\mathrm{d}x + \int_\Omega \nu(\rho_\epsilon)\,|\nabla(\delta u_\epsilon)|^2\,\mathrm{d}x$$
$$+\int_\Omega (\delta u_\epsilon\cdot\nabla)u\cdot\delta u_\epsilon \mathrm{d}x + \int_\Omega (r_\epsilon u_\epsilon^\perp - r u^\perp)\cdot\delta u_\epsilon \mathrm{d}x + \epsilon\left\{\int_\Omega r_\epsilon (u_\epsilon\cdot\nabla)u\cdot\delta u_\epsilon \mathrm{d}x + \int_\Omega r_\epsilon \partial_t u\cdot\delta u_\epsilon \mathrm{d}x\right\}$$
$$= \int_\Omega (b_\epsilon\cdot\nabla)\delta b_\epsilon\cdot\delta u_\epsilon\,\mathrm{d}x + \int_\Omega (\delta b_\epsilon\cdot\nabla)b\cdot\delta u_\epsilon\,\mathrm{d}x.$$

The integral $\int_\Omega \left(\frac{1}{\epsilon}\nabla\pi_\epsilon - \nabla\pi + \frac{1}{\epsilon}u_\epsilon\right)\cdot\delta u_\epsilon \mathrm{d}x$ is equal to zero because δu_ϵ is a divergence-free vector-field. Multiplying the second equation of (294) by δb_ϵ and integrating, we get similarly:

$$\text{(296)}\quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\delta b_\epsilon|^2\,\mathrm{d}x + \int_\Omega (u_\epsilon\cdot\nabla)\delta b_\epsilon\cdot\delta b_\epsilon\,\mathrm{d}x + \int_\Omega (\delta u_\epsilon\cdot\nabla)b\cdot\delta b_\epsilon\,\mathrm{d}x$$
$$+\int_\Omega \big(\mu(\rho_\epsilon)-\mu(1)\big)\nabla b\cdot\nabla\delta u_\epsilon\,\mathrm{d}x + \int_\Omega \mu(\rho_\epsilon)\,|\nabla(\delta b_\epsilon)|^2\,\mathrm{d}x = \int_\Omega (b_\epsilon\cdot\nabla)\delta u_\epsilon\cdot\delta b_\epsilon\,\mathrm{d}x + \int_\Omega (\delta b_\epsilon\cdot\nabla)u\cdot\delta b_\epsilon\,\mathrm{d}x.$$

Finally, multiplying the third equation of (294) by δr_ϵ and integrating over $\mathbb{R}^2$ gives

$$\text{(297)}\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\delta r_\epsilon|^2 \mathrm{d}x = -\int_\Omega (\nabla r\cdot\delta u_\epsilon)\delta r_\epsilon \mathrm{d}x.$$

Our goal is to find an estimate on the "energy" function $e(t) = \|\delta r_\epsilon(t)\|_{L^2}^2 + \|\delta u_\epsilon(t)\|_{L^2}^2 + \|\delta b_\epsilon\|_{L^2}^2$. However, this function appears in no clear way in equation (295). To circumvent this difficulty, we consider instead $E(t) = \|\delta r_\epsilon(t)\|_{L^2}^2 + \|\sqrt{\rho_\epsilon}\delta u_\epsilon(t)\|_{L^2}^2 + \|\delta b_\epsilon\|_{L^2}$ which we make apparent in (295). Using the fact that ρ_ϵ solves the mass equation, we get

$$\text{(298)}\qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho_\epsilon|\delta u_\epsilon|^2 \mathrm{d}x = \int_\Omega \partial_t\rho_\epsilon|\delta u_\epsilon|^2 \mathrm{d}x + 2\int_\Omega \rho_\epsilon\partial_t(\delta u_\epsilon)\cdot\delta u_\epsilon\,\mathrm{d}x$$
$$\text{(299)}\qquad = -\int_\Omega \mathrm{div}(\rho_\epsilon u_\epsilon)|\delta u_\epsilon|^2\,\mathrm{d}x + 2\int_\Omega \rho_\epsilon\partial_t(\delta u_\epsilon)\cdot\delta u_\epsilon\,\mathrm{d}x$$
$$\text{(300)}\qquad = 2\int_\Omega \rho_\epsilon(u_\epsilon\cdot\nabla)\delta u_\epsilon\cdot\delta u_\epsilon\,\mathrm{d}x + 2\int_\Omega \rho_\epsilon\partial_t(\delta u_\epsilon)\cdot\delta u_\epsilon\,\mathrm{d}x$$

and equation (295) rewrites

(301)
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho_\epsilon|\delta u_\epsilon|^2\,\mathrm{d}x + \int_\Omega \nu(\rho_\epsilon)\,|\nabla(\delta u_\epsilon)|^2\,\mathrm{d}x + \int_\Omega \big(\nu(\rho_\epsilon)-\nu(1)\big)\nabla u\cdot\nabla\delta u_\epsilon\,\mathrm{d}x + \int_\Omega (\delta u_\epsilon\cdot\nabla)u\cdot\delta u_\epsilon \mathrm{d}x$$
$$+\int_\Omega (r_\epsilon u_\epsilon^\perp - r u^\perp)\cdot\delta u_\epsilon \mathrm{d}x + \epsilon\left\{\int_\Omega r_\epsilon(u_\epsilon\cdot\nabla)u\cdot\delta u_\epsilon \mathrm{d}x + \int_\Omega r_\epsilon\partial_t u\cdot\delta u_\epsilon \mathrm{d}x\right\}$$
$$= \int_\Omega (b_\epsilon\cdot\nabla)\delta b_\epsilon\cdot\delta u_\epsilon\,\mathrm{d}x + \int_\Omega (\delta b_\epsilon\cdot\nabla)b\cdot\delta u_\epsilon\,\mathrm{d}x.$$

Before starting to bound the dozen of integrals which we have, we note that, thanks to integration by parts, $\int_\Omega (u_\epsilon\cdot\nabla)\delta b_\epsilon\cdot\delta b_\epsilon\,\mathrm{d}x = 0$. Moreover, the first integrals on the righthand side of (295) and (296) are opposite, so that they will cancel each other when summing the three relations (295), (296) and (297) in order to apply Gronwall's lemma.

We will first handle the integrals in which the magnetic field does not appear before giving our attention to those three which do contain magnetic terms.

We start by the last integral in (297). Recall that $r \in L^\infty_T(H^{1+\gamma})$ for $0 \le \gamma < \beta$ and for some positive $\beta > 0$. Making use of HÖLDER's inequality, followed by the GN inequality, we get

$$\left|\int_\Omega (\nabla r \cdot \delta u_\epsilon)\delta r_\epsilon \, \mathrm{d}x\right| \le \|\delta r_\epsilon\|_{L^2}\|\delta u_\epsilon\|_{L^p}\|\nabla r\|_{L^q} \le \|\nabla r\|_{L^q}\|\delta u_\epsilon\|_{L^2}^{2/p}\|\nabla(\delta u_\epsilon)\|_{L^2}^{1-2/p} \tag{302}$$

$$\le C(q)\|\nabla r\|_{L^q}\left(\eta\|\nabla(\delta u_\epsilon)\|_{L^2}^2 + C(\eta)\|\delta r_\epsilon\|_{L^2}^{q'}\|\delta u_\epsilon\|_{L^2}^{2q'/p}\right) \tag{303}$$

where $\eta > 0$ is arbitrarily small and $p, q \ge 2$ are chosen so that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. The exponent q' is associated to q in YOUNG's inequality by $\frac{1}{q} + \frac{1}{q'} = 1$. Using YOUNG's inequality one more time with the exponents $\alpha = \frac{2(q-1)}{q}$ and $\beta = \frac{2(q-1)}{q-2}$ (which satisfy $\frac{1}{\alpha} + \frac{1}{\beta} = 1$),

$$\left|\int_\Omega (\nabla r \cdot \delta u_\epsilon)\delta r_\epsilon \, \mathrm{d}x\right| \le C(q)\|\nabla r\|_{L^q}\left(\eta\|\nabla(\delta u_\epsilon)\|_{L^2}^2 + C(\eta)E(t)\right). \tag{304}$$

Now since $\nabla r \in L^\infty_T(H^\gamma)$, we see that $\nabla r \in L^\infty_T(L^q)$ for q close enough to 2 by SOBOLEV embedding. For such q, it is always possible to find a $p \ge 2$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ so that all of the preceding inequalities are justified. *In fine*, we have the following inequality:

$$\left|\int_\Omega (\nabla r \cdot \delta u_\epsilon)\delta r_\epsilon \, \mathrm{d}x\right| \le C(\|r_0\|_{H^{1+\beta}}, T)\left(\eta\|\nabla(\delta u_\epsilon)\|_{L^2}^2 + C(\eta)E(t)\right). \tag{305}$$

Note that the dependency of the constant $\|r\|_{L^\infty_T(H^{1+\gamma})} = C(\|r_0\|_{H^{1+\beta}}, T)$ on the time T is very bad: we get from proposition 6.1 that (by taking $\gamma = \beta/2$ for instance), for any $h > 0$,

$$C(\|r_0\|_{H^{1+\beta}}, T) = C(\beta)\exp\left\{C(\beta)\left(\int_0^T \|\nabla u\|_{H^1}\,\mathrm{d}t\right)^2\right\} \le C(\beta)\exp\left\{C(\beta, h)(1 + T^h)\right\}. \tag{306}$$

We now look at the first integral on the second line of of (301). This integral is very similar to the previous one: expanding $r_\epsilon u_\epsilon$ into $ru + r\delta u_\epsilon + \delta r_\epsilon \delta u_\epsilon + \delta r_\epsilon u^\perp$, we see that

$$\int_\Omega (r_\epsilon u_\epsilon^\perp - r u^\perp) \cdot \delta u_\epsilon \, \mathrm{d}x = \int_\Omega \delta r_\epsilon u^\perp \cdot \delta u_\epsilon \, \mathrm{d}x \tag{307}$$

so that we can make the same computations by substituting u to ∇r. In fact, since $u \in L^\infty_T(H^1)$, (and not only $L^\infty_T(H^\gamma)$) it suffices to conduct the computations for any value of p and q, say $p = q = 4$. Then,

$$\left|\int_\Omega \delta r_\epsilon u^\perp \cdot \delta u_\epsilon \, \mathrm{d}x\right| \le \|u\|_{L^4}\|\delta r_\epsilon\|_{L^2}\|\delta u_\epsilon\|_{L^4} \le \|u\|_{L^\infty_T(H^1)}\left(\eta\|\nabla(\delta u_\epsilon)\|_{L^2} + C(\eta)\|\delta r_\epsilon\|_{L^2}^{4/3}\|\delta u_\epsilon\|_{L^2}^{2/3}\right) \tag{308}$$

$$\le C(\|u_0\|_{H^1}, \|b_0\|_{H^1}, T)\left(\eta\|\nabla(\delta u_\epsilon)\|_{L^2}^2 + C(\eta)E(t)\right). \tag{309}$$

Note again the growth of the constant with respect to T. This time it is of polynomial order, which is negligible compared to the exponential growth we had in the previous integral.

We focus our attention on the last integral in the first line of (297). Using the GN inequality again,

$$\left|\int_\Omega (\delta u_\epsilon \cdot \nabla) u \cdot \delta u_\epsilon \, \mathrm{d}x\right| \le \|\delta u_\epsilon\|_{L^4}^2\|\nabla u\|_{L^2} \le C\|\nabla(\delta u_\epsilon)\|_{L^2}\|\delta u_\epsilon\|_{L^2}\|\nabla u\|_{L^2} \tag{310}$$

and YOUNG's inequality gives, for any $\eta > 0$,

$$\left|\int_\Omega (\delta u_\epsilon \cdot \nabla) u \cdot \delta u_\epsilon \, \mathrm{d}x\right| \le \eta\|\nabla(\delta u_\epsilon)\|_{L^2}^2 + C(\eta)\|\nabla u\|_{L^2}^2\|\delta u_\epsilon\|_{L^2}^2. \tag{311}$$

Remembering that $\nabla u \in L^2(L^2)$, we obtain an integrable function $M_1(t) = C(\eta)\|\nabla u(t)\|_{L^2}^2 \in L^1(t \geq 0)$ whose $L^1(t \geq 0)$ norm depends on η, $\|u_0\|_{H^1}$ and $\|b_0\|_{H^1}$.

$$\left|\int_\Omega (\delta u_\epsilon \cdot \nabla) u \cdot \delta u_\epsilon \, \mathrm{d}x\right| \leq \eta \|\nabla(\delta u_\epsilon)\|_{L^2}^2 + M_1(t)E(t). \tag{312}$$

We study the first integral in the brackets (that is the second one on the second line of (301)). Using HÖLDER's inequality with $\frac{1}{\infty} + \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$, followed by the GN inequality,

$$\left|\epsilon \int_\Omega r_\epsilon (u_\epsilon \cdot \nabla) u \cdot \delta u_\epsilon \, \mathrm{d}x\right| \leq \epsilon \|r_\epsilon\|_{L^\infty} \|u_\epsilon\|_{L^4} \|\nabla u\|_{L^2} \|\delta u_\epsilon\|_{L^4} \tag{313}$$

$$\leq \epsilon C(\|r_{0,\epsilon}\|_{L^\infty}) \|u_\epsilon\|_{L^2}^{1/2} \|\nabla u_\epsilon\|_{L^2}^{1/2} \|\nabla u\|_{L^2} \|\delta u_\epsilon\|_{L^2}^{1/2} \|\nabla(\delta u_\epsilon)\|_{L^2}^{1/2} \tag{314}$$

Recall that $u_\epsilon \in L^\infty(L^2)$. Using YOUNG's inequality a first time with coefficients $\frac{1}{4} + \frac{1}{4/3} = 1$,

$$\begin{aligned}\left|\epsilon \int_\Omega r_\epsilon (u_\epsilon \cdot \nabla) u \cdot \delta u_\epsilon \, \mathrm{d}x\right| &\leq \eta \|\nabla(\delta u_\epsilon)\|_{L^2}^2 \\ &\quad + \epsilon^{4/3} C(\eta, \|r_{0,\epsilon}\|_{L^\infty}, \|u_{0,\epsilon}\|_{L^2}, \|b_{0,\epsilon}\|_{L^2}) \|\nabla u_\epsilon\|_{L^2}^{2/3} \|\nabla u\|_{L^2}^{4/3} \|\delta u_\epsilon\|_{L^2}^{2/3},\end{aligned} \tag{315}$$

and a second time on the second summand with coefficients $\frac{1}{3} + \frac{1}{3/2} = 1$,

$$\begin{aligned}\left|\epsilon \int_\Omega r_\epsilon (u_\epsilon \cdot \nabla) u \cdot \delta u_\epsilon \, \mathrm{d}x\right| &\leq \eta \|\nabla(\delta u_\epsilon)\|_{L^2}^2 + \|\delta u_\epsilon\|_{L^2}^2 \|\nabla u_\epsilon\|_{L^2}^2 \\ &\quad + \epsilon^2 C(\eta, \|r_{0,\epsilon}\|_{L^\infty}, \|u_{0,\epsilon}\|_{L^2}, \|b_{0,\epsilon}\|_{L^2}) \|\nabla u\|_{L^2}^2.\end{aligned} \tag{316}$$

Since $\nabla u_\epsilon, \nabla u \in L^2(L^2)$, we can write

$$\left|\epsilon \int_\Omega r_\epsilon (u_\epsilon \cdot \nabla) u \cdot \delta u_\epsilon \, \mathrm{d}x\right| \leq \eta \|\nabla(\delta u_\epsilon)\|_{L^2}^2 + M_1(t)E(t) + \epsilon^2 M_1(t). \tag{317}$$

In this last inequality, $M_1(t)$ generically denotes a $L^1(t \geq 0)$ function with $\|M_1\|_{L^1(\mathbb{R}_+)}$ depending on $(\eta, \|r_{0,\epsilon}\|_{L^\infty}, \|u_{0,\epsilon}\|_{L^2}, \|b_{0,\epsilon}\|_{L^2})$.

We study the second integral in the brackets, the last one in the second line of (301).

$$\epsilon \left|\int_\Omega r_\epsilon \partial_t u \cdot \delta u_\epsilon \, \mathrm{d}x\right| \leq \epsilon \|\partial_t u\|_{L^2} \|r_\epsilon\|_{L^\infty} \|\delta u_\epsilon\|_{L^2} \tag{318}$$

$$\leq \epsilon^2 C(\|r_{0,\epsilon}\|_{L^\infty}) \|\partial_t u\|_{L^2}^2 + E(t) \tag{319}$$

$$\leq \epsilon^2 N_1(t) + E(t), \tag{320}$$

where $\|N_1\|_{L^1_T}$ grows at polynomial speed $1 + T^h$ and depends on $(h, \|r_{0,\epsilon}\|_{L^\infty}, \|u_0\|_{H^1}, \|b_0\|_{H^1}, T)$.

We look at the integral containing the viscosity and resistivity terms. Since (ρ_ϵ) is bounded in $L^\infty(L^\infty)$, the C^1 functions ν and μ are Liphschitz on the range of the ρ_ϵ with a constant smaller than

$$L = \sup_{0 \leq \rho \leq \rho^*} |\nu'(\rho)| + \sup_{0 \leq \rho \leq \rho^*} |\mu'(\rho)|, \tag{321}$$

which is independent of ϵ. Hence,

$$\left|\int_\Omega \big(\nu(1 + \epsilon r_\epsilon) - \nu(1)\big) \nabla u \cdot \nabla \delta u_\epsilon \, \mathrm{d}x\right| \leq \epsilon L \|r_\epsilon\|_{L^\infty} \|\nabla u_\epsilon\|_{L^2} \|\delta u_\epsilon\|_{L^2} \tag{322}$$

$$\leq \epsilon^2 C(\eta) L^2 \|r_\epsilon\|_{L^\infty} \|\nabla u_\epsilon\|_{L^2}^2 + \eta \|\nabla \delta u_\epsilon\|_{L^2}^2 = \epsilon^2 M_1(t) + \|\nabla \delta u_\epsilon\|_{L^2}^2$$

and

$$\left| \int_\Omega \big(\mu(1+\epsilon r_\epsilon) - \mu(1)\big) \nabla b \cdot \nabla \delta b_\epsilon \, \mathrm{d}x \right| \leq \epsilon^2 M_1(t) + \|\nabla \delta b_\epsilon\|_{L^2}^2, \tag{323}$$

with $M_1(t) \in L^1(t \geq 0)$ and $\|M_1\|_{L^1(\mathbb{R}_+)}$ depending only on η, L, $\|u_{0,\epsilon}\|_{L^2}$ and $\|b_{0,\epsilon}\|_{L^2}$.

Now for the three integrals which contain magnetic field terms. We bound the last integral in (301) by

$$\left| \int_\Omega (\delta b_\epsilon \cdot \nabla) b \cdot \nabla \delta u_\epsilon \, \mathrm{d}x \right| \leq \|\delta b_\epsilon\|_{L^2} \|\nabla b\|_{L^4} \|\delta u_\epsilon\|_{L^4} \tag{324}$$

$$\leq \|\delta b_\epsilon\|_{L^2}^2 + \|\nabla b\|_{L^2} \|\Delta b\|_{L^2} \|\delta u_\epsilon\|_{L^2} \|\nabla \delta u_\epsilon\|_{L^2} \tag{325}$$

$$\leq \eta \|\nabla \delta u_\epsilon\|_{L^2}^2 + \|\delta b_\epsilon\|_{L^2}^2 + C(\eta) \|\nabla b\|_{L^2}^2 \|\Delta b\|_{L^2}^2 \|\delta u_\epsilon\|_{L^2}^2 \tag{326}$$

$$= \eta \|\nabla \delta u_\epsilon\|_{L^2}^2 + N_1(t) E(t), \tag{327}$$

where $N_1 \in L^1_T$ with $\|N_1\|_{L^1_T}$ being a function of $(\eta, \|u_0\|_{H^1}, \|b_0\|_{H^1}, T)$ and grows at polynomial speed in T.

The third integral in (296) is bounded in a quasi-identical way

$$\left| \int_\Omega (\delta u_\epsilon \cdot \nabla) b \cdot \delta b_\epsilon \, \mathrm{d}x \right| \leq \eta \|\nabla \delta u_\epsilon\|_{L^2}^2 + N_1(t) E(t). \tag{328}$$

Finally, we are left with the last integral in (296).

$$\left| \int_\Omega (\delta b_\epsilon \cdot \nabla) u \cdot \delta b_\epsilon \, \mathrm{d}x \right| \leq \|\nabla u\|_{L^2} \| \|\delta b_\epsilon\|_{L^4}^2 \tag{329}$$

$$\leq \|\nabla u\|_{L^2}^2 \|\delta b_\epsilon\|_{L^2} \|\nabla \delta b_\epsilon\|_{L^2} \tag{330}$$

$$\leq \eta \|\nabla \delta b_\epsilon\|_{L^2}^2 + C(\eta) \|\nabla u\|_{L^2}^2 \|\delta b_\epsilon\|_{L^2}^2 \tag{331}$$

$$= \eta \|\nabla \delta b_\epsilon\|_{L^2}^2 + M_1(t) \|\delta b_\epsilon\|_{L^2}^2, \tag{332}$$

where $M_1(t) \in L^1(t \geq 0)$ and $\|M_1\|_{L^1(\mathbb{R}_+)} = C(\eta, \|u_0\|_{H^1}, \|b_0\|_{H^1})$.

Piecing all these inequalities together with η small enough, say $\eta = \frac{1}{100} \min\{\nu_*, \mu_*\}$, we find a differential inequality on our energy:

$$E'(t) + \frac{1}{2} \int_\Omega \left\{ \nu_* |\nabla \delta u_\epsilon|^2 + \mu_* |\nabla \delta b_\epsilon|^2 \right\} \mathrm{d}x \leq \mathcal{M}_1(t) E(t) + \epsilon^2 \mathcal{N}_1(t), \tag{333}$$

with $\mathcal{M}_1$ and $\mathcal{N}_1$ being locally integrable functions on $\mathbb{R}_+$, the growth of $\|\mathcal{N}_1\|_{L^1_T}$ being at most polynomial in T and

$$\int_0^T \mathcal{M}_1(t) \, \mathrm{d}t \leq C \exp\left\{ C(1 + T^h) \right\}. \tag{334}$$

Use of GRONWALL's lemma on this differential inequality provides the result we covet: for all $0 \leq t \leq T$,

$$\begin{aligned} \|\delta r_\epsilon(t)\|_{L^2}^2 + \|\delta u_\epsilon(t)\|_{L^2}^2 + \|\delta b_\epsilon(t)\|_{L^2}^2 + \frac{1}{2} \int_0^t \left\{ \nu_* |\nabla \delta u_\epsilon|^2 + \mu_* |\nabla \delta b_\epsilon|^2 \right\} \mathrm{d}x \mathrm{d}s \\ \leq C \exp\left(C e^{T^h} \right) \left\{ \|\delta r_{0,\epsilon}\|_{L^2}^2 + \|\delta u_{0,\epsilon}\|_{L^2}^2 + \|\delta b_{0,\epsilon}\|_{L^2}^2 + \epsilon^2 \right\} \end{aligned} \tag{335}$$

where the constant C depends on $(h, L, \nu*, \mu*, \|u_0\|_{H^1}, \|b_0\|_{H^1}, \|r_0\|_{H^{1+\beta}}, \|r_0\|_{L^\infty})$.

□

A Appendix – Functional Inequalities and Useful Lemmas

Lemma A.1 (Gagliardo-Nirenberg inequality). *Let $2 \leq p < +\infty$ such that $\frac{1}{p} > \frac{1}{2} - \frac{1}{d}$. Then,*

$$\|u\|_{L^p} \leq C(p)\|u\|_{L^2}^{2/p}\|\nabla u\|_{L^2}^{1-2/p}. \tag{336}$$

Lemma A.2 (Sobolev Embeddings). *1. For all $p \geq 2$, $H^1 \subset L^p$.*

2. For all $0 \leq s < 1$ and $2 \leq p \leq \frac{2}{1-s}$, $H^s \subset L^p$.

3. For any $s > 1$, $H^s \subset L^\infty \cap C^0$.

Lemma A.3 (Interpolation). *Let $s_1, s_2 \in \mathbb{R}$ and $0 \leq \theta \leq 1$. Then if $s = \theta s_1 + (1-\theta)s_2$,*

$$\forall f \in H^{s_1} \cap H^{s_2}, \quad \|f\|_{H^s} \leq \|f\|_{H^{s_1}}^{\theta}\|f\|_{H^{s_2}}^{1-\theta}. \tag{337}$$

B Appendix – Fourier and harmonic analysis toolbox

We recall here the main ideas of Littlewood-Paley theory, which we exploited in the previous analysis. We refer e.g. to Chapter 2 of [1] for details. For simplicity of exposition, let us deal with the $\mathbb{R}^d$ case; however, the whole construction can be adapted also to the d-dimensional torus $\mathbb{T}^d$.

First of all, let us introduce the so called "Littlewood-Paley decomposition", based on a non-homogeneous dyadic partition of unity with respect to the Fourier variable. We fix a smooth radial function χ supported in the ball $B(0,2)$, equal to 1 in a neighborhood of $B(0,1)$ and such that $r \mapsto \chi(r\,e)$ is nonincreasing over $\mathbb{R}_+$ for all unitary vectors $e \in \mathbb{R}^d$. Set $\varphi(\xi) = \chi(\xi) - \chi(2\xi)$ and $\varphi_j(\xi) := \varphi(2^{-j}\xi)$ for all $j \geq 0$.

The dyadic blocks $(\Delta_j)_{j\in\mathbb{Z}}$ are defined by[3]

$$\Delta_j := 0 \quad \text{if } j \leq -2, \qquad \Delta_{-1} := \chi(D) \qquad \text{and} \qquad \Delta_j := \varphi(2^{-j}D) \quad \text{if } j \geq 0\,.$$

We also introduce the following low frequency cut-off operator:

$$S_j u := \chi(2^{-j}D) = \sum_{k \leq j-1} \Delta_k \qquad \text{for} \qquad j \geq 0\,. \tag{338}$$

The following classical property holds true: for any $u \in \mathcal{S}'$, then one has the equality $u = \sum_j \Delta_j u$ in the sense of $\mathcal{S}'$. Let us also mention the so-called Bernstein *inequalities*, which explain the way derivatives act on spectrally localized functions.

Lemma B.1. *Let $0 < r < R$. A constant C exists so that, for any nonnegative integer k, any couple (p,q) in $[1,+\infty]^2$, with $p \leq q$, and any function $u \in L^p$, we have, for all $\lambda > 0$,*

$$\operatorname{supp}\widehat{u} \subset B(0,\lambda R) \quad \Longrightarrow \quad \|\nabla^k u\|_{L^q} \leq C^{k+1}\lambda^{k+d\left(\frac{1}{p}-\frac{1}{q}\right)}\|u\|_{L^p}\,;$$
$$\operatorname{supp}\widehat{u} \subset \{\xi \in \mathbb{R}^d \,|\, r\lambda \leq |\xi| \leq R\lambda\} \quad \Longrightarrow \quad C^{-k-1}\lambda^k\|u\|_{L^p} \leq \|\nabla^k u\|_{L^p} \leq C^{k+1}\lambda^k\|u\|_{L^p}\,.$$

By use of Littlewood-Paley decomposition, we can define the class of Besov spaces. Let $s \in \mathbb{R}$ and $1 \leq p, r \leq +\infty$. The *non-homogeneous* Besov *space* $B^s_{p,r}$ is defined as the subset of tempered distributions u for which

$$\|u\|_{B^s_{p,r}} := \left\|\left(2^{js}\,\|\Delta_j u\|_{L^p}\right)_{j\geq -1}\right\|_{\ell^r} < +\infty\,.$$

[3] Throughout we agree that $f(D)$ stands for the pseudo-differential operator $u \mapsto \mathcal{F}^{-1}(f\,\mathcal{F}u)$.

Besov spaces are interpolation spaces between Sobolev spaces. In fact, for any $k \in \mathbb{N}$ and $p \in [1, +\infty]$ we have the following chain of continuous embeddings:

$$B^k_{p,1} \hookrightarrow W^{k,p} \hookrightarrow B^k_{p,\infty}\,,$$

where $W^{k,p}$ denotes the classical Sobolev space of L^p functions with all the derivatives up to the order k in L^p. Moreover, for all $s \in \mathbb{R}$ we have the equivalence $B^s_{2,2} \equiv H^s$, with

$$\|f\|_{H^s} \sim \left(\sum_{j \geq -1} 2^{2js} \, \|\Delta_j f\|^2_{L^2} \right)^{1/2} . \tag{339}$$

As an immediate consequence of the first Bernstein inequality, one gets the following embedding result.

Proposition B.2. *The space $B^{s_1}_{p_1,r_1}$ is continuously embedded in the space $B^{s_2}_{p_2,r_2}$ for all indices satisfying $p_1 \leq p_2$ and*

$$s_2 \, < \, s_1 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \qquad \text{or} \qquad s_2 \, = \, s_1 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \quad \text{and} \quad r_1 \, \leq \, r_2\,.$$

We recall also Lemma 2.73 of [1].

Lemma B.3. *If $1 \leq r < +\infty$, for any $f \in B^s_{p,r}$ one has*

$$\lim_{j \to +\infty} \|f \, - \, S_j f\|_{B^s_{p,r}} \, = \, 0\,.$$

Let us now introduce the paraproduct operator (after J.-M. Bony, see [2]). Constructing the paraproduct operator relies on the observation that, formally, any product of two tempered distributions u and v, may be decomposed into

$$u\,v \; = \; T_u(v) \, + \, T_v(u) \, + \, R(u,v)\,, \tag{340}$$

where we have defined

$$T_u(v) \, := \, \sum_j S_{j-1} u \Delta_j v, \qquad \text{and} \qquad R(u,v) \, := \, \sum_j \sum_{|j'-j| \leq 1} \Delta_j u \, \Delta_{j'} v\,.$$

The above operator T is called "paraproduct" whereas R is called "remainder". The paraproduct and remainder operators have many nice continuity properties. The following ones have been of constant use in this paper (see the proof in e.g. Chapter 2 of [1]).

Proposition B.4. *For any $(s,p,r) \in \mathbb{R} \times [1,\infty]^2$ and $t > 0$, the paraproduct operator T maps continuously $L^\infty \times B^s_{p,r}$ in $B^s_{p,r}$ and $B^{-t}_{\infty,\infty} \times B^s_{p,r}$ in $B^{s-t}_{p,r}$. Moreover, the following estimates hold:*

$$\|T_u(v)\|_{B^s_{p,r}} \, \leq \, C \, \|u\|_{L^\infty} \, \|\nabla v\|_{B^{s-1}_{p,r}} \qquad \text{and} \qquad \|T_u(v)\|_{B^{s-t}_{p,r}} \, \leq \, C \|u\|_{B^{-t}_{\infty,\infty}} \, \|\nabla v\|_{B^{s-1}_{p,r}}\,.$$

For any (s_1,p_1,r_1) and (s_2,p_2,r_2) in $\mathbb{R} \times [1,\infty]^2$ such that $s_1 + s_2 > 0$, $1/p := 1/p_1 + 1/p_2 \leq 1$ and $1/r := 1/r_1 + 1/r_2 \leq 1$, the remainder operator R maps continuously $B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$ into $B^{s_1+s_2}_{p,r}$. In the case $s_1 + s_2 = 0$, provided $r = 1$, operator R is continuous from $B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$ with values in $B^0_{p,\infty}$.

As a corollary of the previous proposition, we deduce the following continuity properties of the product in Sobolev spaces, which have been used in the course of the analysis. In the statement, we limit ourselves to the case of space dimension $d = 2$, the only relevant one for this study.

Lemma B.5. *We work in two dimensions of space $d = 2$. For appropriate f and g,*

1. *For $s \in \mathbb{R}$ and $t > 0$, $\|T_f(g)\|_{H^{s-t}} \leq C\|f\|_{H^{1-t}}\|g\|_{H^s}$.*
2. *For $s \in \mathbb{R}$, $\|T_f(g)\|_{H^s} \leq C\|f\|_{L^\infty}\|g\|_{H^s}$.*
3. *For $s_1, s_2 \in \mathbb{R}$ such that $s_1 + s_2 > 0$, $\|R(f,g)\|_{H^{s_1+s_2-1}} \leq C\|f\|_{H^{s_1}}\|g\|_{H^{s_2}}$.*

As a consequence, we see that the space $H^{1+\delta}$ is a Banach *algebra as soon as $\delta > 0$.*

Proof. We start by proving the first point. We get, from the second inequality in proposition B.4 that

$$\|T_f(g)\|_{H^{s-t}} = \|T_f(g)\|_{B^{s-t}_{2,2}} \leq C\|f\|_{B^{-t}_{\infty,\infty}}\|\nabla g\|_{B^{s-1}_{2,2}} = C\|f\|_{B^{-t}_{\infty,\infty}}\|\nabla g\|_{H^{s-1}}. \tag{341}$$

Next, because $d = 2$, proposition B.2 gives the embedding $H^{1-t} = B^{1-t}_{2,2} \subset B^{-t}_{\infty,\infty}$ and we get the first inequality $\|T_f(g)\|_{H^{s-t}} \leq C\|f\|_{H^{1-t}}\|g\|_{H^s}$.

Next, using the first inequality in proposition B.4, we have

$$\|T_f(g)\|_{H^s} = \|T_f(g)\|_{B^{s-t}_{2,2}} \leq C\|f\|_{L^\infty}\|\nabla g\|_{B^{s-1}2,2} \leq C\|f\|_{L^\infty}\|g\|_{H^s}, \tag{342}$$

which proves the second point.

Finally, using proposition B.4 to estimate the remainder term, we get, because we have assumed that $s_1 + s_2 > 0$,

$$\|R(f,g)\|_{B^{s_1+s_2}_{1,1}} \leq C\|f\|_{H^{s_1}}\|g\|_{H^{s_2}}. \tag{343}$$

Proposition B.2 provides the embedding $B^{s_1+s_2}_{1,1} \subset B^{s_1+s_2-1}_{2,2} = H^{s_1+s_2-1}$, which gives the last inequality $\|R(f,g)\|_{H^{s_1+s_2-1}} \leq C\|f\|_{H^{s_1}}\|g\|_{H^{s_2}}$. □

Another useful application of the Bernstein inequalities is an interpolation inequality for the Lebesgue L^∞ space.

Proposition B.6. *Let $f \in L^2 \cap H^2$. There exists a constant $C = C(d) > 0$ and an exponent $\alpha = \alpha(d) = \frac{d/2}{1+d/2}$ such that*

$$\|f\|_{L^\infty} \leq C\|f\|_{L^2}^{1-\alpha}\|\Delta f\|_{L^2}^{\alpha}. \tag{344}$$

If $d = 2$ then $\alpha = 1/2$ and $\|f\|_{L^\infty} \leq C\|f\|_{L^2}^{1/2}\|\Delta f\|_{L^2}^{1/2}$.

Proof. The main idea of the proof is to look separately at the high and low frequencies. Let $N \geq 1$ be an integer to be fixed later on. We write, thanks to the Littlewood-Paley decomposition,

$$\|f\|_{L^\infty} \leq \sum_{j<N}\|\Delta_j f\|_{L^\infty} + \sum_{j\geq N}\|\Delta_j f\|_{L^\infty}. \tag{345}$$

Applying the Bernstein inequalities to, on the one hand, the fist sum gives $\|\Delta_j f\|_{L^\infty} \leq C2^{jd/2}\|\Delta_j f\|_{L^2}$, and, on the other hand, to the second sum gives $\|\Delta_j\|_{L^2} \leq C2^{jd/2}2^{-2j}\|\Delta_j \Delta f\|_{L^2}$. Therefore

$$\|f\|_{L^\infty} \leq C\|f\|_{L^2}\sum_{j<N}2^{jd/2} + C\|\Delta f\|_{L^2}\sum_{j\geq N}2^{-j} \leq C\|f\|_{L^2}2^{Nd/2} + C\|\Delta f\|_{L^2}2^{-Nj}. \tag{346}$$

By choosing N so that $2^{N(\frac{d}{2}+1)} \approx \frac{\|\Delta f\|_{L^2}}{\|f\|_{L^2}}$ (say that N is the largest integer such that $2^{N(\frac{d}{2}+1)}$ is smaller than $\frac{\|\Delta f\|_{L^2}}{\|f\|_{L^2}}$) we get the desired inequality

$$\|f\|_{L^\infty} \leq C\|f\|_{L^2}\left(\frac{\|\Delta f\|_{L^2}}{\|f\|_{L^2}}\right)^{\frac{d/2}{1+d/2}} + C\|\Delta f\|_{L^2}\left(\frac{\|f\|_{L^2}}{\|\Delta f\|_{L^2}}\right)^{\frac{1}{1+d/2}} \leq C\|f\|_{L^2}^{1-\alpha}\|\Delta f\|_{L^2}^{\alpha}. \tag{347}$$

□

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