Spectral Analysis of Non-Relativistic QED

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States of one Electron and Photons

The Hilbert Space $\mathcal{H}$ is the space of sequences

$$\psi = (\psi^{(0)}, \psi^{(1)}, \dots), \qquad \sum_{n=0}^{\infty} \|\psi^{(n)}\|^2 < \infty$$

$\psi^{(0)} = \psi^{(0)}(x)$	one electron and ***zero*** photons
$\psi^{(n)} = \psi^{(n)}(x, k_1, \lambda_1, \dots, k_n, \lambda_n)$	one electron and n photons

where $k_n \in \mathbb{R}^3$ is the wave-vector (momentum) of the n-th photons and $\lambda_n \in \{1, 2\}$ denotes its polarization.

$$\begin{aligned} (\psi^{(0)}, 0, 0, \dots) &= \text{zero-photon state} \\ (0, \dots, \psi^{(n)}, 0, \dots) &= n - \text{photon-state} \end{aligned}$$

Hamiltonian of the Hydrogen Atom

The Hamilton operator $H : D(H) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\begin{aligned} H &= (-i\nabla_x + \alpha^{3/2}A)^2 - \frac{Z}{|x|} + H_f \\ &= \underbrace{\left(-\Delta - \frac{Z}{|x|}\right)}_{H_{\text{el}}} + H_f + \alpha^{3/2}W_\alpha, \end{aligned}$$

where $D(H) = D(-\Delta + H_f)$, $x \in \mathbb{R}^3$ the position of the electron, $\alpha > 0$ fein-structure constant ($\alpha = e^2/\hbar c \simeq 1/137$),

$$\begin{aligned} H_f &: \text{field energy}, \\ A = (A_1, A_2, A_3) &: \text{quantized vector potential} \end{aligned}$$

Field Energy and Vector-Potential

Field energy. Identify $(0, \dots, \psi^{(n)}, 0 \dots)$ with $\psi^{(n)}$. Then

$$\begin{aligned} H_f \psi^{(0)} &= 0 \\ H_f \psi^{(n)}(x, k_1, \dots, k_n) &= \sum_{j=1}^{n} |k_j| \psi^{(n)}(x, k_1, \dots, k_n). \end{aligned}$$

Quant. vector potential. $A_j = a_j + a_j^*$, $j = 1, 2, 3$, where a_j and a_j^* act like shift-operators:

$$\begin{aligned} a_j \psi &= (\tilde{\psi}^{(0)}, \tilde{\psi}^{(1)}, \tilde{\psi}^{(2)}, \dots) \quad \text{annihilation operator} \\ & \qquad \nwarrow \quad \nwarrow \quad \nwarrow \\ \psi &= (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \psi^{(3)}, \dots) \\ & \qquad \searrow \quad \searrow \quad \searrow \\ a_j^* \psi &= (0, \bar{\psi}^{(1)}, \bar{\psi}^{(2)}, \bar{\psi}^{(3)} \dots) \quad \text{creation operator} \end{aligned}$$

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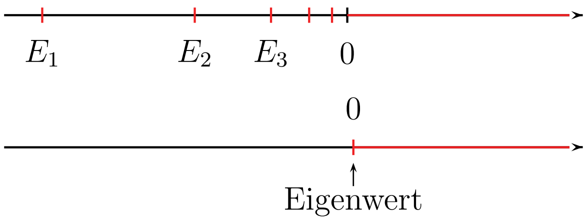
Spectrum of H: $\alpha = 0$.

$\sigma(H_{\mathrm{el}})$

$\sigma(H_f)$



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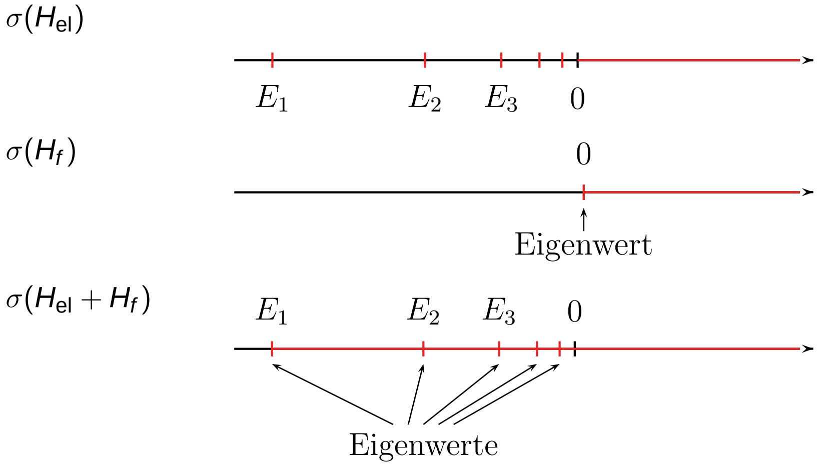

Eigenvectors: $(H_{\text{el}} + H_f)(\psi_n^{\text{Sch}}, 0, \ldots) = E_n(\psi_n^{\text{Sch}}, 0, \ldots)$.

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Self-adjointness. $H = H^*$ on $D(H) = D(-\Delta + H_f)$.
(α small : Kato-Rellich / α arbitrary : via construction von e^{-Ht} by stoch. methods)

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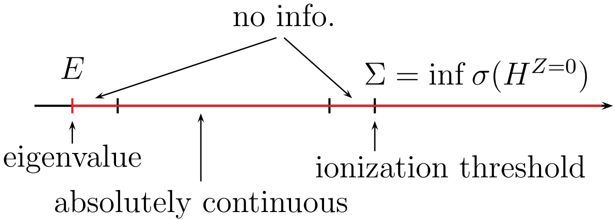

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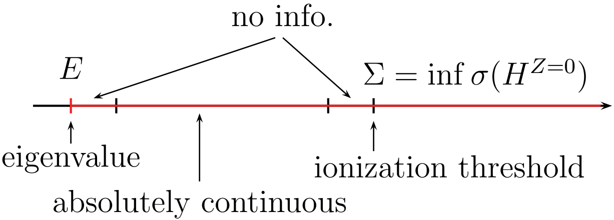

Huebner, Spohn / Bach, Fröhlich, Sigal / Skibsted / Dereziński, Jacksic / Lieb, Loss, Griesemer,...

Conjugate Operator Theory

Amrein, Boutet de Monvel, Georgescu / Sahbani

Assumptions. Let H and B be self-adjoint operators in a Hilbert space $\mathcal{H}$, and let $U \subset \mathbb{R}$ be open.

- H is locally of class $C^2(B)$ in U: The map

$$s \mapsto e^{-iBs} f(H) e^{iBs} \varphi$$

  is twice continuously differentiable for all $\varphi \in \mathcal{H}$ and for all $f \in C_0^\infty(U)$.

- Mourre estimate: For every $\lambda \in U$ there exists a neighborhood $\Delta \ni \lambda$, ($\bar{\Delta} \subset U$), and a number $\beta > 0$ such that

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Theorem (Limiting absorption principle)

For all $s > 1/2$ and all $\varphi, \psi \in \mathcal{H}$, the limit

$$\lim_{\varepsilon \downarrow 0} \langle \varphi, \langle B \rangle^{-s} (H - \lambda \pm i\varepsilon)^{-1} \langle B \rangle^{-s} \psi \rangle$$

exists uniformly for λ in compact subsets of U ($\langle B \rangle = \sqrt{B^2 + 1}$). In particular, the spectrum of H is purely absolutely continuous in U.

The conjugate operator

Bach, Fröhlich, Sigal

$B =$ second quantized dilation generator, that is,

$$B = \mathrm{d}\Gamma(b), \qquad b = \frac{1}{2}(k \cdot y + y \cdot k)$$

where $y := i\nabla_k$. Then

$$[H_f, iB] = H_f > 0 \quad \text{on } [\text{vacuum}]^\perp.$$

With interaction: $H = H_0 + \alpha^{3/2} W_\alpha$

$$\begin{aligned} [H, iB] &= H_f + \alpha^{3/2}[W_\alpha, iB] \\ &\geq \frac{1}{2} H_f + O(\alpha^3) \end{aligned}$$

No positive commutator below $E + O(\alpha^3)$!

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$\hat{B}$ = second quantized radial derivative, that is,

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where $\hat{k} = k/|k|$, $y = i\nabla_k$. Then

$$[H_f, i\hat{B}] = N \geq 1 \quad \text{on [vacuum]}^\perp.$$

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$\hat{B}$ is symmetric but not self-adjoint!

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New Mourre Estimate

Assumptions.

- $e_1 = \inf \sigma(H_{el})$ is simple and isolated.
- $\alpha \ll 1$.

Let $e_2 = \inf \sigma(H_{el}) \setminus \{e_1\}$ and $e_{gap} = e_2 - e_1$.

THEOREM

- *The Hamiltonian H is locally of class $C^2(B)$ on the interval $(-\infty, e_{gap}/3)$.*
- *If $\sigma \leq e_{gap}/2$ and $\Delta = [\sigma/3, 2\sigma/3]$, then*

$$E_\Delta(H-E)[H, iB]E_\Delta(H-E) \geq \frac{\sigma}{10} E_\Delta(H-E).$$

In particular, the spectrum of H is purely absolutely continuous in $U = (E, E + e_{gap}/3)$.

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Ingredients for proving the Mourre estimate

The IR-cutoff Hamiltonian. Let H_σ be the Hamiltonian H with an infrared cutoff at $|k| = \sigma$. Then

$$H_\sigma = H^\sigma \otimes 1 + 1 \otimes H_{f,\sigma}$$

w.r.to $\mathcal{H} = \mathcal{H}^\sigma \otimes \mathcal{F}_\sigma$, where $\mathcal{F}_\sigma$ is the bosonic Fock space over $L^2(|k| \leq \sigma, \mathbb{C}^2)$.

Key ingredient. H^σ has the gap $(E_\sigma, E_\sigma + \sigma)$ in its spectrum above $E_\sigma = \inf \sigma(H_\sigma) = \inf \sigma(H^\sigma)$. It follows that

$$\boxed{f_\Delta(H_\sigma - E_\sigma) = P^\sigma \otimes f_\Delta(H_{f,\sigma})}$$

for every function f_Δ with support in $(0, \sigma)$. P^σ = ground state projection of H^σ.

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Strategy for proving the Mourre estimate

Let f_Δ be a smoothed characteristic function of the interval $[\sigma/3, 2\sigma/3]$.

Step 1.

$$f_\Delta(H_\sigma - E_\sigma)[H, iB]f_\Delta(H_\sigma - E_\sigma) \geq \frac{\sigma}{8} f_\Delta(H_\sigma - E_\sigma)^2.$$

Step 2.

$$\|f_\Delta(H - E) - f_\Delta(H_\sigma - E_\sigma)\| = O(\alpha^{3/2}\sigma).$$

Steps 1 and 2 prove the Theorem for $\alpha \ll 1$.

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Prove of Step 1

We split $B = B_\sigma + B^\sigma$, according to $1 = \chi(|k| \leq \sigma) + \chi(|k| \geq \sigma)$. Then

$$f_\Delta(H_\sigma - E_\sigma)[H, iB^\sigma]f_\Delta(H_\sigma - E_\sigma) = 0.$$

as a consequence of a Virial Theorem, while

$$f_\Delta(H_\sigma - E_\sigma)[H, iB_\sigma]f_\Delta(H_\sigma - E_\sigma) \geq \frac{\sigma}{8} f_\Delta(H_\sigma - E_\sigma)^2.$$

by straightforward estimates using $f_\Delta(H_\sigma - E_\sigma) = P^\sigma \otimes f_\Delta(H_{f,\sigma})$.

Second Approach using Renormalization

(so far only for QED in dipole approximation)

Feshbach-Schur Transform

Let $P^2 = P = P^*$, $\bar{P} = 1 - P$, and $H_{\bar{P}} = \bar{P}H\bar{P}$. If

$$H_{\bar{P}}^{-1} \restriction \bar{P}\mathcal{H}, \qquad \text{exists,}$$

then, with respect to $\mathcal{H} = P\mathcal{H} \oplus \overline{P}\mathcal{H}$,

$$H = \begin{pmatrix} 1 & PH\bar{P}H_{\bar{P}}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{F}_P(H) & 0 \\ 0 & H_{\bar{P}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ H_{\bar{P}}^{-1}\bar{P}HP & 1 \end{pmatrix},$$

where

$$\boxed{\mathcal{F}_P(H) = PHP - PH\bar{P}H_{\bar{P}}^{-1}\bar{P}HP.}$$

Hence, if $(H_{\bar{P}} - z)^{-1} \restriction \bar{P}\mathcal{H}$ exists, then

$$\text{LAP for } \mathcal{F}_P(H - z) \quad \Rightarrow \quad \text{LAP for } (H - z).$$

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Renormalization procedure

Step 1. Choose

$$P = P_{\text{el}} \otimes \chi(H_f \le 1),$$

P_{el} = ground state projection of H_{el}. Since $\text{rank}(P_{\text{el}})) = 1$,

$$H^{(0)}(z) := \mathcal{F}_P(H - z) \qquad \text{on } \mathcal{H}_{\text{red}} = \chi(H_f \le 1)\mathcal{F}.$$

Step 2. Let $P = \chi(H_f \le \rho)$ where $\rho < 1$, and set

$$H^{(1)}(z) := \underbrace{\frac{1}{\rho}\Gamma_\rho \mathcal{F}_P(H^{(0)}(z))\Gamma_\rho^*}_{\mathcal{R}_\rho(H^{(0)}(z))}$$

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Iterating the RG-Transform

Let $H^{(n)}(z) = \mathcal{R}_\rho^n(H^{(0)}(z))$. Then

$$H^{(n)}(z) = \underbrace{T^{(n)}(H_f, z)}_{\text{function of } H_f} + \underbrace{E^{(n)}(z)}_{\langle H^{(n)}(z)\rangle_\Omega} + \underbrace{W^{(n)}(z)}_{\to 0,\ (n\to\infty)} .$$

Mourre est. and LAP for $H^{(n)}(z)$ and $\mathrm{Re}(z) \in \Delta_n \subset (E, \infty)$.

Δ_n is determined by

- existence of $H^{(n)}(z)$ (bound on Δ_n from above)
- positivity of the Mourre constant (bound on Δ_n from below)

For $g \ll 1$ one can achieve that

$$\bigcup_{n=0}^{\infty} \Delta_n = (E, E + e_{\mathrm{gap}}/18).$$

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Hydrogen Atom and Scalar Bosons

Model.

$$H := H_{\text{el}} \otimes 1 + 1 \otimes H_f + g\phi(G_x),$$

$$\phi(G_x) := \int \frac{d^3k}{|k|^{1/2}} \left(\overline{\kappa(k)} e^{ik\cdot x} a(k) + \kappa(k) e^{-ik\cdot x} a(k)^* \right).$$

Assumptions.

- $e_1 = \inf \sigma(H_{\text{el}})$ is simple and isolated.
- There exists $\mu > 0$ such that

$$|\kappa(k)| = O(|k|^{\mu}), \qquad (k \to 0).$$

THEOREM

If $g \ll 1$, then for $\lambda \in (E, E + e_{\text{gap}}/18)$ and $s \in (1/2, 1)$

$$\langle B \rangle^{-s} (H - \lambda \pm i0)^{-1} \langle B \rangle^{-s}$$

exists and is Hölder-continuous of degree $(s - 1/2)$.