Elliptic Optimal Control Problems with Mixed Constraints

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supported by FWF

RICAM
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DMV Bonn
September 21, 2006

Overview

1. Introduction
2. Stability of Optimal Solutions
3. Concluding Remarks

Example Problem

Problem setting

$$\text{Minimize } \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u - u_d\|^2_{L^2(\Omega)}$$

$$\text{s.t. } \begin{cases} -\Delta y = u & \text{on } \Omega \\ y = 0 & \text{on } \Gamma \end{cases}$$

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$$\text{and } \begin{cases} u \ge 0 & \text{on } \Omega \\ y \ge y_c & \text{on } \Omega \end{cases}$$

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$$\text{and } \begin{cases} u \geq 0 & \text{on } \Omega \\ \varepsilon u + y \geq y_c & \text{on } \Omega \end{cases}$$

Mixed control-state constraints

- Lavrentiev-type regularization of state constraints
- $\varepsilon > 0$ fixed in this talk

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$$\text{Minimize } \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u - u_d\|^2_{L^2(\Omega)}$$

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Substitution trick?

new control $v := \varepsilon u + y \quad \rightsquigarrow v \geq y_c$

[Meyer, Tröltzsch]: 12th FGS Conference on Optimization, 2006

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$$\text{Minimize } \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u - u_d\|^2_{L^2(\Omega)}$$

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Substitution trick?

new control $v := \varepsilon u + y \quad \rightsquigarrow v \ge y_c$, but also $v - y \ge 0$

[Meyer, Tröltzsch]: 12th FGS Conference on Optimization, 2006

Known Results

Assumptions

- $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$ has $C^{1,1}$ boundary $\Gamma \Rightarrow y \in H^2(\Omega) \cap H_0^1(\Omega)$
- $y_d \in L^2(\Omega)$, $u_d \in L^\infty(\Omega)$, $y_c \in L^\infty(\Omega)$

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Theorem (Existence of regular Lagrange multipliers)

There exist $\mu_i \in L^\infty(\Omega)$, *such that*

$$\begin{cases} 0 \le \mu_1 \perp u \ge 0 & \text{on } \Omega \\ 0 \le \mu_2 \perp \varepsilon u + y - y_c \ge 0 & \text{on } \Omega \end{cases}$$

[Rösch, Tröltzsch]: SIAM Journal on Control and Optimization 45(2), 2006

Known Results

Assumptions

- $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$ has $C^{1,1}$ boundary $\Gamma \Rightarrow y \in H^2(\Omega) \cap H_0^1(\Omega)$
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Theorem (Existence of regular Lagrange multipliers)

There exist $\mu_i \in L^\infty(\Omega)$, $p \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta p = -(y - y_d) + \mu_2 & \text{on } \Omega \\ p = 0 & \text{on } \Gamma \end{cases}$$

$$\begin{cases} 0 \le \mu_1 \perp u \ge 0 & \text{on } \Omega \\ 0 \le \mu_2 \perp \varepsilon u + y - y_c \ge 0 & \text{on } \Omega \end{cases}$$

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Stability of Solutions under Perturbations

Perturbed problem setting

$$\text{Minimize } \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u - u_d\|^2_{L^2(\Omega)}$$

$$\text{s.t. } \begin{cases} -\Delta y = u & \text{on } \Omega \\ y = 0 & \text{on } \Gamma \end{cases}$$

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Stability of Solutions under Perturbations

Perturbed problem setting

$$\text{Minimize } \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u - u_d\|^2_{L^2(\Omega)} - (y, \delta_1) - (u, \delta_2)$$

$$\text{s.t. } \begin{cases} -\Delta y = u + \delta_3 & \text{on } \Omega \\ y = 0 & \text{on } \Gamma \end{cases}$$

$$\text{and } \begin{cases} u \geq 0 + \delta_4 & \text{on } \Omega \\ \varepsilon u + y \geq y_c + \delta_5 & \text{on } \Omega \end{cases}$$

Question

How does the optimal solution change with δ?

Stability of Solutions under Perturbations

Perturbed problem setting

$$\text{Minimize } \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u - u_d\|^2_{L^2(\Omega)} - (y, \delta_1) - (u, \delta_2)$$

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Motivation

- convergence of discretized solutions
- convergence of iterative methods for nonlinear problems

Standard Approach: Control Constraints Only

Optimality system

$$
\begin{aligned}
(\nabla p_\delta, \nabla v) + (y_\delta - y_d, v) - (\delta_1, v) &= 0 \quad \forall v \in H_0^1(\Omega) \\
\gamma(u_\delta - u_d, v) - (p_\delta, v) - (\mu_\delta, v) - (\delta_2, v) &= 0 \quad \forall v \in L^2(\Omega) \\
(\nabla y_\delta, \nabla v) - (u_\delta, v) - (\delta_3, v) &= 0 \quad \forall v \in H_0^1(\Omega) \\
0 \le \mu_\delta \perp u_\delta - \delta_4 &\ge 0 \quad \text{on } \Omega
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Essential estimate

[Malanowski, Tröltzsch]: Control and Cybernetics 29, 2000

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\begin{aligned}
(\nabla p_\delta, \nabla v) + (y_\delta - y_d, v) - (\delta_1, v) &= 0 \quad v = y_\delta - y_{\delta'} =: \delta y \\
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Essential estimate

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\|\delta y\|_2^2 \qquad + (\nabla \delta p, \nabla \delta y)
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= (\delta_1 - \delta_1', \delta y)
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Essential estimate

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\begin{aligned}
&\|\delta y\|_2^2 + \gamma \, \|\delta u\|_2^2 + (\nabla \delta p, \nabla \delta y) - (\delta p, \delta u) \\
&= (\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) \qquad + (u_\delta - u_{\delta'}, \delta \mu)
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Essential estimate

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\begin{aligned}
&\|\delta y\|_2^2 + \gamma\, \|\delta u\|_2^2 \\
&\le (\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) - (\delta_3 - \delta_3', \delta p) + (\delta_4 - \delta_4', \delta \mu)
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Essential estimate

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\begin{aligned}
&\|\delta y\|_2^2 + \gamma \, \|\delta u\|_2^2 \\
\le\; &(\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) - (\delta_3 - \delta_3', \delta p) + (\delta_4 - \delta_4', \delta \mu) \\
\le\; &\frac{1}{2} \|\delta_1 - \delta_1'\|_2^2 + \frac{1}{2} \|\delta y\|_2^2
\end{aligned}
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&\frac{1}{2} \|\delta y\|_2^2 + \gamma \, \|\delta u\|_2^2 \\
&\le (\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) - (\delta_3 - \delta_3', \delta p) + (\delta_4 - \delta_4', \delta \mu) \\
&\le \frac{1}{2} \, \|\delta_1 - \delta_1'\|_2^2
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Essential estimate

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\begin{aligned}
&\frac{1}{2}\|\delta y\|_2^2 + \gamma\,\|\delta u\|_2^2 \\
&\le (\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) - (\delta_3 - \delta_3', \delta p) + (\delta_4 - \delta_4', \delta \mu) \\
&\le \frac{1}{2}\,\|\delta_1 - \delta_1'\|_2^2 + \frac{1}{2\gamma}\|\delta_2 - \delta_2'\|_2^2 + \frac{\gamma}{2}\|\delta u\|_2^2
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&\frac{1}{2}\|\delta y\|_2^2 + \frac{\gamma}{2}\|\delta u\|_2^2 \\
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$$0 \le \mu_\delta \perp u_\delta - \delta_4 \ge 0 \quad \text{on } \Omega$$

Essential estimate

$$\frac{1}{2}\|\delta y\|_2^2 + \frac{\gamma}{2}\|\delta u\|_2^2$$
$$\le (\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) - (\delta_3 - \delta_3', \delta p) + (\delta_4 - \delta_4', \delta \mu)$$
$$\le \frac{1}{2}\|\delta_1 - \delta_1'\|_2^2 + \frac{1}{2\gamma}\|\delta_2 - \delta_2'\|_2^2 + \frac{1}{4\kappa}\|\delta_3 - \delta_3'\|_2^2 + \kappa\|\delta p\|_2^2$$

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$$\gamma(u_\delta - u_d, v) - (p_\delta, v) - (\mu_\delta, v) - (\delta_2, v) = 0 \qquad v = u_\delta - u_{\delta'} =: \delta u$$
$$(\nabla y_\delta, \nabla v) - (u_\delta, v) - (\delta_3, v) = 0 \qquad v = p_\delta - p_{\delta'} =: \delta p$$
$$0 \le \mu_\delta \perp u_\delta - \delta_4 \ge 0 \quad \text{on } \Omega$$

Essential estimate

$$\frac{1}{2}\|\delta y\|_2^2 + \frac{\gamma}{2}\|\delta u\|_2^2$$
$$\le (\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) - (\delta_3 - \delta_3', \delta p) + (\delta_4 - \delta_4', \delta \mu)$$
$$\le \frac{1}{2}\|\delta_1 - \delta_1'\|_2^2 + \frac{1}{2\gamma}\|\delta_2 - \delta_2'\|_2^2 + \frac{1}{4\kappa}\|\delta_3 - \delta_3'\|_2^2 + \kappa\|\delta p\|_2^2$$

[Malanowski, Tröltzsch]: Control and Cybernetics 29, 2000

Standard Approach: Control Constraints Only

Optimality system

$$\begin{aligned}
(\nabla p_\delta, \nabla v) + (y_\delta - y_d, v) - (\delta_1, v) &= 0 \qquad v = y_\delta - y_{\delta'} =: \delta y \\
\gamma(u_\delta - u_d, v) - (p_\delta, v) - (\mu_\delta, v) - (\delta_2, v) &= 0 \qquad v = u_\delta - u_{\delta'} =: \delta u \\
(\nabla y_\delta, \nabla v) - (u_\delta, v) - (\delta_3, v) &= 0 \qquad v = p_\delta - p_{\delta'} =: \delta p \\
0 \le \mu_\delta \perp u_\delta - \delta_4 &\ge 0 \quad \text{on } \Omega
\end{aligned}$$

Essential estimate

$$\begin{aligned}
&\frac{1}{4}\|\delta y\|_2^2 + \frac{\gamma}{2}\|\delta u\|_2^2 \\
&\le (\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) - (\delta_3 - \delta_3', \delta p) + (\delta_4 - \delta_4', \delta \mu) \\
&\le c_1\|\delta_1 - \delta_1'\|_2^2 + \frac{1}{2\gamma}\|\delta_2 - \delta_2'\|_2^2 + \frac{1}{4\kappa}\|\delta_3 - \delta_3'\|_2^2
\end{aligned}$$

[Malanowski, Tröltzsch]: Control and Cybernetics 29, 2000

Standard Approach: Control Constraints Only

Optimality system

$$
\begin{aligned}
(\nabla p_\delta, \nabla v) + (y_\delta - y_d, v) - (\delta_1, v) &= 0 \qquad v = y_\delta - y_{\delta'} =: \delta y \\
\gamma(u_\delta - u_d, v) - (p_\delta, v) - (\mu_\delta, v) - (\delta_2, v) &= 0 \qquad v = u_\delta - u_{\delta'} =: \delta u \\
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$$

Essential estimate

$$
\begin{aligned}
&\frac{1}{4}\|\delta y\|_2^2 + \frac{\gamma}{2}\|\delta u\|_2^2 \\
&\le (\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) - (\delta_3 - \delta_3', \delta p) + (\delta_4 - \delta_4', \delta \mu) \\
&\le c_1\|\delta_1 - \delta_1'\|_2^2 + \frac{1}{2\gamma}\|\delta_2 - \delta_2'\|_2^2 + \frac{1}{4\kappa}\|\delta_3 - \delta_3'\|_2^2
\end{aligned}
$$

[Malanowski, Tröltzsch]: Control and Cybernetics 29, 2000

Standard Approach: Control Constraints Only

Optimality system

$$(\nabla p_\delta, \nabla v) + (y_\delta - y_d, v) - (\delta_1, v) = 0 \qquad v = y_\delta - y_{\delta'} =: \delta y$$
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$$(\nabla y_\delta, \nabla v) - (u_\delta, v) - (\delta_3, v) = 0 \qquad v = p_\delta - p_{\delta'} =: \delta p$$
$$0 \le \mu_\delta \perp u_\delta - \delta_4 \ge 0 \quad \text{on } \Omega$$

Essential estimate

$$\frac{1}{4}\|\delta y\|_2^2 + \frac{\gamma}{2}\|\delta u\|_2^2$$
$$\le (\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) - (\delta_3 - \delta_3', \delta p) + (\delta_4 - \delta_4', \delta \mu)$$
$$\le c_1\|\delta_1 - \delta_1'\|_2^2 + \frac{1}{2\gamma}\|\delta_2 - \delta_2'\|_2^2 + \frac{1}{4\kappa}\|\delta_3 - \delta_3'\|_2^2$$

[Malanowski, Tröltzsch]: Control and Cybernetics 29, 2000

Standard Approach: Control Constraints Only

Optimality system

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\begin{aligned}
(\nabla p_\delta, \nabla v) + (y_\delta - y_d, v) - (\delta_1, v) &= 0 \qquad v = y_\delta - y_{\delta'} =: \delta y \\
\gamma(u_\delta - u_d, v) - (p_\delta, v) - (\mu_\delta, v) - (\delta_2, v) &= 0 \qquad v = u_\delta - u_{\delta'} =: \delta u \\
(\nabla y_\delta, \nabla v) - (u_\delta, v) - (\delta_3, v) &= 0 \qquad v = p_\delta - p_{\delta'} =: \delta p \\
0 \le \mu_\delta \perp u_\delta - \delta_4 &\ge 0 \quad \text{on } \Omega
\end{aligned}
$$

Essential estimate

$$
\begin{aligned}
&\frac{\gamma}{4}\|\delta u\|_2^2 \\
&\le (\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) - (\delta_3 - \delta_3', \delta p) + (\delta_4 - \delta_4', \delta\mu) \\
&\le c_1\|\delta_1 - \delta_1'\|_2^2 + c_2\ \|\delta_2 - \delta_2'\|_2^2 + \frac{1}{4\kappa}\|\delta_3 - \delta_3'\|_2^2 + \frac{1}{4\kappa}\|\delta_4 - \delta_4'\|_2^2
\end{aligned}
$$

[Malanowski, Tröltzsch]: Control and Cybernetics 29, 2000

Standard Approach: Control Constraints Only

Theorem (Lipschitz stability)

There exists $L > 0$ *such that*

$$\|u_\delta - u_{\delta'}\|_{L^2(\Omega)}$$

$$\leq L \|\delta - \delta'\|_{L^2(\Omega)}$$

Standard Approach: Control Constraints Only

Theorem (Lipschitz stability)

There exists $L > 0$ *such that*

$$\|u_\delta - u_{\delta'}\|_{L^2(\Omega)} + \|y_\delta - y_{\delta'}\|_{H^2(\Omega)} \le L\,\|\delta - \delta'\|_{L^2(\Omega)}$$

Standard Approach: Control Constraints Only

Theorem (Lipschitz stability)

There exists $L > 0$ such that

$$\|u_\delta - u_{\delta'}\|_{L^2(\Omega)} + \|y_\delta - y_{\delta'}\|_{H^2(\Omega)} + \|p_\delta - p_{\delta'}\|_{H^2(\Omega)} + \|\mu_\delta - \mu_{\delta'}\|_{L^2(\Omega)} \le L\,\|\delta - \delta'\|_{L^2(\Omega)}$$

Standard Approach: Control Constraints Only

Theorem (Lipschitz stability)

There exists $L > 0$ *such that*

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Mixed constraints only

- same technique for $\varepsilon u + y \ge y_c + \delta_5$

Standard Approach: Control Constraints Only

Theorem (Lipschitz stability)

There exists $L > 0$ such that

$$\|u_\delta - u_{\delta'}\|_{L^2(\Omega)} + \|y_\delta - y_{\delta'}\|_{H^2(\Omega)} + \|p_\delta - p_{\delta'}\|_{L^2(\Omega)} \le L\,\|\delta - \delta'\|_{L^2(\Omega)}$$

Mixed constraints only

- same technique for $\varepsilon u + y \ge y_c + \delta_5$

State constraints only

- same technique for $y \ge y_c + \delta_5$
- Lagrange multiplier only a measure
- only L^2 estimate for adjoint state

[Griesse]: Journal of Analysis and its Applications 25, 2006

Standard Approach for Mixed and Control Constraints?

Optimality system

$$
\begin{aligned}
(\nabla p_\delta, \nabla v) + (y_\delta - y_d, v) - (\mu_{2,\delta}, v) - (\delta_1, v) &= 0 \quad \forall v \cdots \\
\gamma(u_\delta - u_d, v) - (p_\delta, v) - (\mu_{1,\delta}, v) - \varepsilon\,(\mu_{2,\delta}, v) - (\delta_2, v) &= 0 \quad \forall v \cdots \\
(\nabla y_\delta, \nabla v) - (u_\delta, v) - (\delta_3, v) &= 0 \quad \forall v \cdots \\
0 \le \mu_{1,\delta} \perp u_\delta - \delta_4 &\ge 0 \quad \text{on } \Omega \\
0 \le \mu_{2,\delta} \perp \varepsilon u_\delta + y_\delta - y_c - \delta_5 &\ge 0 \quad \text{on } \Omega
\end{aligned}
$$

Standard Approach for Mixed and Control Constraints?

Optimality system

$$
\begin{aligned}
(\nabla p_\delta, \nabla v) + (y_\delta - y_d, v) - (\mu_{2,\delta}, v) - (\delta_1, v) &= 0 \quad \forall v \cdots \\
\gamma(u_\delta - u_d, v) - (p_\delta, v) - (\mu_{1,\delta}, v) - \varepsilon(\mu_{2,\delta}, v) - (\delta_2, v) &= 0 \quad \forall v \cdots \\
(\nabla y_\delta, \nabla v) - (u_\delta, v) - (\delta_3, v) &= 0 \quad \forall v \cdots \\
0 \le \mu_{1,\delta} \perp u_\delta - \delta_4 &\ge 0 \quad \text{on } \Omega \\
0 \le \mu_{2,\delta} \perp \varepsilon u_\delta + y_\delta - y_c - \delta_5 &\ge 0 \quad \text{on } \Omega
\end{aligned}
$$

Standard Approach for Mixed and Control Constraints?

Optimality system

$$
\begin{aligned}
(\nabla p_\delta, \nabla v) + (y_\delta - y_d, v) - (\mu_{2,\delta}, v) - (\delta_1, v) &= 0 \quad \forall v \cdots \\
\gamma(u_\delta - u_d, v) - (p_\delta, v) - (\mu_{1,\delta}, v) - \varepsilon\,(\mu_{2,\delta}, v) - (\delta_2, v) &= 0 \quad \forall v \cdots \\
(\nabla y_\delta, \nabla v) - (u_\delta, v) - (\delta_3, v) &= 0 \quad \forall v \cdots \\
0 \le \mu_{1,\delta} \perp u_\delta - \delta_4 &\ge 0 \quad \text{on } \Omega \\
0 \le \mu_{2,\delta} \perp \varepsilon u_\delta + y_\delta - y_c - \delta_5 &\ge 0 \quad \text{on } \Omega
\end{aligned}
$$

Essential estimate

$$
\begin{aligned}
&\|\delta y\|_2^2 + \gamma\,\|\delta u\|_2^2 \\
&\le (\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) - (\delta_3 - \delta_3', \delta p) + (\delta_4 - \delta_4', \delta\mu_1) \\
&\quad + (\delta_5 - \delta_5', \delta\mu_2)
\end{aligned}
$$

Standard Approach for Mixed and Control Constraints?

Optimality system

$$(\nabla p_\delta, \nabla v) + (y_\delta - y_d, v) - (\mu_{2,\delta}, v) - (\delta_1, v) = 0 \quad \forall v \cdots$$
$$\gamma(u_\delta - u_d, v) - (p_\delta, v) - (\mu_{1,\delta}, v) - \varepsilon(\mu_{2,\delta}, v) - (\delta_2, v) = 0 \quad \forall v \cdots$$
$$(\nabla y_\delta, \nabla v) - (u_\delta, v) - (\delta_3, v) = 0 \quad \forall v \cdots$$
$$0 \le \mu_{1,\delta} \perp u_\delta - \delta_4 \ge 0 \quad \text{on } \Omega$$
$$0 \le \mu_{2,\delta} \perp \varepsilon u_\delta + y_\delta - y_c - \delta_5 \ge 0 \quad \text{on } \Omega$$

Essential estimate

$$\|\delta y\|_2^2 + \gamma \|\delta u\|_2^2$$
$$\le (\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) - (\delta_3 - \delta_3', \delta p) + (\delta_4 - \delta_4', \delta \mu_1)$$
$$+ (\delta_5 - \delta_5', \delta \mu_2)$$

Standard Approach for Mixed and Control Constraints?

Optimality system

$$(\nabla p_\delta, \nabla v) + (y_\delta - y_d, v) - (\mu_{2,\delta}, v) - (\delta_1, v) = 0 \quad \forall v \cdots$$
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$$(\nabla y_\delta, \nabla v) - (u_\delta, v) - (\delta_3, v) = 0 \quad \forall v \cdots$$
$$0 \le \mu_{1,\delta} \perp u_\delta - \delta_4 \ge 0 \quad \text{on } \Omega$$
$$0 \le \mu_{2,\delta} \perp \varepsilon u_\delta + y_\delta - y_c - \delta_5 \ge 0 \quad \text{on } \Omega$$

Essential estimate

$$\|\delta y\|_2^2 + \gamma\,\|\delta u\|_2^2$$
$$\le (\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) - (\delta_3 - \delta_3', \delta p) + (\delta_4 - \delta_4', \delta\mu_1)$$
$$+ (\delta_5 - \delta_5', \delta\mu_2) \quad \rightsquigarrow \quad \text{dead end}$$

A Partial Explanation: Non-Uniqueness

Example

$$\text{Minimize } \frac{1}{2}\|y\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u - (-\gamma^{-1}(\varepsilon + S1))\|^2_{L^2(\Omega)}$$

$$\text{s.t. } \begin{cases} -\Delta y = u & \text{on } \Omega \\ y = 0 & \text{on } \Gamma \end{cases} \qquad \rightsquigarrow S$$

$$\text{and } \begin{cases} u \geq 0 & \text{on } \Omega \\ \varepsilon u + y \geq 0 & \text{on } \Omega \end{cases}$$

Solution and Lagrange multipliers

$$y = u \equiv 0, \qquad (p, \mu_1, \mu_2) = \begin{cases} (S1, 0, 1) \\ (0, \varepsilon + S1, 0) \end{cases}$$

An Additional Assumption

Adjoint and gradient equations

$$-\Delta p = -(y - y_d) + \mu_2 + \delta_1 \quad \text{on } \Omega, \quad p = 0 \quad \text{on } \Gamma$$

$$\mu_1 + \varepsilon \mu_2 = \gamma\,(u - u_d) - \delta_2 - p \quad \text{on } \Omega$$

An Additional Assumption

Adjoint and gradient equations

$$-\Delta p = -(y - y_d) + \mu_2 + \delta_1 \quad \text{on } \Omega, \quad p = 0 \quad \text{on } \Gamma$$

$$\mu_1 + \varepsilon \mu_2 = \gamma\,(u - u_d) - \delta_2 - p \quad \text{on } \Omega$$

Main idea

Suppose *supp* $\mu_1 \subset S_1$ and *supp* $\mu_2 \subset S_2$ and $S_1 \cap S_2 = \emptyset$.

An Additional Assumption

Adjoint and gradient equations

$$-\Delta p = -(y - y_d) + \mu_2 + \delta_1 \quad \text{on } \Omega, \quad p = 0 \quad \text{on } \Gamma$$

$$\mu_1 \qquad = \gamma\,(u - u_d) - \delta_2 - p \quad \text{on } \Omega$$

Main idea

Suppose *supp* $\mu_1 \subset S_1$ and *supp* $\mu_2 \subset S_2$ and $S_1 \cap S_2 = \emptyset$.

$$\text{on } S_1: \quad \mu_1 = \gamma\,(u - u_d) - \delta_2 - p$$

An Additional Assumption

Adjoint and gradient equations

$$-\Delta p = -(y - y_d) + \mu_2 + \delta_1 \quad \text{on } \Omega, \quad p = 0 \quad \text{on } \Gamma$$
$$\varepsilon \mu_2 = \gamma\,(u - u_d) - \delta_2 - p \quad \text{on } \Omega$$

Main idea

Suppose *supp* $\mu_1 \subset S_1$ and *supp* $\mu_2 \subset S_2$ and $S_1 \cap S_2 = \emptyset$.

$$\text{on } S_1: \quad \mu_1 = \gamma\,(u - u_d) - \delta_2 - p$$
$$\text{on } S_2: \quad \varepsilon \mu_2 = \gamma\,(u - u_d) - \delta_2 - p$$

An Additional Assumption

Adjoint and gradient equations

$$-\Delta p = -(y - y_d) + \mu_2 + \delta_1 \quad \text{on } \Omega, \quad p = 0 \quad \text{on } \Gamma$$

$$\mu_1 + \varepsilon\mu_2 = \gamma\,(u - u_d) - \delta_2 - p \qquad \text{on } \Omega$$

Main idea

Suppose *supp* $\mu_1 \subset S_1$ and *supp* $\mu_2 \subset S_2$ and $S_1 \cap S_2 = \emptyset$.

$$\text{on } S_1: \quad \mu_1 = \gamma\,(u - u_d) - \delta_2 - p$$

$$\text{on } S_2: \quad \varepsilon\mu_2 = \gamma\,(u - u_d) - \delta_2 - p$$

$$\Rightarrow -\Delta p + \varepsilon^{-1}\chi_{S_2} p$$

$$= -(y - y_d) + \varepsilon^{-1}\chi_{S_2}\big(\gamma\,(u - u_d) - \delta_2\big) + \delta_1$$

An Additional Assumption

Adjoint and gradient equations

$$-\Delta p = -(y - y_d) + \mu_2 + \delta_1 \quad \text{on } \Omega, \quad p = 0 \quad \text{on } \Gamma$$
$$\mu_1 + \varepsilon\mu_2 = \gamma\,(u - u_d) - \delta_2 - p \qquad \text{on } \Omega$$

Main idea

Suppose *supp* $\mu_1 \subset S_1$ and *supp* $\mu_2 \subset S_2$ and $S_1 \cap S_2 = \emptyset$.

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$$\Rightarrow -\Delta p + \varepsilon^{-1}\chi_{S_2} p$$
$$= -(y - y_d) + \varepsilon^{-1}\chi_{S_2}\big(\gamma\,(u - u_d) - \delta_2\big) + \delta_1$$

[Malanowski]: Dissertationes Mathematicae (394), 2001

An Additional Assumption

How to define S_1 and S_2?

$$\{x \in \Omega : 0 \le u \quad - \delta_4 \qquad \}$$

$$\{x \in \Omega : 0 \le \varepsilon\, u \quad + y \quad - y_c - \delta_5 \qquad \}$$

An Additional Assumption

How to define S_1 and S_2?

$$S_1^{\sigma} = \{x \in \Omega : 0 \leq u - \delta_4 \leq \sigma\}$$
$$S_2^{\sigma} = \{x \in \Omega : 0 \leq \varepsilon u + y - y_c - \delta_5 \leq \sigma\}$$

An Additional Assumption

How to define S_1 and S_2?

$$S_1^\sigma = \{x \in \Omega : 0 \le u - \delta_4 \le \sigma\} \quad \Rightarrow supp\ \mu_1 \subset S_1^\sigma$$

$$S_2^\sigma = \{x \in \Omega : 0 \le \varepsilon\, u + y - y_c - \delta_5 \le \sigma\} \quad \Rightarrow supp\ \mu_2 \subset S_2^\sigma$$

An Additional Assumption

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$S_1^\sigma \cap S_2^\sigma = \emptyset \Leftrightarrow$ separate the multipliers $\Leftrightarrow$ separate the active sets

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Restrictive assumption?

An Additional Assumption

How to define S_1 and S_2?

$$S_1^\sigma = \{x \in \Omega : 0 \le u_0 - 0 \le \sigma\} \quad \Rightarrow supp\ \mu_1 \subset S_1^\sigma$$

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$S_1^\sigma \cap S_2^\sigma = \emptyset \Leftrightarrow$ separate the multipliers $\Leftrightarrow$ separate the active sets

Restrictive assumption?

Demand this only at $\delta = 0$!

An Additional Assumption

How to define S_1 and S_2?

$$S_1^\sigma = \{x \in \Omega : 0 \le u_0 - 0 \ \le \sigma\} \qquad \Rightarrow supp\ \mu_1 \subset S_1^\sigma$$
$$S_2^\sigma = \{x \in \Omega : 0 \le \varepsilon\, u_0 + y_0 - y_c - 0 \ \le \sigma\} \qquad \Rightarrow supp\ \mu_2 \subset S_2^\sigma$$

$S_1^\sigma \cap S_2^\sigma = \emptyset \Leftrightarrow$ separate the multipliers $\Leftrightarrow$ separate the active sets

Restrictive assumption?

Demand this only at $\delta = 0$!

Immediate consequence of $S_1^\sigma \cap S_2^\sigma = \emptyset$

Multipliers $\mu_{i,0}$ and adjoint state p_0 are unique.

An Auxiliary Problem

Ensure constraints to be separated

$$\text{Minimize } \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u - u_d\|^2_{L^2(\Omega)} - (y, \delta_1) - (u, \delta_2)$$

$$\text{s.t. } \begin{cases} -\Delta y = u + \delta_3 & \text{on } \Omega \\ y = 0 & \text{on } \Gamma \end{cases}$$

$$\text{and } \begin{cases} u \geq \delta_4 \\ \varepsilon u + y \geq y_c + \delta_5 \end{cases}$$

An Auxiliary Problem

Ensure constraints to be separated

$$\text{Minimize } \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u - u_d\|^2_{L^2(\Omega)} - (y, \delta_1) - (u, \delta_2)$$

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$$\text{and } \begin{cases} u \geq \delta_4 & \text{on } S_1^\sigma \\ \varepsilon u + y \geq y_c + \delta_5 & \text{on } S_2^\sigma \end{cases}$$

An Auxiliary Problem

Ensure constraints to be separated

$$\text{Minimize } \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u - u_d\|^2_{L^2(\Omega)} - (y, \delta_1) - (u, \delta_2)$$

$$\text{s.t. } \begin{cases} -\Delta y = u + \delta_3 & \text{on } \Omega \\ y = 0 & \text{on } \Gamma \end{cases}$$

$$\text{and } \begin{cases} u \geq \delta_4 & \text{on } S_1^\sigma \\ \varepsilon u + y \geq y_c + \delta_5 & \text{on } S_2^\sigma \end{cases}$$

Consequence

Multipliers $\mu_{i,\delta}$ and adjoint state p_δ are unique.

Stability for the Auxiliary Problem

Recall the estimate

$$\|\delta y\|_2^2 + \gamma \|\delta u\|_2^2 \leq (\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) - (\delta_3 - \delta_3', \delta p) + (\delta_4 - \delta_4', \delta \mu_1) + (\delta_5 - \delta_5', \delta \mu_2)$$

Stability for the Auxiliary Problem

Recall the estimate

$$\|\delta y\|_2^2 + \gamma \|\delta u\|_2^2$$
$$\le (\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) - (\delta_3 - \delta_3', \delta p) + (\delta_4 - \delta_4', \delta \mu_1)$$
$$+ (\delta_5 - \delta_5', \delta \mu_2) \quad \rightsquigarrow \quad \text{can estimate now}$$

Stability for the Auxiliary Problem

Recall the estimate

$$\|\delta y\|_2^2 + \gamma \|\delta u\|_2^2$$
$$\leq (\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) - (\delta_3 - \delta_3', \delta p) + (\delta_4 - \delta_4', \delta \mu_1)$$
$$+ (\delta_5 - \delta_5', \delta \mu_2) \quad \rightsquigarrow \quad \text{can estimate now}$$

Theorem (Lipschitz stability for the auxiliary problem, Part 1)

There exists $L > 0$ such that

$$\|u_\delta - u_{\delta'}\|_{L^2(\Omega)} + \|y_\delta - y_{\delta'}\|_{H^2(\Omega)} + \|p_\delta - p_{\delta'}\|_{H^2(\Omega)}$$
$$+ \|\mu_{1,\delta} - \mu_{1,\delta'}\|_{L^2(\Omega)} + \|\mu_{2,\delta} - \mu_{2,\delta'}\|_{L^2(\Omega)} \leq L \|\delta - \delta'\|_{L^2(\Omega)}$$

Stability for the Auxiliary Problem

Recall the estimate

$$\|\delta y\|_2^2 + \gamma \|\delta u\|_2^2$$
$$\leq (\delta_1 - \delta_1', \delta y) + (\delta_2 - \delta_2', \delta u) - (\delta_3 - \delta_3', \delta p) + (\delta_4 - \delta_4', \delta \mu_1)$$
$$+ (\delta_5 - \delta_5', \delta \mu_2) \quad \rightsquigarrow \quad \text{can estimate now}$$

Theorem (Lipschitz stability for the auxiliary problem, Part 1)

There exists $L > 0$ *such that*

$$\|u_\delta - u_{\delta'}\|_{L^2(\Omega)} + \|y_\delta - y_{\delta'}\|_{H^2(\Omega)} + \|p_\delta - p_{\delta'}\|_{H^2(\Omega)}$$
$$+ \|\mu_{1,\delta} - \mu_{1,\delta'}\|_{L^2(\Omega)} + \|\mu_{2,\delta} - \mu_{2,\delta'}\|_{L^2(\Omega)} \leq L \|\delta - \delta'\|_{L^2(\Omega)}$$

Question

Estimate in $L^\infty(\Omega)$ for the control?

Stability for the Auxiliary Problem

Projection formula

$$\mu_1 + \varepsilon\,\mu_2 = \max\Big\{0, \gamma\,\big(\max\{\delta_4, \varepsilon^{-1}(y_c + \delta_5 - y)\} - u_d\big) - p - \delta_2\Big\}$$

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$$\delta \in Z = L^2(\Omega) \times L^\infty(\Omega) \times L^2(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)$$

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$$u = \gamma^{-1}\big(p + \mu_1 + \varepsilon\,\mu_2 + \delta_2\big) + u_d \in L^\infty(\Omega)$$

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$$u = \gamma^{-1}\big(p + \mu_1 + \varepsilon\,\mu_2 + \delta_2\big) + u_d \in L^\infty(\Omega)$$

Theorem (Lipschitz stability for the auxiliary problem, Part 2)

There exists $L > 0$ *such that*

$$\|u_\delta - u_{\delta'}\|_{L^\infty(\Omega)} + \|y_\delta - y_{\delta'}\|_{H^2(\Omega)} + \|p_\delta - p_{\delta'}\|_{H^2(\Omega)}$$
$$+ \|\mu_{1,\delta} - \mu_{1,\delta'}\|_{L^\infty(\Omega)} + \|\mu_{2,\delta} - \mu_{2,\delta'}\|_{L^\infty(\Omega)} \le L\,\|\delta - \delta'\|_Z$$

Main Result: Back to Original Problem

Key observation

Owing to $L^\infty(\Omega)$ estimate . . .

Main Result: Back to Original Problem

Key observation

Owing to $L^\infty(\Omega)$ estimate ... for sufficiently small δ,

Main Result: Back to Original Problem

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Owing to $L^\infty(\Omega)$ estimate ... for sufficiently small δ,

$$\begin{aligned} u_\delta &\geq \delta_4 && \text{on } \Omega \\ \varepsilon u_\delta + y_\delta &\geq y_c + \delta_5 && \text{on } \Omega. \end{aligned}$$

Main Result: Back to Original Problem

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Owing to $L^\infty(\Omega)$ estimate ... for sufficiently small δ,

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The solutions of the auxiliary and original problems coincide.

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Key observation

Owing to $L^\infty(\Omega)$ estimate . . . for sufficiently small δ,

$$\begin{aligned} u_\delta &\geq \delta_4 && \text{on } \Omega \\ \varepsilon u_\delta + y_\delta &\geq y_c + \delta_5 && \text{on } \Omega. \end{aligned}$$

The solutions of the auxiliary and original problems coincide.

Theorem (Lipschitz stability)

Let $\|\delta\|$ *and* $\|\delta'\| \leq g(\sigma)$.

[Alt, Griesse, Metla, Rösch]: submitted, 2006

Main Result: Back to Original Problem

Key observation

Owing to $L^\infty(\Omega)$ estimate ... for sufficiently small δ,

$$\begin{aligned} u_\delta &\ge \delta_4 && \text{on } \Omega \\ \varepsilon u_\delta + y_\delta &\ge y_c + \delta_5 && \text{on } \Omega. \end{aligned}$$

The solutions of the auxiliary and original problems coincide.

Theorem (Lipschitz stability)

Let $\|\delta\|$ and $\|\delta'\| \le g(\sigma)$. There exists $L > 0$ such that

$$\begin{aligned} &\|u_\delta - u_{\delta'}\|_{L^\infty(\Omega)} + \|y_\delta - y_{\delta'}\|_{H^2(\Omega)} + \|p_\delta - p_{\delta'}\|_{H^2(\Omega)} \\ &\qquad + \|\mu_{1,\delta} - \mu_{1,\delta'}\|_{L^\infty(\Omega)} + \|\mu_{2,\delta} - \mu_{2,\delta'}\|_{L^\infty(\Omega)} \le L\,\|\delta - \delta'\|_Z \end{aligned}$$

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Concluding Remarks

Main ideas

- optimal control problem with mixed and control constraints
- Lagrange multipliers exist but may be non-unique

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Concluding Remarks

Main ideas

- optimal control problem with mixed and control constraints
- Lagrange multipliers exist but may be non-unique
- idea: separate the active sets
- stability for an auxiliary problem
- stability for original problem (owing to L^∞ estimate)