Problem 1 (6 points). *Let A be a ring, B an A-algebra and $b \in B$. Show the equivalence of the following two conditions:*

- *There are a positive integer n and $(a_i)_{i=0}^{n-1} \in A^n$ such that $b^n = \sum_{i=0}^{n-1} b^i$.*
- *There is a subring $\tilde{B} \subseteq B$ containing b and the image of A in B which is finitely generated as an A-module.*

Remark 1. *In the Noetherian case it is somewhat easier to derive the first condition from the second. This is the only case needed for basic algebraic number theory. Solutions which only work in the Noetherian case and are otherwase correct will be graded with 4 points. The two bonus points available from this exercise sheet are for solutions which also work in the non-Noetherian case.*

Remark 2. *Recall from the lecture that such b are called* integral over *A.*

Problem 2 (2 points). *In the situation of the first problem, let $B_1 \subseteq B$ be a sub-A-algebra which is finitely generated as an A-module, and let $b \in B$ be integral over A. Show that there is a subring $\tilde{B} \subseteq B$ which contains both b and B_1 and is finitely generated as an A-algebra.*

Problem 3 (2 points). *Let B be an A-algebra and let $b_1, \ldots, b_n$, with $n \in \mathbb{N}$, be integral over A. Show that there is a sub-A-algebra $\tilde{B} \subseteq B$ which contains all b_i and is a finitely generated A-module.*

Problem 4 (2 points). *If B is an A-algebra, show that the integral closure of A in B (i. e, the set of all $b \in B$ which are integral over A) is a sub-A-algebra.*

Problem 5 (6 points). *Let B be an A-algebra which is integral in the sense that every $b \in B$ is integral over A. If C is a B-algebra and $c \in C$ integral over B, show that it is also integral over A.*

Problem 6 (4 points). *Let L/K be a finite field extension, V a finite dimensional L-vector space and A an L-linear endomorphism of V. Show that $\mathrm{Tr}_K(A) = \mathrm{Tr}_{L/K}\big(\mathrm{Tr}_L(A)\big)$.*

Two of the 22 points for this sheet are bonus points, as explained in Remark 1. Solutions should be submitted to the tutor by e-mail before Friday October 18 24:00.