Tenth exercise sheet "Algebra II" winter term 2024/5. Let K be a field equipped with a non-trivial and non-Archimedean absolute value $|\cdot|$ and V a K-vector space equipped with an ultrametric norm $\|\cdot\,|V\,\|$.

Definition 1. *A K-linear functional $V \xrightarrow{\ell} K$ is called* bounded *if there is a non-negative real number C such that*

$$|\ell(v)| \leq C\,\|v\,|V\,\| \tag{1}$$

holds for all $v \in V$. The K-vector space of bounded K-linear functionals is denoted V^* and the smallest C for which (1) *holds is denoted $\|\ell\,|V^*\,\|$.*

If K is complete and $\dim_K V < \infty$ it follows from Proposition 3.2.1 of the lecture that all K-linear functionals on V are bounded.

Problem 1 (4 points)**.** *Assume that $\dim_K V < \infty$ and that for every $v \in V$ there is a linear functional $\ell \in V^*$ such that $\|v\,|V\,\| \cdot \|\ell\,|V^*\,\| = |\ell(v)|$. Show that V has an orthogonal base!*

In view of the Hahn-Banach theorem from classical functional analysis one could thus hope that the equivalent conditions of Proposition 3.3.1 from the lecture hold for all finite field extensions of a complete non-Archimedean field K. However this is not the case as the Hahn-Banach theorem does not always hold in non-Archimedean functional analysis. The following problems serve as an introduction to this matter. Recall that the notion of a ball in K was introduced in the last exercise sheet.

Problem 2 (3 points)**.** *For a non-empty set $\mathfrak{M}$ of balls in a field K equipped with a non-Archimedean absolute value, show that the following conditions are equivalent:*

- *The intersection of any finite subset of $\mathfrak{M}$ is non-empty.*
- *The intersection of two elements of $\mathfrak{M}$ is non-empty.*
- *If $B_{1,2} \in \mathfrak{M}$ then $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.*

Definition 2. *We call K* spherically complete *if for every $\mathfrak{M}$ satisfying the equivalent conditions from Problem 2, the intersection of all elements of $\mathfrak{M}$ is non-empty.*

Remark 1. *An equivalent condition is that for every sequence $(B_i)_{i=1}^{\infty}$ of balls in K such that $B_1 \supseteq B_2 \ldots$, the intersection $\bigcap_{i=1}^{\infty} B_i$ is non-empty. The fact that this is an equivalent characterization of sperical completeness is rather obvious and can be used without proof in the solutions to the following problems.*

Problem 3 (3 points)**.** *If K is spherically complete, show that it is complete!*

Problem 4 (2 points)**.** *If K is complete and $|K^\times|$ a discrete subgroup of $(0,\infty)_{\mathbb{R}}$, show that K is spherically complete!*

Remark 2. *If $|K^\times|$ is not discrete and K contains a countable dense subset $\{x_i \mid i \in \mathbb{N}\}$ then the following argument shows that K cannot be spherically complete: Chose a sequence $(\rho_i)_{i=0}^\infty$ of elements of K such that the sequence of real numbers $r_i = |\rho_i|$ is strictly monotonically decreasing and has a positive limit. We chose a sequence $B_0 \supseteq B_1 \dots$ of $\leq$-balls of radius r_i in K as follows. Chose K_0 such that does not contain x_0. If $n > 0$ and the $(B_i)_{i=0}^{n-1}$ of radius r_i not containing x_i have already been chosen, select $B_n \subseteq B_{n-1}$ of radius r_n not containing x_n. This is possible since B_{n-1} contains more than one $\leq$-ball of radius r_n and two such balls are disjoint if they are different.*

It is easy to see that $\bigcap_{i=0}^\infty B_i = \emptyset$ if the balls are chosen in this way. For instance, the field $\mathbb{C}_p$ is not spherically complete.

The relation between spherical completeness and the Hahn-Banach theorem results from the following fact:

Problem 5 (4 points)**.** *Let V be K-vector space equipped with a ultrametric norm $\|\cdot|V\|$. Let $W \subseteq V$ be a subspace of codimension 1 and let $\|\cdot|W\|$ be the restriction to W of $\|\cdot|V\|$. Let $\ell \in W^*$ and $C = \|\ell\,|W^*\|$, and let $v \in V \setminus W$. For $t \in K$, consider the linear functional ℓ_t on V defined by*

$$\ell_t(w + \lambda v) = \ell(w) + t\lambda$$

for $w \in W$, $\lambda \in K$. For $w \in W$, let

$$B_w = \{t \in K \mid |\ell_t(w+v)| \leq C\,\|w+v\,|V\|\}.$$

Show that B_w is a ball in K and show that $\mathfrak{M} = \{B_w \mid w \in W\}$ satisfies the equivalent conditions from Problem 2!

Chosing t from the intersecion of all B_w, if this is $\neq \emptyset$, allows us to extend ℓ to V preserving its norm. Combining this with a Zorn lemma argument shows that the Hahn-Banach theorem holds in non-Archimedean functional analysis over spherically complete non-Archimedean fields. On the other side the Hahn-Banach theorem allows one to construct linear functionals on $\ell_\infty(K)$ with properties making them "pseudo-limits" of bounded sequences in K which do not necessarily converge, and in the situation of 1 one can apply such a pseudlimit to the a sequence x_i where $x_i \in B_i$. Fully working this out gives

Problem 6 (7 points). *For a field K equipped with a non-Archimedean absolute value, the following conditions are equivalent:*

- *K is spherically complete.*
- *The Hahn-Banach theorem holds over K: If V is a K-vector space equipped with an ultrametric norm $\|\cdot|V\|$, $W \subseteq V$ a subspace and $\|\cdot|W\|$ the restriction to W of $\|\cdot|V\|$ and $\ell \in W^*$ then there is $\lambda \in V^*$ such that $\lambda|_W = \ell$ and $\|\lambda|V^*\| = \|\ell|W^*\|$.*
- *Let ℓ_∞ be the K-vector space of sequences $x = (x_i)_{i=1}^{\infty}$ from K for which $\|x|\ell_\infty\| = \sup_{1\leq i<\infty}|x_i|$ is finite. Then there is $\lambda \in \ell_\infty^*$ such that $\|\lambda|\ell_\infty^*\| = 1$ and such that $\lambda(x) = \xi$ if $\xi \in K$ and the sequence satisfies that $x_i = \xi$ for all sufficiently large ξ.*

Remark 3. *Using this in Problem 1 shows that the equivalent conditions of Proposition 3.3.1 from the lecture hold for all finite field extensions of a spherically complete field K. In particular this applies to the case where $|K^\times|$ is discrete and thus finishes the alternative proof of Proposition 3.3.2 in that case, which was only hinted at in the lecture.*

Problem 7 (3 points). *Assume that K is spherically complete with respect to $|\cdot|$, that the residue field $\mathfrak{k} = K^o/K^{oo}$ is algebraically closed and the group $|K^\times| \subseteq (0,\infty)_{\mathbb{R}}^\times$ divisible. Show that K is algebraically closed!*

Since the assumptions imply that $|K^\times|$ is not discrete the previous result is rather hard to apply, the easiest application being to the large field of Hahn series. However, instead of Hahn-Banach the trace $\mathrm{Tr}_{L/K}$ can also be used to construct the linear functional needed for an application of Problem 1.

Problem 8 (4 points). *Assume that K is complete with respect to $|\cdot|$ and that L/K is a finite field extension of K of degree prime to the characteristic of the residue field $\mathfrak{k}$ of K. Show that L/K satisfies the equivalent conditions of Proposition 3.3.1 from the lecture!*

Remark 4. *For an example where K is complete and has a finite field extension L without an orthonormal base see subsection 3.6.1 of Bosch/Güntzer/Remmert, Non-Archimedean Analysis.*

Problem 9 (3 points). *Assume that K is complete with respect to $|\cdot|$, that its residue field $\mathfrak{k}$ is algebraically closed and has characteristic zero and that the group $|K^\times|$ is divisible. Show that K is algebraically closed!*

For instance, let $\mathfrak{k}$ be an algebraically closed field of characteristic 0 and K the field of formal series with coefficients $f_l \in \mathfrak{k}$

$$f(T) = \sum_{l \in \mathbb{Q}} f_l T^l \tag{2}$$

where for all $n \in \mathbb{N}$, only finitely many of the f_l with $l < n$ are $\neq 0$. Then K is algebraically closed. This result was at least in principle known to Newton, whose approach to calculus was by clever manipulations with power series.

Another approach to a related result is by using Hensel's lemma.

Problem 10 (3 points)**.** *Let K be complete with respect to $|\cdot|$ and let n be a positive integer which is prime to the characteristic of the residue field. Let $x \in K^o$ such that $\xi = x \bmod K^{oo}$ is not zero and an n-th power in $\mathfrak{k}$. Show that x is an n-th power in K^o!*

If $|K^\times|$ is discrete, K^o is a discrete valuation ring. In the case where $\mathfrak{k}$ is algebraically closed and of characteristic prime to n it easily follows that $K(\sqrt[n]{\pi})$ does not depend on the choice of a uniformizer π of K^o. Moreover in view of the following result this is the only extension of degree n of K.

Problem 11 (4 points)**.** *Let $|K^\times|$ be discrete and L/K a finite field extension of degree d prime to the characteristic of the residue field $\mathfrak{k}$. Assume that $\mathfrak{k}$ is algebraically closed. Show that there is a uniformizing element π of the discrete valuation ring L^o such that π^d is a uniformizing element of K^o!*

Let $\mathfrak{k}$ be an algebraically closed field of charactersitic 0. It follows from the previous result that the field of Puiseux series, namely of series (2) for which the set of $l \in \mathbb{Q}$ with $f_l \neq 0$ is bounde from below and there is a positive integer n such that $nl \in \mathbb{Z}$ for all such l, is algebraically closed.

Twenty of the forty points available from this exercise sheet are bonus points which are disregarded in the calculation of the 50%-limit for passing the exercises.

Solutions should be submitted to the tutor by e-mail before Friday January 3 24:00.