Problem sheet 11 Rigid analytic geometry Winter term 2024/25

Recall the notions of filter, ultrafilter and limit from sheet 1. In the case of a topological space which is not Hausdorff, ultrafilters may have more than one limit. But it is easy to see that the set of limits of an ultrafilter $\mathfrak{F}$ is always closed. In particular, every specialization of a limit of $\mathfrak{F}$ is a limit of $\mathfrak{F}$. We say[1] that x is a *generic limit* of $\mathfrak{F}$ if it is a limit of $\mathfrak{F}$ and every limit of $\mathfrak{F}$ is a specialization of x. We say that ultrafilters on X *have unique generic limits* if every ultrafilter on X has precisely one generic limit.[2] If X has this property, then a subset $A \subseteq X$ is called *closed under generic limits* iff the generic limit of every ultrafilter containing A belongs to A. Obviously the set of such A is closed under finite unions and arbitrary intersections and contains all closed subsets of X.

Problem 1 (6 points)**.** *Show that a topological space is spectral if and only if it has unique generic limits of ultrafilters and the set of open subsets which are closed under generic limits of ultrafilters in X is a topology base.*

In this case it turns out that an open subset is quasicompact if and only if it is closed under generic limits. More generally, a subset of X is closed under generic limits if and only if it is closed for the constructible topology.

Let Γ be an ordered abelian group. It will be written multplicatively and 0 is assumed to be a non-element of every such Γ occuring in the following considerations, and $0 < g$ for all $g \in \Gamma$. By a *valuation* on a ring R we understand a map

$$R \xrightarrow{v} \Gamma \cup \{0\}$$

such that

- $v(xy) = v(x)v(y)$ for $x, y \in R$.
- $v(x+y) \leq \max\bigl(v(x), v(y)\bigr)$.
- We have $v(0) = 0$ and $v(1) = 1$.

If follows that $\mathrm{supp} v = \bigl\{r \in R \bigm| v(r) = 0\bigr\}$ is a prime ideal of R. In the following we will always assume that Γ is generated by $v\bigl(R \setminus \mathrm{supp} v\bigr)$, although this assumption is inconventient for some purposes and is often not made. Making this assumption we say that v is *equivalent* to a valuation $\tilde{v}$ if there is an order preserving isomorphism $\Gamma \xrightarrow{\iota} \tilde{\Gamma}$ such that $\tilde{v}(r) = \iota\bigl(v(r)\bigr)$ for all $r \in R$, where we put $\iota(0) = 0$. The set of

[1] This is my terminology. I do not know whether it is in common use

[2] For instance, this the case for compact spaces.

equivalence classes valuations of R is denoted Spv$R$. Valuations will often be written $|r|$, whith the distinction between different valuations indicated by a subscript like $|r|_x$.

We will primarily be interested in the case of a ring A which is equipped with some nat-topology.

Problem 2 (2 points). *Show that the following conditions are equivalent:*

- *For all $a \in A \backslash \mathrm{supp} v$, $\{\alpha \in A \mid v(\alpha) < v(a)\}$ is a neighbourhood of 0 in A.*
- *For all $\gamma \in \Gamma$, $\{\alpha \in A \mid v(\alpha) < \gamma\}$ is a neighbourhood of 0 in A.*

We say that v is continuous if this holds. Obviously, continuity of v depends only on the equivalence class of v. The set of equivalence classes of continuous valuations on A is denoted $\mathrm{Cont}(A)$. If $f \in A$ and $(g_i)_{i=1}^m \in A^m$ we put

$$R(f|g_1, \dots, g_m) = \\ = \{x \in \mathrm{Cont}(A) \mid |f|_x \neq 0 \text{ and } |g_i|_x \leq |f|_x \text{ for } 1 \leq i \leq m\}$$

We equip Cont$A$ with the topology generated by the subbase containing all $R(f|g_1, \dots, g_m)$ with the property that the ideal of A generated by f and the g_i is open. Since adding f to the list of g_i does not change R, the topology does not change if we impose the sharper condition that the ideal generated by the g_i must be open. When applied in the case of the discrete topology on a ring R this also defines a topology on $\mathrm{Spv}(R)$.

If $x \in \mathrm{Cont}(A)$ and $a, b \in A$, we put $a \preceq_x b$ if $|a|_x \leq |b|_x$. Obviously this is a transitive and reflexive relation on A. Let $a \prec_x b$ if $|a|_x < |b|_x$, or, equivalently, $a \preceq_x b$ but not $b \preceq_x a$. Throughout the next four problems we always fix x and drop the subscript x.

Problem 3 (1 point). *Show that $a \not\preceq 0$ if and only if $A_{\preceq a} = \{b \in A \mid b \prec a\}$ is a neighbourhood of 0 in A.*

Problem 4 (2 points). *Show that $b \preceq c$ implies $ab \preceq ac$ for all $a \in A$ where the opposite implication holds if $a \not\preceq 0$.*

Problem 5 (1 point). *For $a \in A$, show that $A_{\preceq a}$ is a subgroup of $(A, +)$.*

Problem 6 (1 point). *For $a, b \in A$, at least one of $a \preceq b$ or $b \preceq a$ always holds.*

Problem 7 (8 points)**.** *Let $\preceq$ be a transitive and reflexive relation on A such that $1 \not\preceq 0$. Put $a \prec b$ iff $a \preceq b$ and $b \not\preceq a$, assume that the conditions from Problems 3 to 6 hold. Show that there is a unique $x \in \operatorname{Cont}(A)$ such that $\preceq = \preceq_x$.*

Let $\mathfrak{F}$ be an ultrafilter on $\operatorname{Cont}(A)$. We say that $a \in A$ is $\mathfrak{F}$-essential if

$$D_a = \Big\{ b \in A \;\Big|\; \big\{ x \in \operatorname{Cont}(A) \mid |b|_x < |a|_x \big\} \in \mathfrak{F} \Big\}$$

is a neighbourhood of 0 in A. Otherwise we say that a is $\mathfrak{F}$-inessential. Moreover, we put $a \preceq b$ iff at least one of the two conditions

$$a \text{ is } \mathfrak{F}\text{-inessential} \tag{1}$$

or

$$\big\{ x \in \operatorname{Cont}(A) \mid |a|_x \le |b|_x \big\} \in \mathfrak{F} \tag{2}$$

holds.

Problem 8 (1 point)**.** *Show that $\preceq$ is transitive.*

It is obvious that $\preceq$ is reflexive. From our definition of "$\mathfrak{F}$-inessential" it is also obvious that the condition of Problem 3 holds for $\preceq$.

Problem 9 (1 point)**.** *If b is $\mathfrak{F}$-essential and $a \preceq b$, show that* (2) *holds.*

Problem 10 (2 points)**.** *Show that the set of $\mathfrak{F}$-inessential elements of A is an ideal in A.*

Problem 11 (1 point)**.** *Show that $\preceq$ satsfies the condition of Problem 5*

Problem 12 (1 point)**.** *Show that $\preceq$ satisfies the condition of Problem 6.*

To proceed with the proof that $\operatorname{Cont}(A)$ is a spectral space we need some additional assumptions on A, like

(A) If U is a neighbourhood of 0 in A then so is

$$\Big\{ \sum_{i=1}^{m} u_i v_i \;\Big|\; m \in \mathbb{N} \text{ and } \vec{u}, \vec{v} \in U^m \Big\},$$

(B) Every open ideal $I \subseteq A$ contains a finitely generated open ideal

or

(C) For every neighbourhood U of 0 in A

$$\text{there are } m \in \mathbb{N} \text{ and } \vec{u} \in U^m \text{ such that } A^m \xrightarrow{\cdot \vec{u}} A \text{ is open.}$$

as well as

(N) A^{oo} is a neigbourhood of 0 in A.

Obviously each of these conditions holds for A if A has an open subring satisfying this condition. Moreover (C) implies (A) and (B).

Problem 13 (3 points). *If A satisfies (A), show that $\preceq$ satisfies the condition of Problem 4.*

Problem 14 (1 point). *If A satisfies (N), show that $1 \not\preceq 0$.*

From now on we assume that A satisfies (A) and (N). Then it follows from Problem 7 that there is a unique $l \in \mathrm{Cont}(A)$ such that $\preceq = \preceq_l$. For showing that this is a generic limit of $\mathfrak{F}$ we study rational open subsets $\Omega = R(f|g_1, \ldots, g_m) \subseteq \mathrm{Cont}(A)$, assuming that the ideal $\langle f, g_1, \ldots, g_m \rangle_A$ is open.

Problem 15 (2 points). *If $l \in \Omega$, show that $\Omega \in \mathfrak{F}$.*

If follows that l is a limit of $\mathfrak{F}$. From now on we assume that A also satisfies (C).

Problem 16 (4 points). *If $\Omega \in \mathfrak{F}$, show that $l \in \Omega$.*

Problem 17 (1 point). *Show that l is a generic limit of $\mathfrak{F}$.*

Problem 18 (1 point). *Show that* $\mathrm{Cont}(A)$ *is a spectral space.*

Nineteen of the 39 points from this sheet are bonus points which are not counted in the calulation of the 50%-threshold for passing the exams. Solutions should be e-mailed to my institute e-mail address (my second name (franke) at math dot uni hyphen bonn dot de) before Wednesday January 25.

The remaining part of this text contains no problems and may be skipped without endangering ones chances to solve the above problems, understand the rest of the lecture or pass the exam.

If A is an ordinary ring and $I \subseteq A$ an ideal, the I-adic topology on A is the topolgy for which $\{I^n \mid n \in \mathbb{B}\}$ is a neighbourhood base of 0. A topological ring is called *adic* if there is an ideal I such that the topology conincides whith the I-adic topology. In this case every such ideal is called an *ideal of definition* for A. It is clear that any adic ring satisfies (A) and (N) above. If a finitely generated ideal of definition can be found, then (B) and (C) also hold.

I do not know whether there a good characterization of nat-rings for which $\mathrm{Cont}(A)$ is a spectral space exists in the literature. As of today (2025-01-08) the non-Archimedean Scottish Problem Book of Kedlaya contains to entry which seems to be equivalent to this question.

Huber calls a nat-ring *f-adic* if it contains an open subring (called a *ring of definition* which is adic and has a finitely generated ideal of definition. Nowadays most people call such rings *Huber rings*. By the remark made after formulating (N), Huber rings satisfy all conditions (A), (B), (C) and (N). Therefore the results from this sheet imply the theorem of Huber that $\mathrm{Cont}(A)$ is a spectral space when A is an adic ring. If A is a Tate ring with pair of definition $(A^\sharp, s)$ then $A^\sharp$ is adic and $sA^\sharp$ an ideal of definition for $A^\sharp$. Therefore A is a Huber ring. It is easy to see that every ring of definition of a Huber ring is bounded. Therefore our old notion of "ring of definition" in the Tate case is a special case of this new notion "ring of definition" for Huber rings.

Let A be an affinoid K-algebra where K is complete and non-discrete with non-Archimedean absolute value $|\cdot|$. Also, let always $s \in K^\times \cap K^{oo}$. For every $x \in \mathrm{Sp}(A)$, $a \to |a(x)|$ defines a continuous valuation of A. But in general not every element of $\mathrm{Cont}(A)$ is obtained in this way. For instance, if $A = \mathbb{T}_n$ then $a \to \|a\,|\mathbb{T}_n\|$ defines an element of $\mathrm{Cont}(A)$. This is not given by an element of $\mathbb{B}^n = \mathrm{Sp}\mathbb{T}_n$ but rather corresponds to a van der Put-point ξ of $\mathbb{B}^n$. To define it, let $\mathfrak{k} = K^o/K^{oo}$ be the residue field of K and recall the map

$$\mathbb{B}^n \xrightarrow{\rho} \mathrm{mSpec}\mathfrak{k}[X_1, \ldots, X_n] =: \mathcal{X}$$

from Definition 2.4.7 in the lecture. If $\Omega = R_{\mathrm{Sp}\mathbb{T}_n}(f|g_1, \ldots, g_m)$ is a rational open subset[3] with $\|f\,|\mathbb{T}_n\| \geq \|g_i\,|\mathbb{T}_n\|$ for all i, then by Fact 2.4.14 there is an open Zariski-dense $U \subseteq \mathcal{X}$ such that $\rho^{-1}U \subseteq \Omega$. For the same reason, if $\|g_i\,|\mathbb{T}_n\| < \|f\,|\mathbb{T}_n\|$ for some i then there is a proper Zariski-closed subset $Z \subseteq \mathcal{X}$ such that $\Omega \subseteq \rho^{-1}Z$. One easily concludes that the set ξ of open subsets U containing an Ω of the first kind is a van der Put-point of $\mathbb{B}^n$.

In general, if $\xi \in X^*$ with $X = \mathrm{Sp}(A)$ we put $a \preceq b$ if and only if $R_X(b|a, s^n) \in \xi$ for all $n \in \mathbb{N}$. By Problem 7 this defines an element $x = x_\xi$ of $\mathrm{Cont}(A)$. If $a \in A^o$ then $R_X(1, a) = X$ by Corollary 2.4.3 from the lecture. Thus it follows that $|a|_x \leq 1$ for all $x \in A^o$. But for most affinoid K-algebras A, not every element of $\mathrm{Cont}(A)$ has this property. For instance, let $A = \mathbb{T}_1$ and $\Gamma = |K^\times| \times \mathbb{Z}$ ordered lexicographically and with the first factor being more significant. For $f = \sum_{j=0}^\infty f_j T^j \in A \setminus \{0\}$, define $|f|_x \in \Gamma$ by

$$|f|_x = \Big(\|f\,|\mathbb{T}_1\|, \max\big\{j \in \mathbb{N} \bigm| |f_j| = \|f\,|\mathbb{T}_1\|\big\}\Big).$$

[3]The subscript $\mathrm{Sp}\mathbb{T}_n$ is to distinguish rational open subsets of $\mathrm{Sp}\mathbb{T}_n$ as introduced in the lecture from the rational open subsets of $\mathrm{Cont}(A)$ introduced earlier

Then $T \in A^o$ but $|T|_x > 1$. Intuitively, this continuous valuation would correspond to a van der Put-point of $\mathbb{B}^1$ capturing the behaviour of $|f(x)|$ if $x \in K$ is outside of but close to the unit ball $\mathbb{B}^1$. But of course no such van der Put-point exists since the general element of $\mathbb{T}_1$ is simply undefined outside $\mathbb{B}^1$.

Because of this, one considers

$$\begin{aligned} \mathrm{Spa}(A^{\rhd}, A^+) &= \left\{ x \in \mathrm{Cont} A \;\middle|\; |a|_x \leq 1 \text{ for all } a \in A^+ \right\} \\ &= \bigcap_{a \in A^+} R_{\mathrm{Cont} A}(1|a) \end{aligned}$$

where the condition for a *Huber pair*[4] is that $A^{\rhd}$ must be a Huber ring and $A^+ \subseteq A^o$ a subring which is open and integrally closed in A. By the second line of the above definition this is a proconstructible subset of Cont$A$ hence a spectral space.

If A is an affinoid K-algebra, $X = \mathrm{Sp}(A)$, $\xi \in X^*$ and x_ξ as above then $x_\xi \in \mathrm{Spa}(A, A^o)$. The inverse construction associates to $x \in \mathrm{Spa}(A, A^o)$ the van der Put-point $\xi = \xi_x$ defined by the condition that a rational subset $\Omega = R_X(f|g_1, \ldots, g_m)$ is $\in \xi$ if and only if $|f|_x \geq |f_i|_x$ for all i. This condition does not depend on the particular representation of Ω as it is equivalent to the condition that $|\cdot|_x$ has a (unique) continuous extension to the affinioid K-algebra $\mathcal{O}_X(\Omega)$.

[4]Called "affinoid ring" by Huber himself