GROMOV'S H-PRINCIPLE AND ITS APPLICATIONS RESEARCH SEMINAR, SUMMER TERM 2010

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An *h-principle* is a method to reduce existence problems in differential geometry to homotopy-theoretic problems. For example, let us consider the following problems in differential geometry on the existence of structures on a given manifold M.

- Problem 0: Can we classify submersions $M \to W$ up to regular homotopy?
- Problem 1: Does M admit a Riemannian metric of positive/negative sectional curvature?
- Problem 2: Does M admit a symplectic structure?
- Problem 3: Can we classify immersions $M \to W$ up to regular homotopy?
- Problem 4: Does M admit a foliation of a given codimension?

All these structures can be described as sections in certain fibre bundles on M that obey a certain partial differential equation (Problems 2,4) or partial differential inequality (Problems 0,1,3).

If the manifold M is *open*, i.e. M has no closed connected component, then a very general theorem of Gromov reduces the problem of solving a partial differential inequality of the type of Problems 0,1,3 to a purely homotopy theory problem provided. Consider the first problem. If $f : M \to W$ is a submersion, then the differential Tf is a bundle epimorphism $TM \to TW$. The classification of bundle epimorphisms is a purely homotopy-theoretic problem, because a bundle epimorphism is a section into a certain fibre bundle over M. Gromov's theorem states in this case that any such bundle-epimorphism is homotopic to the differential of a submersion. This was proven originally by Phillips.

The condition of having positive sectional curvature is a condition on the second derivative of the metric. Gromov's theorem implies then that any open manifold has a metric of positive sectional curvature (the same statement is true for negative sectional curvature).

Recall that a symplectic structure on M is a closed nondegenerate 2-form ω. It determines two data: a cohomology class $[\omega] \in H^2(M;\mathbb{R})$ and a homotopy class of almost complex structures on TM (these are endomorphisms J of TM and one requires that $\omega(\dot{,}J)$ is positive definite). Although the existence of symplectic structures is a problem on a differential equation, but it can be reduced to a differential inequality and then solved by the h-principle. The result is that if $x \in H^2(M;\mathbb{R})$ and an almost complex structure J are given, then there exists a symplectic form ω (i.e. a nondegenerate closed 2-form), such that $[\omega] = x$ and such that J is compatible with ω.

These examples show that similar statements are false for closed manifolds:

- No closed nonempty manifold M admits a submersion onto $\mathbb{R}$ (undergraduate calculus), but a bundle epimorphism $TM \to T\mathbb{R}$ exists if $\chi(M) = 0$ (Poincaré-Hopf).

- If the closed manifold M has a metric of positive sectional curvature, then $\pi_1(M)$ is finite (Bonnet-Myers).
- If the closed manifold M has a metric of negative sectional curvature, then $M \simeq B\pi_1(M)$ (Cartan-Hadamard).
- If $\omega \in H^2(M;\mathbb{R})$ is symplectic structure on the closed $2n$-manifold M, then $[\omega]^n \neq 0 \in H^{2n}(M)$.

For immersions, the answer is stronger: we will see that any bundle monomorphism $TM \to TW$ can be homotoped to the derivative of an immersion, even if M is closed. This is the *Smale-Hirsch theorem* [16], [30]. This follows from Gromov's theorem by reducing it to a problem on an open manifold.If $M = \mathbb{S}^2$ and $W = \mathbb{R}^3$, it has the amusing consequence that a sphere can be "turned inside out".

For Problem 4, the answer is more complicated. Naively, after seeing the solutions to the first four problems, one would expect that any subbundle $V \subset TM$ of codimension is homotopic to a foliation. This, however, is false. There is a necessary condition on the Pontrjagin classes of TM/V (Bott's vanishing theorem). Haefliger constructed a fibration $B\Gamma_q \to B\,\mathrm{GL}_q(\mathbb{R})$, such that the classifying map of the normal bundle of a foliation admits a unique lift to $B\Gamma_q$. He also proved the converse, for open manifolds M: if $V \subset TM$ is a subbundle of codimension q and the classifying map of TM/V admits a lift to $B\Gamma_q$, then V is homotopic to a foliation. Gromov's theorem is a crucial ingredient for the proof.

The h-principle is proven in a much more general framework and it was successfully applied to problems which do not look even remotely related to differential inequalities. Examples are:

- Mc Duff's work on configuration spaces [18], [19], which leads to an isomorphism $H_k(C^r(M)) \cong H_k(\Gamma(M;\mathrm{Fr}(M) \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{S}^n))$, where $C^r(M)$ is the configuration space of r points in the open n-manifold M, $\mathrm{Fr}(M)$ is the frame bundle of M and Γ is the notation for "space of sections" and $r > 2k$.
- Galatius' work on the stable homology of automorphism groups of free groups [5]. He showed that the natural map $\Sigma_r \to \mathrm{Aut}(F_r)$ is an isomorphism in homology in degrees less than $r/2$.
- A new proof of the Galatius-Madsen-Tillmann-Weiss theorem on the homotopy type of the cobordism category [6], asserting a homotopy equivalence between $B\mathbf{Cob}_d$, the classifying space of the d-dimensional cobordism category and the infinite loop space of a certain Thom spectrum $\mathrm{MTO}(d)$. The new proof was sketched in [5] and worked out later in [7] and [27] (the original proof in [6], as well as the closely related proof of the Mumford conjecture by Madsen and Weiss [22] rely on h-principles as well.

We will discuss the first and the third result, while the second result is beyond the scope of the seminar. Instead of stating something close to precise definitions, we give a hint how the problems above might be parallel to the differential relations mentioned earlier by means of the following table.

Submersion theory	Configuration spaces	Spaces of submanifolds	General notion
$\mathrm{Sub}(M;W)$	$C(M)$	space of d-dimensional submanifolds of M	micro-flexible sheaf $\mathcal{F}$ on M
$\mathrm{Epi}(TM;TW)$	$\mathrm{map}(M;\mathbb{S}^n)$	space of maps from M to a certain Thom space	space of sections in the fibre space $\mathcal{F}_M^{fib} \to M$
differential map	scanning map	Pontrjagin-Thom construction	h-principle comparison map
Gromov-Phillips Submersion theorem	Barratt-Priddy-Quillen-Segal	Galatius-Madsen-Tillmann-Weiss	

The seminar has of three parts, each consisting of three talks.

Part 1: Talks 1 and 2 introduce the language in which the h-principle is formulated: Jet bundles, partial differential relations, microflexible sheaves Talk 3 is the proof of the h-principle.

Part 2: Talk 4 discusses immediate applications, for example submersion and immersion theory. Talks 5 and 6 are about foliations.

Part 3: The last part of the seminar consists in applications to configuration spaces (talk 7) and spaces of manifolds (talks 8 and 9).

If you wish to give a talk, send an e-mail to both of us (ebert@math.uni-bonn.de, andresangelalumni@googlemail.com) no later than March 12th. In order to faciliate the distribution of the talks, you should name a first and a second choice. Needless to say: each of the talks should be finished within half an afternoon.

Talk 1. (**Jet bundles and differential relations**, 22.4.)

The goal of this talk is to give the precise statement of Gromov's h-principle for open invariant differential relations (this is the "main theorem" of [10], p. 129). The statement of the theorem involves sections of jet bundles of fibre bundle, so you need to discuss the definition and some important properties of jet bundles. You also have to explain the notion of *natural fibre bundles*. Jet bundles are discussed in [15], [1], [4], [9]. Don't forget that the h-principle is about *spaces* of smooth maps, so you need to discuss the topologies. Needless to say: give a few examples (maybe other ones than immersions and submersions which are discussed later in detail, for example curvature relations, [9], p.109 f).

Talk 2. (**Microflexible sheaves**, 22.4.)

The goal of this talk is to state Gromov's h-principle for microflexible sheaves. The statement can be found in the middle of page p.79 [9] and the relevant notions are explained on p.74-75 loc. cit. There are alternative and more intuitive descriptions of the sheaf Φ^* and the h-principle comparison morphism $\Delta : \Phi \to \Phi^*$ as a "scanning map". The notion goes back to G. Segal [28], but an explanation that is closer to our topic can be found in [27], section 6. See also [2], chapter 2, for more details.

The second goal is to explain why the h-principle for microflexible sheaves is a generalization of the h-principle for differential relations. This is sketched in [9], p.76, Remarks A' and A" and it is your job to work out that sketch. Of course, you should discuss some examples: differential relations and something apart from that. You can look into the sources of the talks 7, 8 and 9 to get an impression of what sheaves the h-principle can be applied to. Gromov's book [9] probably contains a lot of examples and see also [2], but not all sheafs that are discussed there are microflexible. Do not go into the details of the definition of the topologies on these sheaves, let alone the proof that they are microflexible - this is done later.

Talk 3. (**Proof of Gromov's h-principle**, 6.5.)

The proof is written down in [10] p. 133-140 for differential relations and it can be generalized to microflexible sheaves by a mere change of notation and you should present this generalization.

Remarks: 1.) The proof is only 7 pages long (with every detail spelled out!) and elementary. 2.) Proposition 3 loc.cit. is somewhat technical. You might be tempted to skip its proof, but *do not* dare to do so: it is the heart of the proof! 3.) There are many formulae but few pictures in [10], so you have to develop your own visualization of the situation, which won't be easy. To get an idea of what is going on, it might be helpful to consult also the relevant pages in [1], [8], [26], but they can also be a source of confusion. 4.) The translation into the more abstract setting of microflexible sheaves will be easy once you understand the proof in [10].

Talk 4. (**Immediate applications of the h-principle**, 6.5.)

Submersions, symplectic structures ([10], p.130-133). For some background on symplectic structures, see the introductory chapters of [20]. Proof of the immersion theorem from the submersion theorem, see [25], p.196. The sphere eversion [4]. Show one of the movies that circulate in the web, but do some serious maths as well. Note that the movies do *not* show the homotopy provided by the proof we discussed, but another one, so they do not help to illuminate the proof discussed in talk 3. Question: in which dimensions is the sphere eversion possible? Browse through [25], [26], [1], [4] to find other examples of differential relations.

You should also give some definite examples that show the (complete) breakdown of the h-principle for closed manifolds (see the introduction)

Talk 5. (**Foliations I**, 10.6.)

Main sources: [11], [3]. Definition of foliations and distributions, the integrability condition and the Frobenius theorem (without proof). This is a basic result covered in many introductory textbooks on smooth manifolds, consult [17], [29]. Bott's vanishing theorem [3]. Maybe you sketch the proof, other sources are [9], p. 100 and [23]. Haefligers topological groupoid Γ_q. Then you should discuss briefly Γ-structures for general groupoids Γ [11]. Then any foliation determines a Γ_q-structure. Discuss the map $\Gamma_q \to \mathrm{GL}_q(\mathbb{R})$. Note: the notions of a "Γ-structure" ([10]), a "Γ-torsor" (e.g. [24]), and a "Γ-principal bundle" on as space X ([14]) are all equivalent. The latter perspective also gives enough credibility to the result that $[X, B\Gamma_q]$ is in bijection with concordance classes of Γ-structures on X without a detailed proof.

Remark: we suggest to restrict to the C^∞ case throughout. This might be too much material for 75 minutes. If there is not enough time, then decide whether you want to emphasize the proof of Bott's theorem or the details of the classification theorem for Γ-structures. The theory is also discussed in [1].

Talk 6. (**Foliations II**, 10.6.)

Haefligers classification theorem: the set of concordance classes of foliations on an open manifold is in bijection with $[X, B\Gamma_q]$, [11], p. 148. You can also consult [3]. The two main ingredients are the Gromov-Phillips theorem for maps transverse to a foliation (straightforward application of Gromov's theorem, see loc.cit., p. 147) and a technical result which is hidden in [12], [13]. These sources are written in French, but maybe you'll skip the proof anyway. If you do that, you have time to discuss some interesting results on the topology of $B\Gamma_q$; they are surveyed in [3] and [11]: the map $B\Gamma_q \to B\,\mathrm{GL}_q$ is q-connected and 0 on the Pontrjagin ring in large degrees (Bott). There is a surjective homomorphism $\pi_3(\Gamma_1) \to \mathbb{R}$ [31]. See also the relevant chapter of [23] and [1].

Talk 7. (**Configuration spaces**, 24.6.)

The goal is to prove Theorem 1 of [18], intended as a warm-up for the following two talks. Instead of using the h-principle, Mac Duff rather tries to avoid it and she proves it by hands for the sheaf under consideration. In the context of this research seminar, there is of course a better way to to it. Prove that the sheaf $\tilde{C}$ of loc. cit. is microflexible, which is not difficult at all. The second part is an argument with group completions and homology fibrations ([21] and homological stability for configuration spaces (see appendix of [28]).

Talk 8. (**Spaces of submanifolds I**, 24.6.)

The goal of this talk and the next one is to discuss a new proof of the main result of [6], which determines the homotopy type of the d-dimensional cobordism category as the infinite loop space of a certain Thom spectrum.

Start with the detailed statement of the result, i.e. Theorem A in [27]. Show how this implies the main result of [6]. Suggestion: forget about the "tangential structures" throughout, for the sake of simpler notation and to save time.

First part of the proof. Definition of the sheaf Ψ_d of d-dimensional submanifolds of M^n, [7], section 2 and [27], section 3. Then discuss [27], chapter 4, which compares the embedded cobordism category with a sheaf.

Note: $d = 0$ is the case of configuration spaces, which was discussed in talk 7. Emphasizing the analogy might be helpful for the understanding, both for the speaker and the audience!

Talk 9. (**Spaces of submanifolds II**, 15.7.)

Second part of the proof. Show that the sheaf Ψ_d is microflexible [27], section 5 and identify the homotopy type of $\Psi_d(\mathbb{R}^n)$ [7], Theorem 3.22. Conclude the main result of [6]. If there is enough time, you can how the Mumford conjecture on the homology of the stable mapping class group follows from this result, see [6], chapter 7.

References

[1] Adachi: *Embeddings and immersions*. Translated from the 1984 Japanese original by Kiki Hudson. Translations of Mathematical Monographs, 124. American Mathematical Society, Providence, RI, 1993.

[2] Ayala: *Geometric cobordism categories*, arXiv:0811.2280
[3] Bott: *Lectures on characteristic classes and foliations* Lectures on algebraic and differential topology (Second Latin American School in Math., Mexico City, 1971), pp. 1–94. Lecture Notes in Math., Vol. 279, Springer, Berlin, 1972.
[4] Eliashberg, Michachev: *Introduction to the h-principle*, Graduate Studies in Mathematics, 48. American Mathematical Society, Providence, RI, 2002.
[5] Galatius: *Stable homology of automorphisms of free groups*, arXiv0610216.
[6] Galatius, Madsen, Tillmann, Weiss: *The homotopy type of the cobordism category*, Acta Math. 202 (2009), 195-239.
[7] Galatius, Randal-Williams: *Monoids of moduli spaces of manifolds*, arXiv:0905.2855
[8] H. Geiges: *h-principles and flexibility in geometry.* Mem. Amer. Math. Soc. 164 (2003), no. 779.
[9] Gromov: *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete 9, Springer-Verlag, Berlin, 1986.
[10] Haefliger: *Lectures on the Theorem of Gromov*, Proceedings of Liverpool Singularities Symposium, II (1969/1970), p. 128-141. Lecture Notes in Math. 209, Springer, Berlin, 1971.
[11] Haefliger: *Homotopy and integrability*, Manifolds-Amsterdam 1970 (Proc. Nuffic Summer School) p. 133-163 Lecture Notes in Mathematics 197 Springer, Berlin 1971.
[12] Haefliger: *Feuilletages sur les varietes ouvertes*, Topology 9 (1970), p.183-194.
[13] Haefliger: *Structures feuilletées et cohomologie á valeur dans un faisceau de groupoides.* Comm. Math. Helv. 32 (1958), p. 248-329.
[14] J. Heinloth: *Some notes on differentiable stacks*, Mathematisches Institut, Seminars 2004705, Universität Göttingen (2005), p. 1-32.
[15] M. W. Hirsch: *Differential topology*, Graduate Texts in Mathematics, No. 33. Springer-Verlag, New York-Heidelberg, 1976.
[16] M. W. Hirsch: *Immersions of manifolds.* Transactions A.M.S. 93 (1959), 242-276.
[17] A. Kosinski: *Differential manifolds*, Differential manifolds. Pure and Applied Mathematics, 138. Academic Press, Inc., Boston, MA, 1993.
[18] D. Mac Duff: *Configuration spaces*, K-theory and operator algebras (Proc. Conf., Univ. Georgia, Athens, Ga., 1975), p. 88-95. Lecture Notes in Math. 575, Springer, Berlin, 1977.
[19] D. Mac Duff: *Configuration spaces of positive and negative particles.* Topology 14 (1975), 91-107.
[20] D. Mac Duff, D. Salamon: *Introduction to symplectic topology*, Oxford Science Publications, Clarendon Press, Oxford, 1994.
[21] D. Mac Duff, G. Segal: *Homology fibrations and the group completion theorem.* Invent. Math. 31 (1975/76), 279-284.
[22] I. Madsen; M. Weiss: *The stable moduli space of Riemann surfaces: Mumford's conjecture.* Ann. of Math. 165 (2007), 843-941.
[23] S. Morita: *Geometry of characteristic classes.* Translated from the 1999 Japanese original. Translations of Mathematical Monographs, 199. Iwanami Series in Modern Mathematics. American Mathematical Society, Providence, RI, 2001.
[24] B. Noohi: *Homotopy types of stacks*, preprint, arXiv:0808.3799.
[25] Phillips: *Submersions of open manifolds*, Topology 6 (1967), p. 171-202.
[26] V. Poenaru: *Homotopy theory and differentiable singularities*, Manifolds-Amsterdam 1970 (Proc. Nuffic Summer School) p. 106-132 Lecture Notes in Mathematics 197 Springer, Berlin 1971.
[27] O. Randal-Williams: *Embedded Cobordism Categories and Spaces of Manifolds*, arXiv:0912.2505
[28] G. Segal: *The topology of the spaces of rational functions*, Acta Math. 143 (1979), 39-72.
[29] R. W. Sharpe: *Differential geometry. Cartan's generalization of Klein's Erlangen program.* Graduate Texts in Mathematics, 166. Springer-Verlag, New York, 1997.
[30] S. Smale: *The classification of immersions of spheres in Euclidean spaces.* Ann. Math. 69 (1959), 327-344.
[31] Thurston: *Noncobordant foliations on* $\mathbb{S}^3$, Bull. Amer. Math. Soc. 78 (1972), 511-514.