Index Theory and Spin Geometry

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Many of the familiar (and not-so-familiar) invariants in the algebraic topology of manifolds may be phrased as an index of an elliptic differential operator. What is an elliptic differential operator? A differential operator D is linear map between the smooth sections of two vector bundles on a manifold which looks locally like a linear combination of partial derivatives (of course, there are also more "functorial" descriptions). Ellipticity is a condition on the coefficients of the highest order terms, which ensures that both kernel and cokernel of the differential operator are finite-dimensional (i.e. if we consider the differential equation $Df = g$, there are only finitely many linear independent solutions for f and given g and only finitely many restrictions for what g the equation possesses a solution). The difference between the dimensions of kernel and cokernel is called the *(analytical) index*.

At first sight, it seems that it requires the computation of the kernel and cokernel of a differential operator D to compute its index and the computation of the kernel means to find all solutions to the above differential equation, which is, of course, in general hopeless. But this is not true: the index is much easier to compute than the dimensions of the kernel and cokernel. For example, it is a homotopy invariant, which gives the hope, we can find a formula for it in terms of algebraic topology. This is exactly the content of the Atiyah-Singer index theorem. For every differential operator D, Atiyah and Singer define a topological index and their theorem states:

Theorem 1 (Atiyah-Singer index theorem)**.** *The topological index of a differential operator equals the analytic index.*

The most natural form of the topological index uses K-theory (which is also the formulation, which we use in the proof), but their is also a concrete formula in characteristic classes in ordinary cohomology.

Of course, the Atiyah-Singer index theorem (ASIT) can be applied to get information about differential equations on manifolds, but a remarkable thing is that it can be used also the other way round. This is already apparent in Hirzebruch's signature theorem, one of the basic corollaries of the ASIT: it seems at first that it is a formula for the signature in terms of characteristic classes. But often one can use the knowledge of the signature to compute the characteristic classes or at least to conclude that the complicated characteristic class expression is an integer. The same happens in the ASIT: the characteristic class formula is a priori only a rational number, but must be an integer because the analytic index is an integer. This approach can be applied to show that some manifolds have no differentiable structure or that most spheres have no complex structure (see talk 11).

The plan for the seminar is roughly this: In the first two talks, we collect some analysis needed for the seminar, especially we fill the words 'Index of an Elliptic Differential Operator' with precise content and proof its basic properties. In the third talk, we recall and introduce some concepts of K-theory, introduce the topological index and can give the precise statement of the Atiyah-Singer index theorem. Talks 4 to 7 are then primarly concerned with the proof of the index theorem and deducing its cohomological form. Talk 8 will give some basic applications, e.g. the signature theorem above, but also to the Hirzebruch-Riemann-Roch formula, which is a basic tool in complex geometry. For deeper applications, often the Dirac operator is of fundamental importance. Therefore, the talks 9 and 10 are dedicated to the basics for Clifford modules and spin geometry and the construction of the Dirac operator. The last two talks then reconnect to the index theory. Talk 11 includes some basic study of the index of the Dirac operator and gives the applications mentioned above and talk 12 is a survey and outlook to the question which exotic spheres admit metrics of positive scalar cuvature (the surprising thing is that not all exotic spheres admit such metrics!). Our main source will be [L-M] and all non-specified references will be to this book.

Talk 1 (Partial differential operators): Give a basic introduction to partial differential operators. These are linear maps between the sections of two vector bundles which locally look like a linear combination of partial derivatives (page 167 in [L-M]); however, this local coordinate definition is awkward to work with, so we will choose a more algebraic approach, following Chapter 10.1 in [Nic]. For smooth vector bundles E and F over a manifold M, let $\Gamma(E)$ resp. $\Gamma(F)$ denote the space of smooth sections of the vector bundle. These spaces are modules over $C^\infty(M)$; review the fact that $C^\infty(M)$-linear functions $\Gamma(E) \to \Gamma(F)$ are the same as vector bundle morphisms $E \to F$. After this, define (partial) differential operators as on page 446 of [Nic] and give some basic examples (for example 10.1.2 in [Nic]). Then prove that differential operators are local (10.1.3) and explain 10.1.4 about their compositions. Next, define the symbol as on page 450 of [Nic] as a morphism $\pi^*(E) \to \pi^*(F)$ with $\pi : T^*M \to M$ the cotangent bundle projection. This requires most of the material covered on the previous two pages; in particular, you should prove Lemma 10.1.8. (This is actually tricky if you don't know the trick; googling for "Hadamard Lemma" and looking at the Wikipedia entry might be helpful.) The symbol of a differential operator is the basic object of study throughout the seminar, so you should explain it in as much detail as possible. Then at least motivate Proposition 10.1.10 and define elliptic operators (10.1.13). Finally, explain (as in the exercises 10.1.15 - 10.1.17) why our algebraic definition agrees with the local coordinate one.

Talk 2 (Properties of elliptic operators). This talk will be rather sketchy, but involves lots of functional analysis. The first aim of the talk is to explain why elliptic differential operators are Fredholm, i.e., have finite-dimensional kernel and cokernel. A complete prove will not be possible, but some of the functional analysis involved should be covered. You probably will

have to introduce formal adjoints (10.1.3 in [Nic]), but you should not prove anything about them - just assume whatever is necessary. Explain some of the things in 10.2 and 10.3 of [Nic] and how they lead to the Fredholm property of elliptic operators (10.4.7). In particular, the point where ellipticity actually enters into the proof should be covered. There are many possible other sources for this, including [L-M]; feel free to pick any other.
In the second part of the talk, discuss inhowfar the Fredholm index depends on the symbol. The aim is Theorem 10.4.13 of [Nic] respectively 7.10 in [L-M]: the index is locally constant on the space of elliptic operators, i.e. it is homotopy invariant. Either follow [Nic] or [L-M] and explain the result in as much detail as your time permits.

Talk 3 (K-Theory and the Topological Index): Recall some basic definitions and properties of (complex) K-theory. For the choice what is important for our purposes, you can orient yourself at [AS1], p. 488-494. As sources [A1] and chapter I.9 of [L-M] are recommended, but you can, of course, use other sources as well. Then define the topological index of a (non-equivariant) differential operator as in [AS1], §3, or [L-M], §13, and state the Atiyah-Singer index theorem. If time permits, sketch the proof of Bott periodicity as in [A2]

Talk 4 (The analytic index revisited): The main goal of this talk is to reformulate the topological and the analytic index to get two maps $K_{cpt}(T^*X) \to \mathbb{Z}$, which are easier to compare than the integer numbers before. For this it is necessary to introduce pseudo-differential operators. Two sources for pseudo-differential operators are [L-M], III.3, and [AS1], §5 (you have, of course, to omit much of the hard analysis). For the actual index map, you can use the sources [L-M], p. 244-247, or [AS1], §6.

Talk 5 (Reductions and Equivariant K-Theory): Reduce the index theorem to the two formal properties stated on [L-M], p. 247. Then prove property 1 and the excision property ([L-M] III.13.4 or [AS1], §8 (where you have to ignore the G-equivariance)). Next introduce equivariant K-theory and and the equivariant index as in [L-M], III.9, or [AS1], §6.

Talk 6 (Multiplicative Properties): The goal of this talk is to investigate the properties of the analytic index in fibre bundles. See propositions III.13.5 and III.3.16 in [L-M] (compare AS1, §9).

Talk 7 (Finishing the Proof and Deducing the Cohomological Form): The final part of the proof essentially consists of computing the index map on one conrete element. This is the statement III.13.7 in [L-M] or (B2) in §4 of AS1. The proof consists of proposition 4.4, 4.7 and the end of §8 of [AS1] and a short version is also presented in [L-M]. For time reasons you may ignore some sign issues. Then you need do transform our K-theory formula into a characteristic class formula. This is III.13.8 in [L-M] and proposition 2.12 in [AS3]. The method of computation is in some sense more important than the actual formula.

Talk 8 (Basic Applications): The aim of this talk is to give some basic applications of the index theorem. Especially interesting corollaries are the Hirzebruch signature theorem and the Hirzebruch-Riemann-Roch formula (the latter is of fundamental importance in complex geometry), but start with the easy application [L-M], II.13.12 (see also the remark after proposition 9.2 of [AS3]). You should then first prove proposition 2.17 of [AS3] and then explain the relevant parts of §4 and §6. For the signature theorem, you will need to explain some basic Hodge theory, for which [Nic], 10.4.3, is a source. Alternative sources for these applications are the theorems 13.9, 13.13, 13.14 in [L-M] and (for a more extensive, but older account) chapters III, V and XIX in [Pal]. For motivation of the Hirzebruch-Riemann-Roch you may have a look at [Huy]. If time permits, you can also briefly allude to G-equivariant theorems (but don't state the concrete formulas!).

Talk 9 (Clifford Algebras): Following the first chapter of [L-M], define Clifford algebras, concentrating on the real and complex case from the outset. Explain the $\mathbb{Z}_2$-grading and the relationship with the graded tensor product (Prop. 1.5), define the Spin group and explain how it is the double cover of $SO(n)$ (2.10). Then motivate the periodicity isomorphisms (4.10) which leads to the classification of Clifford algebras as certain matrix algebras. Then say something about the representation theory of Clifford algebras, maybe concentrating on complex representations. Both ungraded and graded representations and the relationship between the two should be mentioned. At some point in the talk, you should also mention that Clifford algebras are "nearly" group rings and the ramifications of this for the representation theory (5.16). If time permits, you could also mention the Atiyah-Bott-Shapiro construction (9.27 or the original paper [ABS]).

Talk 10: Spinor bundles I Give a quick introduction to Spin manifolds (II.1 and II.2 in [L-M]). Define the Clifford bundle associated to a Riemannian manifold (or any vector bundle with an inner product) (II.3.4) and Spinor bundles (II.3.6). Since any module over a Clifford algebra carries a compatible scalar product (I.5.16), real Spinor bundles carry a Riemannian structure and complex ones a Hermitian structure. Then define the Dirac operator (II.5.0) of a Spinor (or, more generally, a Dirac bundle, see II.5.2) and compute its symbol. You probably need to explain some basic facts about connections on vector bundles and the Levi-Civita connection on a Riemannian manifold.
For the remainder of the talk, you should concentrate on the even-dimensional case. In even dimensions, it follows from the representation theory of Clifford algebras that there is a unique irreducible complex ungraded Spinor bundle. Show how to split this bundle using the complex volume element (the discussion after II.3.10) and explain why the Dirac operator is an odd operator (II.6). Finally, introduce the operator D^+ to which we want to apply the Atiyah-Singer Index theorem (II.6).

Talk 11: Spinor bundles II Prove Theorem III.13.10 in [L-M]. Again, the method of computation is maybe more important than the actual result. Give the first few terms of the $\hat{A}$-genus and an example of a non-Spin manifold where it fails to be an integer. It follows that the $\hat{A}$-genus is an integer for Spin manifolds; by a closer examination, one sees that it is an even integer in dimensions $= 4(8)$ (IV.1.1). Rohlin's theorem (IV.1.2) immediately follows from this and the Hirzebruch signature theorem. The discussion after (IV.1.2) about Freedman's non-smooth Spin manifolds should also be mentioned. Then

you should prove (IV.1.4) and explain that this means that no high-dimensional sphere carries a complex structure.
If time permits, you could mention the results about immersions of manifolds (IV.2.1), or any other application of the Atiyah-Singer Index theorem.

Talk 12: The Cl_k-index theorem, positive scalar curvature and exotic spheres This is more of an outlook/overview talk. Explain what scalar curvature is and inhowfar there can be topological obstructions to prescribing scalar curvature (Theorem 0.1 in [Ros]). You should also explain that there are no obstructions in the non-spin case (IV.4.4). Then explain the Lichnerowicz formula (II.8.8), which shows that the $\hat{A}$-genus has to vanish on a manifold with positive scalar curvature. This is only an obstruction in dimensions $= 4(8)$; however, one can do slightly better. Explain the basics of Cl_k-linear operators and the corresponding index theory (II.7), leading to the $\hat{\alpha}$-genus. Since the Dirac operator is injective on a Spin manifold of positive scalar curvature, it follows that also this refined index vanishes, which gives additional information in dimensions $= 1$ and $= 2(8)$. By a result of Stolz, this is the only obstruction for positive scalar curvature in the simply-connected case. Finally, explain that only half of the exotic spheres in these dimensions can carry positiv scalar curvature (II.2.18 and the discussion before).

References

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