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From Euclidean Field Theory to K-theory I

Juan Wang

1-EFT's and the real K-theory spectrum

Theorem .1. $K^{-n}(B) \cong [B, \mathcal{EFT}_n^{\mathbb{C}}]$ and $KO^{-n}(B) \cong [B, \mathcal{EFT}_n^{\mathbb{R}}]$.

In other words, the space susy $\mathcal{EFT}_n^{\mathbb{F}}$ form an Ω-spectrum representing complex and real K-theory, resp. i.e. $\mathcal{EFT}_n^{\mathbb{R}} \simeq KO_{-n}$, the $(-n)$-th space in the real K-theory spectrum KO.

Note: it implies that the connected components of the space of susy EFT's of degree n are the homotopy groups of the spectrum:

$$\pi_0(\mathcal{EFT}_{\boldsymbol{n}}) = \pi_0(KO_{-n}) = KO^{-n}(pt) = KO_n(pt)$$

recall

Theorem .2. (Stolz, Teichner) Let $\mathcal{EFT}(H_n)$ be the set of 1-dim supersymmetric euclidean field theory functors on some fixed Hilbert space H_n with compactible right C_n module structure. Then there is a bijection

$$\mathcal{EFT}(H_n) \xleftrightarrow{1-1} \mathrm{ssgh}(\mathbb{R}^{1|1}_{>0}, HS^{sa}_{C_n}(H_n))$$

$$E \longleftrightarrow (t,\theta) \longmapsto E(I_{t,\theta})$$

<u>Thm</u>: every homomorphism of super semi-groups as above is of the shape.

$$\mathbb{R}^{1|1}_{>0}(S) \longrightarrow HS^{sa}_{C_n}(H_n)(S) : (t,\theta) \longmapsto e^{-tG^2+\theta G}$$

for some operator G on H_n.

$$\begin{aligned} HS^{sa}_{C_n}(H_n)(S) &:= (HS^{sa}_{C_n}(H_n) \otimes \Gamma(\mathcal{O}_S))^{ev} \\ &= HS^{sa,ev}_{C_n}(H_n) \otimes \Gamma(\mathcal{O}_S)^{ev} \oplus HS^{sa,odd}_{C_n}(H_n) \otimes \Gamma(\mathcal{O}_S)^{odd} \end{aligned}$$

call G the generator of the field theory.

$$\mathcal{EFT}(H_n) \xleftrightarrow{R} \mathrm{ssgh}(\mathbb{R}^{1|1}_{>0}, HS^{sa}(H_n)) :$$

$T_{t,\theta} : \mathbb{R}^{1|1} \longrightarrow \mathbb{R}^{1|1}$ preserves the metric structure

$(z,\eta) \longmapsto (t+z+\theta\eta, \theta+\eta)$ given by even 1-form $\omega = dz + \eta d\eta$.

$I_{t,\theta} := [0,t] \times \mathbb{R}^{0|1} \subset \mathbb{R}^{1|1}$

(1/1)-dim smfd with metric st. ω

$I_{t,\theta}(\Lambda) = \Lambda^{ev}_{>0} \times \Lambda^{odd}$
$(z \,,\, \eta)$

$z = \underset{\in [0,t] = \Lambda^0_{>0}}{z_B} + z_S \in \bigoplus_{p=1}^{\infty} \Lambda^p$

consider $I_{t,\theta} \in S\mathcal{EB}^1_n(\underset{\underset{0\times\mathbb{R}^{0|1}}{\shortparallel}}{spt}, spt)$ by identifying $\{0\}\times\mathbb{R}^{0|1} \sim \{t\}\times\mathbb{R}^{0|1}$ by the translation $T_{(t,\theta)}$

composition in $S\mathcal{EB}^1_n$: $I_{t_2,\theta_2} \circ I_{t_1,\theta_1} = I_{t_1+t_2+\theta_1\theta_2, \theta_1+\theta_2}$ $\quad \theta_1 = 0 + \theta_1$ body soul

$[0,t_2] \times \mathbb{R}^{0|1} \underset{\sim}{\overset{T_{t_1,\theta_1}}{}} [t_1, t_1+t_2] \times \mathbb{R}^{0|1}$

$\Rightarrow$ $R(E) : \mathbb{R}^{1|1}_{>0} \longrightarrow HS(E(pt)) \subset HS(H_n)$
$(t,\theta) \longmapsto E(I_{t,\theta})$

is a super homomorphism.

$E(I_{t,\theta})$ is s.a., but in general <u>not</u> <u>even</u>.

<u>in $\mathcal{EB}^1_n$</u>, canonical spin involution ε on I_t

$\Rightarrow \varepsilon \circ I_t = I_t \circ \varepsilon$

$\Rightarrow E(\varepsilon) \circ E(I_t) = E(I_t) \circ E(\varepsilon) \Rightarrow E(I_t)$ even

<u>in $S\mathcal{EB}^1_n$</u>, $\boxed{\varepsilon \circ I_{t,\theta} \circ \varepsilon = I_{t,-\theta}}$

$\Sigma = \{0\}$, $S(\Sigma) = \{0\} \times \mathbb{R}^{0|1} \subset \mathbb{R}^{1|1}$

$\varepsilon \circ T_{t,\theta} \circ \varepsilon = T_{t,-\theta}$ $\quad \varepsilon : (z,\eta) \to (z,-\eta)$

$\varepsilon_\Sigma = id : \Sigma \longrightarrow \Sigma$

$\varepsilon_\Sigma = -1 : S(\Sigma_x) \longrightarrow S(\Sigma_x)$

fibre of spinor bundle $S(\Sigma) \to \Sigma$

recall:

Susy EFT $:\Leftrightarrow$ ssgh $\phi: \mathbb{R}^{1|1}_{>0} \longrightarrow HS^{sa}_{C_n}(H_n)$

ssg structure

$((t_1,\theta_1),(t_2,\theta_2)) \longmapsto (t_1+t_2+\theta_1\theta_2, \theta_1+\theta_2)$ ssg structure

consists of smooth maps composition of operators.

$A(t) + \theta B(t)$

where $A: \mathbb{R}_{>0} \longrightarrow HS^{sa,ev}_{C_n}(H_n)$

$B: \mathbb{R}_{>0} \longrightarrow HS^{sa,odd}_{C_n}(H_n)$

example: D C_n-linear Dirac operator

$(t,\theta) \longmapsto e^{-tD^2+\theta D} = e^{-tD^2} + \theta D e^{-tD^2}$

generally, $\forall$ C_n-submodule $V_\infty \subset H_n$

$\forall$ odd, densely defined, sa D on $V_\infty^\perp$.

$\exists!$ ssgh $\phi = A + \theta B : \mathbb{R}^{1|1}_{>0} \longrightarrow K^{sa}_{C_n}(H_n)$

$A(t) = e^{-tD^2}$, $B(t) = De^{-tD^2}$ on $V_\infty^\perp$

and $A(t) = B(t) = 0$ on V_∞.

check: s sg h

D defines an EFT $\Longleftrightarrow$ A, B are H.S. $\forall t$
$\Uparrow$
ev. $D \longrightarrow \infty$

e^{-D} is compact $\not\Longrightarrow$ D compact / bounded

$$\text{ev.}(e^{-D}) = e^{-(\text{ev. } D)}$$

$$\rightsquigarrow e^{-\lambda} = \frac{1}{e^{\lambda}} \text{ ev. } e^{-D} .$$
$\searrow 0.$

note: λ small $\Rightarrow e^{-D}$ not bounded!

ex: D Dirac operators $\checkmark$

Lemma: Let $A, B : \mathbb{R}_{>0} \longrightarrow K^{sa}(H)$ be smooth families of s.a. compact operators and satisfying $\forall s, t > 0$

1) $A(s+t) = A(s)\,A(t)$
2) $B(s+t) = A(s)\,B(t)$
3) $A'(s+t) = -B(s)\,B(t)$.

Then H decomposes uniquely into orthogonal subspaces H_λ, $\lambda \in \mathbb{R} \cup \{\infty\}$ s.t. on H_λ $A(t) = e^{-\lambda t^2}$ and $B(t) = \lambda e^{-\lambda t}$ (set $e^{-\infty} = 0$, $\infty \cdot 0 = 0$). And $\{\lambda \in \mathbb{R} \mid H_\lambda \neq 0\}$ is discrete

Pf.: 1), 2) $\Rightarrow$ all operators $A(s)$, $B(t)$ commute.
apply the spectral thm for self-adjoint, compact operato[rs] to obtain a decomposition of H into simult. eigenspaces H_λ of $A(s)$ and $B(t)$ with $\lambda \in \mathbb{R} \cup \{\infty\}$.

Define functions $A_\lambda, B_\lambda : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$ by

$$A(t)\,x = A_\lambda(t)\,x \quad , \quad B(t)\,x = B_\lambda(t)\,x \quad , \forall x \in H_\lambda$$

$\Rightarrow$ A_λ, B_λ are smooth and satisfy 1) – 3).

$\overset{1)}{\Rightarrow}$ $A_\lambda \geq 0$ $\quad \overset{3)}{\Rightarrow} A'_\lambda \leq 0 \quad \Rightarrow A_\lambda$ is decreasing.

$\overset{1)}{\Rightarrow}$ $A_\lambda(\frac{1}{n}) = \sqrt[n]{A_\lambda(1)} \quad \Rightarrow$

$$A_\lambda(0) := \lim_{t\to 0} A_\lambda(t) \text{ exists and equals } 0 \text{ or } 1.$$

in the first case $\Rightarrow$ $A_\lambda \equiv 0$ and thus also $B_\lambda \equiv 0$; the label of the corresponding subspace is $\lambda = \infty$.

in the second case $A_\lambda(1) \neq 0$, using 1) compute

$$A'_\lambda(s) = \frac{A_\lambda(1)}{A_\lambda(1)} \lim_{t\to 0} \frac{A_\lambda(s+t) - A_\lambda(s)}{t}$$

$$= \frac{A_\lambda(s)}{A_\lambda(1)} \lim_{t\to 0} \frac{A_\lambda(1+t) - A_\lambda(1)}{t} = -\lambda^2 A_\lambda(s)$$

where $\lambda^2 := -\dfrac{A'_\lambda(1)}{A_\lambda(1)}$

Because solutions of ODEs are unique

$$\Rightarrow A(t) = e^{-t\lambda^2}$$

$$\overset{3)}{\Rightarrow} \quad B_\lambda(t) = \sqrt{\lambda^2 e^{-2t\lambda^2}} = \lambda e^{-t\lambda^2}.$$

∎

Given a ssgh $\phi = A + \theta B$,

exploit the homomorphism property of ϕ for $U = \mathbb{R}^{0|2}$.

Let θ, η be the odd coordinates on $\mathbb{R}^{0|2}$, and let $s, t \in \mathbb{R}_{>0}$, considered as constant (even) functions on $\mathbb{R}^{0|2}$.

$$\begin{aligned}\phi(s+t+\eta\theta, \eta+\theta) &= A(s+t+\eta\theta) + (\eta+\theta)\, B(s+t+\eta\theta) \\ &= A(s+t) + A'(s+t)\,\eta\theta + (\eta+\theta)\big(B(s+t) + B'(s+t)\eta\theta\big) \\ &= \underline{A(s+t)} + \eta\, \underline{B(s+t)} + \theta\, B(s+t) + \eta\theta\, \underline{\underline{A'(s+t)}}\end{aligned}$$

equal to

$$\begin{aligned}\phi(s,\eta)\,\phi(t,\theta) &= \big(A(s) + \eta\, B(s)\big)\big(A(t) + \theta\, B(t)\big) \\ &= \underline{A(s)\,A(t)} + \eta\, \underline{B(s)\,A(t)} + \theta\, A(s)\,B(t) \underline{\underline{- \eta\theta\, B(s)\,B(t)}}\end{aligned}$$

$\Longrightarrow$ 1) – 3).

$\blacksquare$

<u>remark</u>: ① A super morphism $f: \mathbb{R}^{1|1}_{>0} \longrightarrow K^{sa}(H)$

$f = A(t) + \theta B(t)$ need to satisfy

$\forall$ finitely generated exterior $\mathbb{R}$-algebra Λ, the map

$$\Lambda(\mathbb{R}^{1|1}_{>0}) := \Lambda^{ev}_{>0} \times \Lambda^{odd} \longrightarrow (\Lambda \otimes K^{sa}(H))^{ev} =: \Lambda(K^{sa}(H))$$
$$(t, \theta) \longmapsto A(t) + \theta B(t)$$

is a sgh.

② f is an element in $\mathcal{SM}(\mathbb{R}^{1|1}_{>0}, K^{sa}(H))$.

Think of an element in $\mathcal{SM}(\mathbb{R}^{1|1}_{>0}, K^{sa}(H))$ as a pair of ∞-dim matrix with values in $\mathcal{O}^{ev}_{\mathbb{R}^{1|1}_{>0}}$ and $\mathcal{O}^{odd}_{\mathbb{R}^{1|1}_{>0}}$. But $\mathcal{O}_{\mathbb{R}^{1|1}_{>0}} = C^\infty(\mathbb{R}_{>0})[\theta]$

These are exactly the "$A(t)$ & $B(t)$" in the def.

③ $A(t), B(t)$ for $t = t_B + t_S$ (t_S nontrivial) are defined by Taylor expansion. $A(t) = \sum_{n=0}^{\infty} \frac{1}{n!} A^{(n)}_{(t_B)} t_S^n$

($t_S^2 = 0$ in case $C^\infty(\mathbb{R}_{>0})[\theta]$)

Define $\mathcal{EFT}_n = \{ \text{ssgh}(\mathbb{R}^{1|1}_{>0} \longrightarrow HS^{sa}_{C_a}(\mathcal{H}_n) \}$ with the topology of pointwise convergence in A, B.

property of G:

1) e^{-tG^2} are even, s.a., Clifford-linear, H.S.

$\Rightarrow$ G is odd, Clifford-linear.

2) G has an eigenspace decomposition with real eigenvalues which are discrete but accumulate at ∞.

$\Rightarrow$ G is unbounded and can in general only be defined on a dense subset $D(G)$, i.e. the algebraic sum of its eigenspaces.

Def: Let H_n be a Hilbert space with C_n-module structure.

$$\mathrm{Sym}(H_n) := \{ \text{linear operator } G: D(G) \longrightarrow \overline{D(G)} \mid$$

$D(G) \subseteq H_n$, G is C_n-linear, has compact resolvent,

$G|_{\overline{D(G)}}$ is self-adjoint, e^{-G^2} is Hibert-Schmidt$\}$

Then $\quad \mathcal{EFT}(H_n) \xleftrightarrow{1-1} \mathrm{Sym}(H_n)$

$\forall\, G \in \mathrm{Sym}(H_n)$ has discrete, real e.v. (which can accumulate at ∞) and its domain splits into pairwise orthogonal, finite-dim eigenspaces (spectral thm for self-adjoint operators.)

Configurations and their Topology

Let H be a Hilbert space and (X, A), $A \subset X$ be a pair of spaces. Define the space of configurations $Conf(X, A; H)$ indexed by subspaces of H to be the space of maps $c: X \longrightarrow Proj(H)$ s.t.

- $C(x) \perp C(y)$ if $x \neq y$
- $\dim c(x) < \infty$, $\forall\, x \in X \setminus A$
- $\{x \in X \setminus A \mid c(x) \neq 0\}$ is a discrete subset of $X \setminus A$.
- H is equal to the Hilbert sum of $c(x)$, $x \in X$.

Consider finest topology on this space s.t. following things can happen:

- We want the labels to add in direct sum, whenever two points meet.
- As long as the nonzero labels $x \in X$ don't collide, the corresponding subspaces $C(x)$ inherit their topology from that of the Grassmannian.

if we have an involution s on the pair (X, A) and $\mathbb{Z}/2$-grading inv α on H.

Define <u>odd</u> configurations $Conf^{odd}(X, A; H) = \{ c \in Conf(X, A;$
$: \quad c(s(x)) = \alpha(c(x)) \}$

<u>finite</u> $Conf^{f}(X, A; H) = \{ c \in Conf(X, A; H) \mid pts(c)$ are finite. $\}$

Fix for each $n \geq 0$, a separable Hilbert space H_n with an action of C_{n-1} s.t. each generator e_i acts as a bounded, skew-adjoint operator and s.t. each irreducible representation of C_{n-1} appears with infinite multiplicity.

$\Rightarrow$ $\mathbb{Z}/2$-graded C_n-module

$$\mathcal{H}_n := H_n \otimes_{C_{n-1}} C_n \qquad C_{n-1} \longrightarrow C_n^{ev} \subset C_n$$

Fix: the configurations are indexed by subspaces of $\mathcal{H}_n$.

<u>Prop</u>: Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. $\forall n$.

$$\{ \text{ssgh}(\mathbb{R}^{1|1}_{>0} \longrightarrow K^{sa}_{C_n}(\mathcal{H}_n) \} \cong \text{Conf}^{\text{odd}}_{C_n}(\overline{\mathbb{R}}, \infty)$$

inv on $\overline{\mathbb{R}}$ by $s(x) = -x$.

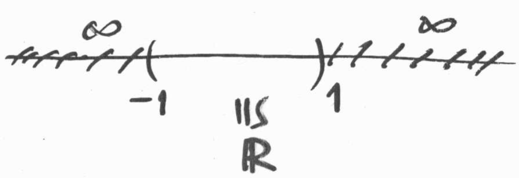
<u>note</u>: for fixed H, $(X, A) \longmapsto \text{Conf}(X, A)$ is a functor:

Given continuous $f: (X, A) \longrightarrow (Y, B)$

induces a continuous $f_*: \text{Conf}(X, A) \longrightarrow \text{Conf}(Y, B)$

<u>Cor</u>: $\mathcal{EFT}_n \simeq \text{Conf}^{f,\text{odd}}_{C_n}(\overline{\mathbb{R}}, \infty)$

<u>Pf</u>: Take $h_1 : \overline{\mathbb{R}} \xrightarrow{\cong} \overline{\mathbb{R}}$



$h_1|_{(-1,1)} \cong \mathbb{R}$

$h_1|_{(-1,1)^c} = \infty$.

$\exists$ homotopy $h_1 \sim id = h_0$.

induces a deform retract $\mathrm{Conf}^{\mathrm{odd}}_{C_n}(\overline{\mathbb{R}},\infty) \longrightarrow \mathrm{Conf}^{f,\mathrm{odd}}_{C_n}(\overline{\mathbb{R}},\infty)$

And h_t preserve the subspace $\mathcal{EFT}_n \subset \mathrm{Conf}^{\mathrm{odd}}_{C_n}(\overline{\mathbb{R}},\infty)$ $\forall t$.

$\Rightarrow$ $\mathrm{Conf}^{f,\mathrm{odd}}_{C_n}(\overline{\mathbb{R}},\infty)$ is a deformation retract of $\mathcal{EFT}_n$. $\boxtimes$

Susy and non-susy $\mathcal{EFT}$.

non-susy $\mathrm{EFT}_n = \{ \mathrm{sgh}(\mathbb{R}_{>0}, \mathrm{HS}^{sa,ev}_{C_n}(\mathcal{H}_n)\}$

$$\begin{array}{l} \text{IS} \\ * \end{array} \quad = \{ t \longmapsto E(I_t) := e^{-tA}\}$$

by $t \longmapsto s^t E(I_t)$, $\forall s \in [0,1]$

in Configuration:

$\sigma(A)$ need to be left-bounded, i.e. only finitely many negative eigenvalues.

$\exists$ smallest e.v. λ_{min}. one can contract $[\lambda_{min}, \infty)$ $\Rightarrow$ induces a contraction on Conf.

Susy $\mathcal{EFT}_n = \{ \mathrm{ssgh}(\mathbb{R}^{1|1}_{>0}, \mathrm{HS}^{sa}_{C_n}(\mathcal{H}_n)\}$

$$(t,\theta) \longmapsto E(I_{t,\theta}) := e^{-tG^2+\theta G}$$

with G odd operator. Configurations symmetric about the origin, not possible to do a similar contraction move while preserving symmetry.

recall (Definition of EFT_n)

susy EFT of dim 1|1 and degree n

ssgh $\phi : \mathbb{R}^{1|1}_{>0} \longrightarrow HS^{sa}_{C_n}(H_n)$

twisted super semi group structure

$((t_1,\theta_1),(t_2,\theta_2)) \longmapsto (t_1+t_2+\theta_1\theta_2, \theta_1+\theta_2)$

with super semi-group structure induced by composition of operators.

consists of smooth maps. $A(t)+\theta B(t)$

where $A : \mathbb{R}_{>0} \longrightarrow HS^{sa,ev}_{C_n}(H_n)$

$B : \mathbb{R}_{>0} \longrightarrow HS^{sa,odd}_{C_n}(H_n)$.

<u>example</u>: If D is the C_n-linear Dirac operator on a spin mfd X, then there is a corresponding field theory

$$(t,\theta) \longmapsto e^{-tD^2+\theta D} = e^{-tD^2} + \theta D e^{-tD^2}$$

More generally, given any C_n-submodule $V_\infty \subset H_n$ and any odd, densely defined, self-adjoint operator D on $V_\infty^\perp$ with compact resolvent, ~~there is a~~ $\exists!$ ssgh into the C^*-algebra of compact operators, self-adjoint and Clifford linear

$$\phi = A+\theta B : \mathbb{R}^{1|1}_{>0} \longrightarrow K^{sa}_{C_n}(H_n).$$

defined by (functional calculus)

$$A(t) = e^{-tD^2} \text{ and } B(t) = De^{-tD^2} \text{ on } V_\infty^\perp.$$

and $A(t) = B(t) = 0$ on V_∞.

Check: ssgh.

D defines an EFT $\Longleftrightarrow$ A, B are Hilbert Schmidt, $\forall t$.

$\mathcal{B}(H)$

C^*- algebra is a subalgebra of the algebra of bounded operators on some Hilbert space which is closed under the operation $a \longmapsto a^*$ of taking adjoints and which is a closed subset of all bounded operators w.r.t. the operator norm.

The C^*-algebra $K(H)$ of compact operators on a Hilbert space H; if H is graded, $K(H)$ is a graded C^*-algebra.

If A, B are graded C^*-algebra, let $C^*(A,B)$ be the space of grading preserving $*$-homomorphism $f: A \longrightarrow B$ (i.e. $f(a^*) = f(a)^*$,) equipped with the topology of pointwise convergence, i.e. a sequence f_n converges to f iff $\forall a \in A$, $f_n(a) \longrightarrow f(a)$

$$(e^{-t_1x^2}+\theta_1 x e^{-t_1x^2})(e^{-t_2x^2}+\theta_2 x e^{-t_2x^2})$$

$$= e^{-(t_1+t_2)x^2} + \theta_1 x e^{-t_1x^2}\cdot\theta_2 x e^{-t_2x^2} + (\theta_1+\theta_2) x e^{-(t_1+t_2)x^2}$$

$$= e^{-(t_1+t_2)x^2} - \theta_1\theta_2 x^2 e^{-(t_1+t_2)x^2} + (\theta_1+\theta_2) x e^{-(t_1+t_2)x^2}$$

commute the odd element x
past the odd element θ_2

$$= e^{-(t_1+t_2+\theta_1\theta_2)x^2} + (\theta_1+\theta_2) x e^{-(t_1+t_2)x^2}$$

Take the Taylor expansion
around the pt t_1+t_2.

$$= e^{-(t_1+t_2+\theta_1\theta_2)x^2} + \underline{(\theta_1+\theta_2)}\, x e^{-(t_1+t_2+\theta_1\theta_2)x^2}$$

the higher terms of the expansion are
annihilated by multi by $\theta_1+\theta_2$

Super semi-groups of self-adjoint compact operators & k-theory

1. Def : (super semi-group of operators).

Let $H = H_+ \oplus H_-$, a graded Hilbert space (where $H_\pm$ are ∞-dim & separable)

A <u>super semi-group of self-adjoint compact operators</u> on H ($\Leftrightarrow$ a super semi-group homomorphism $\mathbb{R}^{1|1}_{>0} \longrightarrow K^{sa}(H)$) consists of smooth maps

$A : \mathbb{R}_{>0} \longrightarrow K^{sa}(H)^{even}$, $B : \mathbb{R}_{>0} \longrightarrow K^{sa}(H)^{odd}$ s.t.

① $\forall$ finitely generated exterior $\mathbb{R}$-algebras $\Lambda = \mathbb{R}[\theta_1, \cdots, \theta_n]$ $n \geq 0$, the map

$$\Lambda(\mathbb{R}^{1|1}_{>0}) := \Lambda^{ev}_{>0} \times \Lambda^{odd} \longrightarrow (\Lambda \otimes K^{sa}(H))^{ev} =: \Lambda(K^{sa}(H))$$

$$(t, \theta) \longmapsto A(t) + \theta B(t)$$

is a ~~group~~ homomorphism. Furthermore, these maps are functorial in Λ (in the obvious way).

① $\searrow$ see below ② As $t \longrightarrow 0$, $A(t)$ converges strongly to a projection operator on H

<u>remark</u> : ① A 'super morphism' $f : \mathbb{R}^{1|1}_{>0} \longrightarrow K^{sa}(H)$ is an element in $\mathcal{SM}(\mathbb{R}^{1|1}_{>0}, K^{sa}(H))$ (see prop 2.4 in [DM]) we should think of an element in $\mathcal{SM}(\mathbb{R}^{1|1}_{>0}, K^{sa}(H))$ as a pair of ∞-dim matrices ($\Leftarrow$ operator) with values in $\mathcal{O}^{ev}_{\mathbb{R}^{1|1}_{>0}}$ & $\mathcal{O}^{odd}_{\mathbb{R}^{1|1}_{>0}}$. But $\mathcal{O}_{\mathbb{R}^{1|1}_{>0}} = C^\infty(\mathbb{R}_{>0})[\eta]$. These are exactly the "$A(t)$" & "$B(t)$" from the definition. The group homomorphism condition on "Λ-points" reflects the fact that we look at a "super semi-group homomorphism

② $\boxed{Q}$: Is the second condition in the Def implied by the first one? Yes! See Lemma / Prop on page 9/10.

③ $A(t)$, $B(t)$ for t with <u>non-trivial super ('soul') part</u> are defined per Taylor expansion, as usual in super analysis. (see [DM], [Lei]

④ Note that since $A(t)$, $B(t)$ are required to be smooth, we can consider the derivation of the super semi-group homomorphism $(t, \theta) \longmapsto A(t) + \theta B(t)$ w.r.t. vector fields on $\mathbb{R}^{1|1}_{>0}$.