HOMOTOPY & COHOMOLOGY
GK 1150

GRK 1150, Mathematisches Institut, Universität Bonn, 53115 Bonn

Winter School

"From Field Theories to Elliptic Objects"

February, 28th till March, 4th 2006
Schloss Mickeln, Düsseldorf

GRK-Sprecher:
Prof. Dr. C.-F. Bödigheimer
Arbeitsgruppe Topologie
Beringstrasse 1
D- 53115 Bonn
Email: grk1150@math.uni-bonn.de

GRK-Sekretariat:
Frau S. George
Tel: 0228-73 77 88
Fax: 0228-73 64 90

Talk No. 13

Speaker: Elke Markert

universitätbonn
Rheinische Friedrich-Wilhelms-
Universität Bonn

From Field Theories to Elliptic Objects?

Plan: I Review: Field theories

II Outlook: Field theory objects

I. Review.

Euclidean ($d=1$) Field Theories.

Definitions: - Clifford-linear EFT's of degree n:

$\mathcal{EB}^1_n \to \mathrm{Hilb}$

SUSY Clifford-linear EFT's (degree 0):

$\mathcal{SEB}^1 \to \mathrm{Hilb}$.

- Spaces $\mathfrak{EFT}_n$ (EFT_n) of susy (non-susy) Cliff.-linear EFT's in each degree n.

Results: - $\boxed{\mathfrak{EFT}_n \simeq KO(-n).}$ ($\star$)

($EFT_n \simeq *$).

- Partition function:

$$\boxed{Z_E : \pi_0\, \mathcal{E}\mathfrak{FT}_* \to \mathbb{Z}} \quad (\star\star)$$

$$[\mathcal{E}_G] \mapsto \mathrm{sdim}_{C_n}(\ker G)$$

$n=4k$: $[\mathcal{E}_{{}_p\not{D}_{M^n}}] \mapsto \mathrm{sdim}_{C_n}(\ker \not{D}_M)$

(susy standard ex.) $= \hat{A}(M) \in \mathbb{Z} \cong KO_n(pt)$

Results: One would wish for analogous results as for EFT's, i.e.:

- $\boxed{\mathcal{C}\mathfrak{FT}_n \simeq TMF(-n)}$ $(\star)'$

- Partition function:

$$\begin{tikzcd}
\mathcal{Z}_C \colon \pi_0 \mathcal{C}\mathfrak{FT}_n \arrow[r] \arrow[dr, dashed] & MF_n := \mathbb{Z}[c_4, c_6, \Delta^{\pm 1}] \big/ (c_4^3 - c_6^2 - 12^3 \\
& TMF_n(pt) \arrow[u]
\end{tikzcd}$$

$(\star\star)'$

Analogon of susy standard example for $\mathfrak{EFT}$s : $\mathcal{C}_M$ coming from string mfld M (Heuristically: "$H = L^2(LM, S_{LM})$", operators given via "$\not{D}_{LM}$"), s.th.

$\boxed{\mathcal{Z}_C([\mathcal{C}_M]) = \varphi_W(M)}$ Witten genus. $(\star\star)''$

Remark: $(\star)'$, $(\star\star)''$: Open Problems.

FIELDS MEDAL

$(\star\star)'$: Definition of $\mathcal{Z}_C$ into modular forms seems likely to work, however:

Need to define SUSY version of CFT's.

In part.: "susy conformal structure"

(analogous: "metric structure").

Conformal ($d=2$) Field Theories.

<u>Definitions:</u> – Clifford-linear CFT's of degree n:

<u>Def.:</u> ~ are functors $\underline{\mathcal{E}: \mathcal{CB}^2_n \to \mathrm{Hilb}}$, satisfying the usual axioms, where:

$\mathcal{CB}^2_n$: objects: Y: 1-dim.l closed conformal spin mfld w/ germ of (2-dim.l) conf. spin collars $\mathcal{U}(Y)$

$\mathcal{CB}^2_n(Y_1, Y_2)$: two types of morphisms:

(f,c): $f: Y_1 \to Y_2$ conf. spin diffeom. extending over collars

$c \in C(Y_1)^{-n}$.

(Σ, ψ): (equiv. class of) Σ: 2-dim.l conformal spin mfld w/ collar $\mathcal{U}(\Sigma)$, and bdry embeddings $\mathcal{U}(\overline{Y}_1) \amalg \mathcal{U}(Y_2) \hookrightarrow \mathcal{U}(\Sigma)$, $\overline{Y}_1 \amalg Y_2 \cong \partial\Sigma$ conf. spin diffeo.

$\psi \in F_{alg}(\Sigma)^{-n}$.

(equiv. relation: conf. spin diffeo rel. collared bdry, respecting ψ)

Composition: Glue conf. structures on overlapping collars.

Remark: (susy conf. structure) Let $\mathcal{M}^{2|1}$ smfld.

Then a susy conf. str. on $\mathcal{M}$ should consist of two distributions $\mathcal{V}, \mathcal{D}$:

$\mathcal{V}$ - a rank $(1|0)$ (loc. free) submodule of the $\mathcal{O}_{\mathcal{M}}$-module $\mathcal{D}er(\mathcal{O}_{\mathcal{M}}) =: \mathcal{T}\mathcal{M}$,

$\mathcal{D}$ - a rank $(0|1)$ submodule of $\mathcal{T}\mathcal{M}$,

sth. $\{\mathcal{V}, \mathcal{D}, \mathcal{D}^2\}$ span $\mathcal{T}\mathcal{M}$.

Talk (15): Presents "reasonable" assumption on susy confomal str., using which one can prove that Z_C returns elements in MF_*.

II. Outlook: Field Theory Objects and (elliptic) Euler class.

"Field theory object": This should be a field theory (EFT/CFT) $\mathcal{E}_X$ over a space X, representing an element $[\mathcal{E}_X]$ in the corresponding cohomology $FT^*(X)$.

I.e. want direct construction of cohomology functors

$$X \mapsto FT^*(X) := \left\{ \begin{array}{c} \text{equiv. classes of FT-objects} \\ \text{over } X \end{array} \right\}.$$

In particular: $E^n \to X$ v.bdle (spin/string), want to construct $\mathbb{E}_E$ in $FT^n(X)$ (Euler class).

Note: Could use $FT^*(X) = [X, \mathcal{FT}_*]$, if we have description of spectrum $\mathcal{FT}_*$.

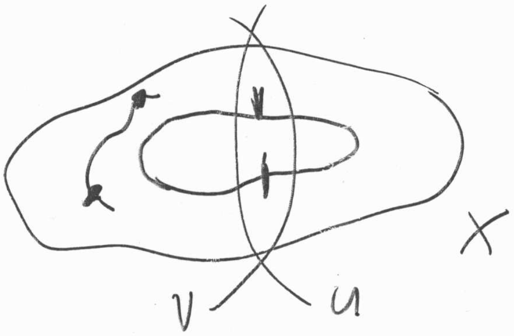
Def.: A Clifford-linear d-dim.l FT-object of degree n over X is a functor $\mathcal{B}^d_n(X) \to \text{Hilb}$, satisfying the usual axioms. $\mathcal{B}^d_n(X)$ is $\mathcal{B}^d_n$ where objects and mor.'s are equipped w/ piecew. smooth maps into X.

Remark: [Susy versions of $\mathcal{EFT}$'s over X; Work in progress.

2-dim.l CFT-object $\mathcal{E}: \mathcal{CB}^2_n(X) \to \text{Hilb}$: "pre-elliptic object".

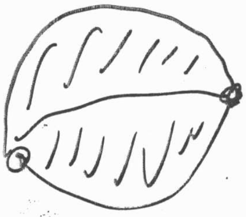
Why? Excision Problem:

$\mathcal{E}$ is not "local".



Stolz/Teichner Solution: "Enrichment"

$\mathcal{E}$ needs to associate data to



- 0 - mfld
- 1 - mflds : $\overset{\text{cpt.}}{\text{open}}$ & closed
- 2 - mflds

Gluing: along arcs of bdry.

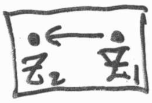
Def.: An enriched elliptic object $\mathcal{E}$ of degree n over X is a bi-functor between bi-categories $\mathcal{D}_n(X) \to vN$, satisfying certain axioms.

$\mathcal{D}_n(*)$:

obj.:

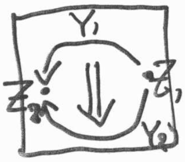
- 0-dim spin mflds w/ 2-dim conf. collars & orient

1-mor.:



- 1-dim.l conf. spin mflds w/ collar and bdry embeddings (collared)
- conf. spin diffeo's of collared 0-mfl

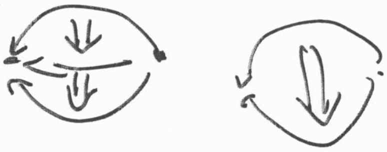
2-mor.:



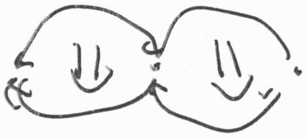
- conf. spin diffeom. of (collared) 1-mflds, together w/ $c \in C(Y_1)$
- (equiv. classes) 2-dim.l conformal spin mflds Σ (w/ collars), bdry embeddings etc. as before, together w/ $\psi \in F_{als}(\Sigma)^{-n}$.

Compositions: 1-mor.: $\cdot \leftarrow \cdot \leftarrow \cdot$

2-mor.: vertical:



horizontal:



vN: bi-category of von Neumann algebras, bimodules, and intertwiners.

Talk (14)!

objects: ∼ weak op. closure of *-subalgebras of Bdd(H).

Bimodules: Hilbert spaces

Product: Connes fusion

Intertwiners: Bdd operators.

Note: There is a problem w/ adjunction on vN.

$$vN(\mathbb{C}, \boxed{F_2 \boxtimes_A F_1}) \rightarrow vN(\bar{F}_2, F_1).$$

Use l.h.s. for FT-objects.

EFT - Euler class.

Recall: $\mathbb{E} : \Omega_{spin} \to KO$

$$[M_{spin}] \longrightarrow [\mathbb{E}_{TM}] \in KO(M).$$

Let in general $E^n \to X$ be n-dim.l v.bdle (w/ Riem. metric) with spin structure $S(E)$.

$S(E)$: bdle of irred. graded bimodules over Clifford-alg. bundle $C(E) - C_n$. $S(E_x) =: S($

Def.: For $E \to X$ as above, X mfld, E w/metric connection, a spin connection S on E assigns to each pw. sm. path

$\gamma : I \to X$ an isomorpism

$\gamma(0) = x_1$
$\gamma(1) = x_2$

$$S(\gamma) : F(\gamma) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{R}}(S(x_1), S(x_2))$$

of left $C(\partial\gamma) \doteq C(x_1)^{op} \otimes C(x_2)$ modules.

technical assumptions: cont. dep. of $S(\gamma)$ on γ

paramete. invar. of $S(\gamma)$ wrt φ

gluing condition:

$$S(\gamma_2 \cup_{x_2} \gamma_1) \cong S(\gamma 2) \otimes S(\gamma_1).$$

$$F(\gamma) := F(\gamma^* E) \otimes F(I)^{-n}$$

$$\nwarrow = C_{-n}.$$

$$C(x) = C(E_x) \otimes C_{-n}.$$

Remark: $S(\gamma)(\Omega_\gamma \otimes \mathrm{id}) : S(x_1) \to S(x_2)$

defines parallel transport in $S(E)$.

($\Omega_\gamma \in F(\gamma^* E)$ vacuum: exist for any v.bdle E w/ metric and connection.)

Def.: The EFT - Euler class associated to $E \to X$ as above w/ spin str. given by $S(E)$ and spin connection S, is the functor $\mathbb{E}_E : \mathcal{EB}_n(X) \to \mathrm{Hilb}$ defined by

0-dim: $\mathbb{E}_E(\underline{x} : pt \to x) := S(x)$.

1-dim: $\mathbb{E}_E(\gamma : I \to X, \underset{\substack{\cap \\ F_{alg}(I)^{-n}}}{\psi}) := S(\gamma)(\Omega_\gamma \otimes \psi) : S(x_1) \to S(x_2)$

Concept: 0-dim.l data for Euler class:

From (top) spin structure $\rightsquigarrow \begin{array}{c} H \\ \downarrow \\ X \end{array}$

1-dim.l data for Euler class:

From spin connection.

Generalize: 0-dim.l data for elliptic Euler class:

From (top) String structure $\rightsquigarrow \begin{array}{c} A \\ \downarrow \\ X \end{array}$

1,2-dim.l data for elliptic Euler class:

From "string connection".

Elliptic Euler class. $(\Omega^*_{String} \longrightarrow tmf)$

Problem: As opposed to Spin case, for String bdles one does not inherit a connection from connection on E.

$\Rightarrow$ A string connection imitates data associated by spin connection in d=1 case, but without the geometric interpretation.

Two conceptual steps:

1. Obtain bdle of vN algs $A \to X$ from String structure.

This is possible even in a more gen. framework.

Thm: Let G cpt, simply conn. Lie grp., $\ell \in H^4(BG)$ a level. Then one can associate a canonical vN alg. $A_{G,\ell}$ which is hyperfinite of type III_1.
There is an extension of top. grps

$$1 \to PU(A_{G,\ell}) \xrightarrow{i} G_\ell \to G \to 1$$

and a monomorphism

$\Phi : G_\ell \hookrightarrow Aut(A_{G,\ell})$, s.th.
- $\pi_3 G \to \pi_2 PU(A) \cong \mathbb{Z}$ is given by $\ell \in H^4(BG$ $\cong Hom(\pi_3 G, \mathbb{Z})$
- $\Phi \circ i$ is incl. of $Inn(A$

Cor.: A G_ℓ-structure on a principal G-bdle $E \to X$ gives a bdle of vN algs $A \to X$.

$$\begin{tikzcd}
G_\ell \arrow[r] & E_\ell \arrow[d] \arrow[r, squiggly, "\Phi"] & E_\ell \times_{G_\ell} (A_{G,\ell}) \arrow[d] \\
 & X & X
\end{tikzcd}$$

.

In particular: $G = Spin(n)$, $\ell = p_1/2 \in H^4(BSpin)$

$\Rightarrow G_\ell = String(n)$.

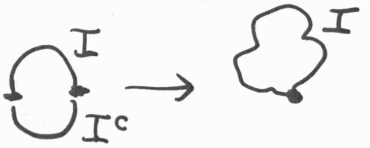
Sketch of construction:

$$\begin{tikzcd}
 & & (\text{contr.}) & (K(\mathbb{Z},2)) & \\
1 \arrow[r] & S^1 \arrow[r] & U(H) \arrow[r] & PU(H) \arrow[r] & 1 \\
 & S^1 \arrow[u, hook] \arrow[r] & \widetilde{LG} \arrow[u, "\hat{\ell}"] \arrow[r] & LG \arrow[u, "\ell"] & \\
 & & \widetilde{L_I G} \arrow[u, hook] \arrow[r] & L_I G \arrow[u, hook] &
\end{tikzcd}$$

$\ell \in H^4(BG)$
$\|$
$H^2(\Omega G)$
$\|$
$[\Omega G, K(\mathbb{Z},2)]$

$\leftarrow$ free loop space (pw. sm. loops)

$\leftarrow$ "half loops"



Define:

$$A_{G,\ell} := \overline{\tilde{\rho}(\widetilde{L_I G})}^{wo} \subset \mathcal{B}dd(H).$$

$$G_\ell := \left(PU(A_{G,\ell}) \rtimes P_1^I G\right) / \underbrace{r(L_I G)}$$ String(n)-model.

s.th. for $\phi : (\Sigma, \Gamma) \to (\Sigma', \Gamma')$ conf. spin diffeo

$$\begin{tikzcd}
\mathbf{F}(\Gamma) \arrow[r, "S(\Gamma)", "\cong"'] \arrow[d, "F(\phi)"'] & \mathbb{S}(\gamma) \arrow[d, "S(\phi|_{\partial\gamma})"] \\
\mathbf{F}(\Gamma') \arrow[r, "\cong", "S(\Gamma')"'] & \mathbb{S}(\gamma')
\end{tikzcd}$$

and gluing / continuity conditions.

Then, roughly (!), the "<u>elliptic Euler class</u>" assoc. to $E \to X$ as before w/ vN alg. bundle $A \to X$ given by string structure and string connection S on E is the functor $\underline{\mathbb{E}ll_E : \mathcal{D}_n(X) \to vN}$ defined by:

<u>0-dim:</u> $\mathbb{E}ll_{E}(\underline{x}) := S(\underline{x}) = \overline{\bigotimes} A(x_j)$

<u>1-dim:</u> $\mathbb{E}ll_E(\gamma) := S(\gamma)$ – as $\overset{C(\gamma) = \underline{C(\gamma^*E)} \otimes C(Y)^{-n}}{C(Y)^{-n}}$ – $S(\partial\gamma)$-bimodule (reducible!)

<u>2-dim:</u> $\mathbb{E}ll_E(\Gamma, \psi) := S(\Gamma)(\Omega_\Gamma \otimes \psi) \in S(\gamma) = \mathbb{E}ll_E(\gamma)$.

<u>Recall:</u> $F(\Gamma) = F(\Gamma^*E) \otimes \boxed{F(\Sigma')^{-n}}$

$\psi \in F_{alg}(\Sigma')^{-n}$, $\Omega_\Gamma \in F(\Gamma^*E)$.

<u>Get:</u>

$$\begin{tikzcd}
\mathbb{E}ll_E(\Gamma, \psi) : \mathbb{E}ll_E(\emptyset) \arrow[r] \arrow[d, phantom, "\parallel"] & \mathbb{E}ll_E(\gamma) \arrow[d, phantom, "\parallel"] \\
S(\emptyset) \arrow[d, phantom, "\parallel"] & S(\gamma) \arrow[d, phantom, "\ni"] \\
\mathbb{C} : \mathbb{1} \arrow[r, mapsto] & S(\Gamma)(\Omega_\Gamma \otimes \psi).
\end{tikzcd}$$

Results: Thm 5.0.2: $A \to X$ can be defined (proof).

It seems to be possible to obtain graded vN alg's as well. (no proof).

Thm. 5.3.5: Claims existence of string connections. (no proof.)

Person to ask: A. Wassermann.

Thm. 5.3.6: Describes the data of a string connection in geometric terms using "trivialization of extended Chern-Simons theory CS_E". (no proof).

Papers to read: D. Freed, Classical Chern-Simons theory, Part I (& II ?)

A. Wassermann; Operator algebras and Conformal field theory (I-III?).