Von Neumann - algebras
and the Connes fusion tensor product

H complex Hilbert space

$\mathcal{B}(H)$ bounded op's on H

Def.: $M \subset \mathcal{B}(H)$ is a von Neumann-algebra iff it is a unital $*$-subalgebra of $\mathcal{B}(H)$ that is closed in the weak operator topology.

⌐ Note: weak topology is generated by seminorms $a \mapsto |\langle \xi, a\eta \rangle| \quad \forall \xi, \eta \in H$ ⌙

Def.: Given $S \subset \mathcal{B}(H)$, the commutant S' of S is given by

$$S' := \{ a \in \mathcal{B}(H) \mid as = sa \quad \forall s \in S \}$$

Thm.: (von Neumann - Bicommutant theorem):

$$M \subset \mathcal{B}(H) \text{ von Neumann-algebra} \iff A'' = A .$$

(„Every von Neumann - alg. arises as a commutant of something.")

Ex.: $M = \mathcal{B}(H)$ is a von Neumann - algebra.

Ex.: Let G be a discrete, countable group. Take $H = \ell^2(G)$, then

$$M = (\mathbb{C}G)'' \quad \text{is a von Neumann-algebra}$$

↑ taken in the bounded op's on $\ell^2(G)$

Ex.: (X, μ) measure space

$H = L^2(X, \mu)$

$M = L^\infty(X, \mu)$ acts on H by left multiplication and is a commutative von Neumann-algebra

Thm: Every commutative vNa arises in that way for some measure space X.

philosophy: Think of vNa's as a non-commutative version of measure theory.

Factors and Type Classification

<u>Def.</u>: The <u>center</u> Z of a vNa M is given by $Z = M \cap M'$ and is itself a commutative vNa.

A von Neumann-algebra with trivial center is called a <u>factor</u>.

Why are factors interesting?

- Z is a comm. vNa $\Rightarrow$ $Z \simeq L^\infty(X,\mu)$

  If μ is discrete, then ...

  $$M \simeq \bigoplus_{i \in I} M_i \quad \text{where all } M_i \text{ are factors}$$

  ... else use direct integral decomposition ...

  $$M \simeq \int_X^\oplus M_x \, d\mu(x) \quad \text{where } M_x \text{ is a factor for all } x \in X$$

  $\rightsquigarrow$ Factors are "building blocks" of general vNa's

- consider projections $p \in M$ with $p = p^* = p^2$
- Take $x \in M$ self-adjoint element and let

  $$x = \int_a^b \lambda \, dE_\lambda \quad \text{be its spectral decomposition, then } E_\lambda \in M \quad \forall \lambda \in [a,b]$$

  $\Rightarrow$ M always contains projections

(Indeed: Every vNa is the norm-closed linear span of its projections.)

Idea: Instead of subspaces of H, consider projections in M and generalize the theory of dimension.

equivalence of projections: Let $e, f \in M$ be proj.

$$e \sim f \quad \text{iff} \quad \exists\, u \in M \quad \text{with} \quad u^* u = e, \; u u^* = f$$

(↑ u: partial isometry)

$\longrightarrow$ induce partial ordering on projections

$$e \prec f \quad \text{iff} \quad \exists\, e_0 \sim e \quad \text{s.th. } e_0 \text{ is a subproj. of } f$$

<u>Thm.</u>: If M is a factor, this is an ordering.

Def.: A projection $e \in M$ is called <u>finite</u>, if e is not equivalent to any proper subprojection of e.

Def.: A factor M is of

- type I, if there exists a non-zero minimal projection in M,
- type II, if M contains non-zero finite projections and is not of type I,
- type III, if no non-zero projection in M is finite.

A factor M is called <u>finite</u>, if $1 \in M$ is finite.

Thm: M finite factor $\Rightarrow$ $\exists!$ faithful, normal (i.e. weakly continuous), tracial state on M (for short: a trace on M)

- type I_n: trace takes <u>discrete values</u> on the proj.

$$tr(e) \in \{0, \dots \underset{n}{\underset{\shortparallel}{\dim_{\mathbb{C}} H}}\} \quad (n = \infty \text{ allowed})$$

- type II_1: trace takes <u>continuous values</u> on the proj.

$$tr(e) \in [0, 1]$$

- For type II_∞ there still is a replacement for tr, that fulfills

$$tr(e) \in [0, \infty]$$

- For type III: no trace at all! But finer classification via modular theory. $\leadsto$ leads to type III_λ with $\lambda \in [0, 1]$

Def.: A factor M is called <u>hyperfinite</u> if

$$M = \left(\bigcup_{i=1}^{\infty} M_i\right)''$$

for an increasing sequence $M_1 \subset M_2 \subset \dots$ of finite dimensional von Neumann-algebras

All hyperfinite factors have been classified:

- type I_n: $M = \mathcal{B}(H)$ with $n = \dim_{\mathbb{C}} H$
- type II_1: Group von Neumann algebras. All isomorphic!
- type II_∞: $I_\infty \otimes II_1$
- type III_0: the Krieger factor
- type III_λ for $\lambda \in]0, 1[$: the Powers factor
- type III_1: The local fermions defined by Wassermann. All isomorphic

construction of the hyperfinite II_1-factor

start with $M_{1\times 1}(\mathbb{C})$, embed it into $M_{2\times 2}(\mathbb{C})$ via

$$x \longmapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

s.th. the trace on $M_{2\times 2}(\mathbb{C})$ is given by $tr_2 \circ i = \frac{1}{2} tr_1$

continue like that embedding $M_{2^n \times 2^n}$ into $M_{2^{n+1} \times 2^{n+1}}$

Take weak closure on direct limit

$\rightsquigarrow$ trace takes values in $[0,1]$

Modular Theory
(aka: Tomita – Takesaki theory)

<u>Def.</u>: $M \subset B(H_0)$ vNa. H_0 is called <u>vacuum representation</u> or <u>standard form</u>, if $\exists\, \Omega \in H_0$ that is cyclic for M and for M'.

- Consider the (unbounded), anti-linear op. given by

$$S_0\, a\Omega = a^*\Omega \rightsquigarrow \text{closable} \quad S = \overline{S_0}$$

Take polar decomposition, since S is <u>anti</u>-linear, this looks like

$$S = J \cdot \Delta^{\frac{1}{2}} \quad \text{with } J \text{ <u>anti</u>-unitary}$$

and Δ positive, self-adjoint.

<u>Ex.</u>: For M being a type II_1-factor, $\exists\, tr: M \longrightarrow \mathbb{C}$.

tr yields a vacuum repr. via the GNS-construction, vacuum vec. Ω

$a, b \in M$
$$\langle b\Omega, S^* S a\Omega\rangle = \langle b^*\Omega, a^*\Omega\rangle = tr(ba^*) = tr(a^*b)$$
$$= \overline{tr(b^*a)} = \overline{\langle b\Omega, a\Omega\rangle}$$

$\Rightarrow S$ is anti-unitary

$\Rightarrow \Delta = 1$ by uniqueness of polar decomp.

<u>Rem.</u>: $J^2 = 1$, functional calculus lets you define Δ^{it} and Δ^{-it}, $t \in \mathbb{R}$

Now...

<u>Thm.</u> (Tomita – Takesaki): M vNa with vac. rep. (H_0, Ω), then

$$JMJ = M'$$
$$\Delta^{it} M \Delta^{-it} = M \quad \forall t \in \mathbb{R}$$

<u>Rem.</u>:
- J turns H_0 into an M-M-bimodule. Let π be the vac. rep. of M, then: $\pi^{op}(a) = \underbrace{J\pi(a)^* J}_{\in M^{op}}$
- $J\Omega = \Delta\Omega = \Omega$
- For a dense subset of M (the entire analytic elements)

$$\sigma(a) = \Delta^{\frac{1}{2}} a \Delta^{-\frac{1}{2}} \in M$$

Now for $\varphi_\Omega(a) = \langle \Omega, a\Omega \rangle$ one has

$$\varphi_\Omega(ba) = \varphi_\Omega(\sigma^{-1}(a)\sigma(b))$$

(φ_Ω is called <u>vacuum state</u>)

Thus the modular operator measures "how much" the vacuum state differs from a trace state.

- Thm: Iff multiples of Ω are the only vectors that are fixed by the modular flow, then M is a type III_1-factor.

- In analogy to the commutative case, the vac. rep. shall be denoted by $H_0 = L^2(M)$

<u>Tensor products of vNa's</u>

- two Hilbert spaces: H_1, H_2

  Hilbert space tensor product $H_1 \otimes H_2$ is the completion of the algebraic tensor product $H_1 \odot H_2$ w.r.t. the norm...

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle_{H_1} \cdot \langle \eta_1, \eta_2 \rangle_{H_2}, \quad \xi_i \in H_1, \ \eta_i \in H_2$$

- Consider two vNa's N and M, now $N \odot M$ inherits a $*$-algebraic structure

$$n_1 \otimes m_1 \cdot n_2 \otimes m_2 = n_1 \cdot n_2 \otimes m_1 \cdot m_2 \qquad n_i \in N, \ m_i \in M$$

$$(n \otimes m)^* = n^* \otimes m^*$$

Now consider $\pi : N \odot M \longrightarrow \mathcal{B}(H_N \otimes H_M)$

$$\pi(n \otimes m)(\xi \otimes \eta) = n\xi \otimes m\eta$$

with $N \subset \mathcal{B}(H_N)$ and $M \subset \mathcal{B}(H_M)$

Now define: $N \otimes M = \pi(N \odot M)''$ – the so-called spatial tensor product of vNa's

The many faces of Connes' fusion

Goal: Find the "right tensor product" for vNa-bimodules

Remember: H is called an M-N-bimodule (with two vNa's M, N) if it is a left module over M and a right module over N, where the module actions are given by weakly continuous, $*$-preserving, unital homomorphisms:

$$\pi_M : M \longrightarrow B(H)$$

$$\pi_N^{op} : N^{op} \longrightarrow B(H)$$

↑ vNa with the opposite multiplication

Tensor product of an M-N-bimod. H_1 and an N-L-bimod. H_2 should have "nice properties"

ex. of a nice property: $\xi \cdot n \boxtimes \eta = \xi \boxtimes n \cdot \eta$, $\xi \in H_1$, $\eta \in H_2$, $n \in N$

First construction by Jones et al. for type II_1-factors ⇝ relative tensor product

Def.: Let H_1 be an M-N-bimodule with actions π_M, π_N^{op}.

then $\mathcal{H}_1 = \{ t : L^2(N) \longrightarrow H_1 \mid t \cdot \pi_{N,0}^{op} = \pi_N^{op} \cdot t \}$

↑ vacuum rep. of N^{op}

denote the Intertwiners between $\pi_{N,0}^{op}$ and π_N^{op}.

Ex.: $\mathcal{L}^2(N) = \{ t : L^2(N) \longrightarrow L^2(N) \mid t\, \pi_{N,0}^{op} = \pi_{N,0}^{op} \cdot t \} = N'' = N$

$$\pi_{N,0}^{op}(n) = J\, \pi_{N,0}(n)^* J \in N'$$

Note that $\mathcal{H}_1$ can be turned into a right Hilbert module over N with inner product ...

$$(t, s) = t^* s \in \mathcal{L}^2(N) = N$$

Def.: Given an M-N-bimod. H_1 and an N-L-bimod. H_2, then the Connes fusion of the two is given by the completion of $\mathcal{H}_1 \odot H_2$ w.r.t. the inner product

$$\langle t \otimes \xi, s \otimes \eta \rangle := \langle \xi, (t, s) \cdot \eta \rangle_{H_2}. \qquad s, t \in \mathcal{H}_1, \; \xi, \eta \in H_2$$

It is denoted by $H_1 \boxtimes H_2$.

Note that:

$$\langle tn\otimes\xi - t\otimes n\cdot\xi,\ tn\otimes\xi - t\otimes n\xi\rangle$$

$$= \langle \xi, (tn, tn)\xi\rangle - \langle n\xi, (t, tn)\xi\rangle - \langle \xi, (tn, t)n\xi\rangle + \langle n\xi, (t,t)n\xi\rangle$$

$$= 0$$

$$\Rightarrow\ tn\otimes\xi = t\otimes n\xi$$ Connes fusion has "nice property".

Ex.: Take weakly cont., unital, $*$-preserving homomorphism of vNa's

$$\rho: L \longrightarrow N.$$

$L^2(\rho)$ is N-N-bimodule $L^2(N)$ considered as L-N-bimodule with left action...

$$\rho(l)\cdot\zeta \qquad \text{for } l\in L,\ \zeta\in L^2(N).$$

Let H be an N-M-bimodule and $\tilde{H}$ be the corresponding L-M-bimodule with left action

$$\rho(l)\cdot\eta \qquad \text{for } l\in L,\ \eta\in H.$$

Thm: $L^2(\rho)\boxtimes H \simeq \tilde{H}$.

Proof: $\mathcal{L}^2(\rho) = \mathcal{L}^2(N) = N$

$$\varphi: L^2(\rho)\boxtimes H \longrightarrow \tilde{H}$$
$$n\otimes\eta \longmapsto n\cdot\eta$$

$$\psi: \tilde{H} \longrightarrow L^2(\rho)\boxtimes H$$
$$\eta \longmapsto 1\otimes\eta$$

extend to an isomorphism.

Therefore:
- $L^2(\rho)\boxtimes L^2(\sigma) = L^2(\sigma\circ\rho)$ for $\sigma: N\longrightarrow M$, $\rho: L\longrightarrow N$
- Taking $\rho = id$, $L = N$ $\Rightarrow$ $L^2(N)\boxtimes H \simeq H$ for any N-M-bimodule H

Problem: Identify vectors in $H_1\boxtimes H_2$ in terms of the tensor product $H_1\odot H_2$.

Symmetric form of Connes fusion

H_1 $M-N$-bimodule

H_2 $N-L$-bimodule

$\tilde{\mathcal{H}}_2 = \{ s : L^2(N) \longrightarrow H_2 \mid s\pi_{N,o} = \pi_N s \}$ therefore: $s_1^* s_2 \in N^{op}$

Now take completion of ...

$\mathcal{H}_1 \odot \tilde{\mathcal{H}}_2$ w.r.t. inner product

$\langle t_1 \otimes s_1 , t_2 \otimes s_2 \rangle = \langle t_2^* t_1 \cdot \Omega \cdot s_2^* s_1 , \Omega \rangle_{L^2(N)}$

... using the inclusion i_Ω one gets...

$\mathcal{H}_1 \Omega \odot \tilde{\mathcal{H}}_2 \Omega$ with symmetric Connes relation...

$\xi \Delta^{-\frac{1}{4}} n \Delta^{\frac{1}{4}} \otimes \eta = \xi \otimes \Delta^{\frac{1}{4}} n \Delta^{-\frac{1}{4}} \cdot \eta$ for $\xi \in \mathcal{H}_1 \Omega$
$\eta \in \tilde{\mathcal{H}}_2 \Omega$
n entire element in N

A. Wassermann "four point-formula"

Connes fusion and the algebraic tensor product

- H_1 M-N-bimodule, left action π_M, right action π_N^{op}

Choose cyclic and sep. (vacuum) vector $\Omega \in L^2(N)$

inclusion $i_\Omega : \mathcal{H}_1 \longrightarrow H_1$, $t \longmapsto t\Omega$ (not canonical, depends on choice of Ω)

Now: $\pi_N^{op}(x)\, t\Omega \underset{\substack{\uparrow \\ \text{intertwining prop.}}}{=} t\, \pi_{N,0}^{op}(x)\Omega \underset{\substack{\uparrow \\ \text{Def.}}}{=} t\, J \pi_{N,0}(x)^* J\Omega$

$$\underset{\substack{\uparrow \\ \text{invariance of } \Omega}}{=} t\, \underbrace{J\Delta^{\frac{1}{2}}}_{S}\, \Delta^{-\frac{1}{2}}\, \pi_{N,0}(x)^*\, \Delta^{\frac{1}{2}}\Omega$$

$$= t\, S\left(\Delta^{\frac{1}{2}}\pi_{N,0}(x)\,\Delta^{-\frac{1}{2}}\right)^*\Omega$$

$$= t\, \Delta^{\frac{1}{2}}\pi_{N,0}(x)\,\Delta^{-\frac{1}{2}}\Omega \qquad \text{for every entire element } x \in N$$

$\Rightarrow \quad i_\Omega(t)\cdot x = i_\Omega(t\sigma(x)) \qquad$ with $\sigma(x) = \Delta^{\frac{1}{2}} x \Delta^{-\frac{1}{2}}$

So, instead of above definition, take $\underset{\substack{\cap \\ H_1}}{\mathcal{H}_1\Omega} \odot H_2$ with ($\mathcal{H}_1\Omega = \operatorname{im} i_\Omega$)

Connes relation: $\xi\cdot x \otimes \eta = \xi \otimes \sigma(x)\cdot\eta \qquad x\in N,\ \xi\in \operatorname{im} i_\Omega,\ \eta\in H_2$

Remark:
- $\mathcal{H}_1\Omega$ is <u>not</u> H_1, but the set of ω-bounded vectors for
  $\omega(n) = \langle \Omega, n\Omega\rangle_{L^2(N)}$
  $\xi \in \operatorname{im} i_\Omega$ is ω-bounded iff $\exists\, C>0$, s.th. $\|\xi x\|_{H_1}^2 \le C\cdot\omega(n^*n)$
- If $\Delta = 1$ (for type I or type II_1-factors), then $\sigma = \mathrm{id}$
  $\Rightarrow$ Connes fusion reduces to an algebraic tensor product of bimodules.

Remember the bi-category $\mathcal{D}_n$ (sketchy)

- objects: 0-dim spin mfds. Z
- morphisms: - spin diffeo. $Z_1 \longrightarrow Z_2$
  - one-dim. spin mfd. Y s.th. $\partial Y = \overline{Z_1} \amalg Z_2$
- 2-morphisms: - either spin diffeo. f rel. boundary with element $c \in C(Y_1)^{\otimes n}$
  - conf. spin surface Σ with $\psi \in F_{alg}(\Sigma)$

enriched
elliptic object should functor this to...

The bicategory vN of von Neumann-algebras

- objects: von Neumann-algebras
- morphisms: A morphism from N to M is an M-N-bimodule
  - composition: given by Connes fusion

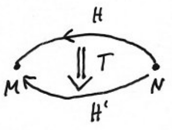
$$M \underset{H_1}{\longleftarrow} N \underset{H_2}{\longleftarrow} L \quad = \quad M \underset{H_1 \boxtimes H_2}{\longleftarrow} L$$

  - trivial element given by $L^2(N)$
  - Connes' fusion is associative up to isomorphisms.

- 2-morphisms:

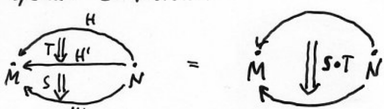
Given by intertwiners: H, H' M-N-bimodules

$$T \in \mathcal{B}_{M,N}(H, H') = \{ T : H \longrightarrow H' \text{ bounded} \mid T\pi_M = \pi'_M T \text{ and } T\pi_N^{op} = \pi_N^{op'} T \}$$



composition of intertwiners

by operator composition

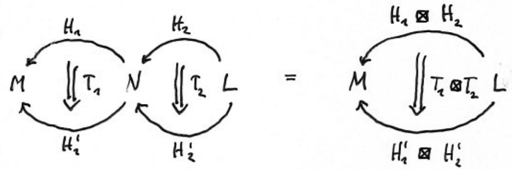


extended gluing lemma: There is a unique unitary isometry of $C(Y_3)$ - $C(Y_1)$ - bimodules

$$F(\Sigma_2) \boxtimes_{A(Y_2)} F(\Sigma_1) \xrightarrow{\ \simeq\ } F(\Sigma_3)$$

mapping $\Omega_2 \otimes \Omega_1$ to Ω_3

... further more ...



$x \otimes \xi \in H_1 \boxtimes H_2$

$(T_1 \boxtimes T_2)(x \otimes \xi)$

$= T_1 \cdot x \otimes T_2\, \xi$

Adjunction tranformations in vN

- three involutions

$N \mapsto N^{op}$ on objects

$H \mapsto \overline{H}$ on morphisms, where $\overline{H}$ is the conjugate bimodule with module actions

(H: M-N-bimod., $\overline{H}$: N-M-bimod.)

$$n \cdot \overline{\xi} \cdot m = \overline{m^* \cdot \xi \cdot n^*} \qquad \xi \in H$$

$T \mapsto T^*$ on 2-morphisms with the usual adjunction

In view of the adjunctions in the geometric category we would like to have...

$$vN(\mathbb{C}, A_1 \otimes A_2) \longrightarrow vN(A_1^{op}, A_2) \quad \text{on morphisms}$$

$$vN(\mathbb{C}, F_2 \boxtimes_A F_1) \longrightarrow vN(\overline{F}_2, F_1) \quad \text{on 2-morphisms}$$

For F_1 a $A_1 \otimes A_2 - \mathbb{C}$-bimodule

F_2 a $\mathbb{C} - (A_1 \otimes A_2)^{op}$-bimodule, both lying in the pre-image of the first map

set $A := A_1 \otimes A_2$

Consider: (intertwiners of the A-action)

$$\Theta : \mathcal{F}_2 \odot F_1 \longrightarrow \mathcal{B}_A(\overline{\mathcal{F}}_2, F_1)$$

$$x \otimes \eta \longmapsto \vartheta_{x,\eta}$$

with $\vartheta_{x,\eta}(\overline{y}) = (y, x)\eta$ and $\overline{x} = x \cdot J$

Take x that fulfills

$x\, \pi_{A,o}^{op}(a) = \pi_A^{op}(a)\, x$, then...

$$\overline{x}\, \pi_{A,o}(a) = xJ \cdot J \pi_{A,o}^{op}(a)^* J = \pi_A^{op}(a)^* xJ$$

$$\Rightarrow \overline{x}\, \pi_{A,o}(a) = \pi_A^{conj}(a)\, \overline{x} \quad \Rightarrow \quad (x \in \mathcal{F}_2 \Rightarrow \overline{x} \in \overline{\mathcal{F}}_2)$$

- $\vartheta_{x,e}(\bar{y})$ is A-linear map (simple comp., using definitions)

  $\Rightarrow$ Θ is well-defined

- Θ is an isometry (shown in Stolz-Teichner for type III-factors

$$\text{so} \ldots \quad \Theta : F_2 \boxtimes F_1 \xrightarrow{\;\cong\;} \mathcal{B}_A(\bar{\mathcal{F}}_2, F_1) \qquad \text{(?)}$$

Interesting subcategories of vN

- Fix an object $N \in obj(vN)$, type III_1-factor

Consider (weakly cont., unital, $*$-preserving) endomorphisms of N

$$\rho : N \longrightarrow N$$

Each ρ induces via $L^2(\rho)$ — another N-N-bimodule

fusion $\rightsquigarrow$ composition

direct sum decomp. $\rightsquigarrow$ "direct sums" of endomorphisms

leads to $\rightsquigarrow$ monoidal or tensor categories

Leads to fusion rules: $\lambda \circ \sigma = \bigoplus_{\rho} N^{\rho}_{\lambda\sigma} \cdot \rho$
↑ multiplicities

If you take a net of factors instead of a single and demand localizability of endomorphisms you get so-called "fusion rules" of superselection sectors from algebraic quantum field theory.

- Jones extension

Take two factors $A \subset B$, where B arises from A by the "Jones basic construction"

all morphisms generated by iterated fusion of $L^2(B)$, which is an A-B-bimodule

subfactor has finite Jones index $\Leftrightarrow$ $F \boxtimes_B \overline{F}$ and $\overline{F} \boxtimes_A F$ contain the vacuum rep. only once.

$\longrightarrow$ important for
- Classification of CFTs
- invariants of 3-mfds.

Local fermions (sketchy)

- H complex Hilbert space

  $\mathrm{Cliff}(H)$ generated by $a(f)$, $f \in H$

  $$a(f)a(g) + a(g)a(f) = 0$$

  $$a(f)a(g)^* + a(g)^* a(f) = (f,g)$$

  acts on ΛH $\quad \pi(a(f))\xi = f \wedge \xi$

  $$c(f) = a(f) + a(f)^*$$

  fulfill $c(f)c(g) + c(g)c(f) = 2\,\mathrm{Re}(f,g)$

  Take projection P into H

  representation $\pi_P(a(f)) = \frac{1}{2}\left(c(f) - i c(i(2P-1)f)\right)$ on ΛH

  is again irreducible

  Now take $H = L^2(S^1) \otimes V$, $V = \mathbb{C}^N$

  P orthog. proj. onto the Hardy space $H^2(S^1) \otimes V$

  π_P corr. irr. rep.

  ... then $M(I) = \pi_P(a(f))''$ with $f \in L^2(I, V)$

  is a (net of) von Neumann-algebra(s)

properties: $I^c = S^1 \setminus \overline{I}$

- vacuum vector Ω is cyclic and sep. for each $M(I)$
- modular group acts geometrically

  Let I be upper semi-circle,

  $$(u_t f)(z) = (z \sinh \pi t + \cosh \pi t)^{-1} \cdot f\left(\frac{z \cdot \cosh \pi t + \sinh \pi t}{z \cdot \sinh \pi t + \cosh \pi t}\right)$$

  "Möbius flow"

  $$\Delta^{it} \pi_P(a(f)) \Delta^{-it} = \pi_P(a(u_t f)) \quad \forall f \in H$$

- modular conjugation acts geometrically
  - F is "flip" $F(f(z)) = z^{-1} f(z^{-1})$
  - K Klein transformation (?)

  $$J \pi_P(a(f)) J = K^{-1} \pi_P(a(Ff)) K \quad , \quad J M(I) J = M(I^c)$$