HOMOTOPY & COHOMOLOGY
GK 1150

GRK 1150, Mathematisches Institut, Universität Bonn, 53115 Bonn

Winter School

"From Field Theories to Elliptic Objects"

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Talk No. 2

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Introduction to quantum theory

PLAN

- Warning
- classical mechanics
- quantum mechanics
- fermions
(- field theories)

Warning

- simplified, but still a lot left
- suitable units
- Einstein convention

Classical mechanics

Lagrange:
Given mf M, function $L : TM \to \mathbb{R}$, look at paths $q : [a,b] \to M$ which extremize the action $S[q(t)] = \int_a^b L(q, \dot q)\, dt$, i.e. solutions to Euler-Lagrange eq'ns:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} = \frac{\partial L}{\partial q} \qquad 2^{nd} \text{ order ODE}$$

$$L = \frac{1}{2}\langle \dot q, \dot q\rangle \quad M = \mathbb{R}^3 \quad \to \quad \ddot q = 0$$

Hamilton:

def momenta $p_i = \frac{\partial L}{\partial \dot{q}^i}$ and construct

function $H: T^*M \to \mathbb{R}$, $H(p,q) = p_i \dot{q}^i - L$

1^{st} order ODEs $\dot{p}_i = -\frac{\partial H}{\partial q^i}$, $\dot{q}^i = \frac{\partial H}{\partial p_i}$

symplectic geometry:

def Poisson brackets $\{f,g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}$

$f,g : T^*M \to \mathbb{R}$

esp.: $\{p_i, q^j\} = \delta_i{}^j$, $\{H, f\} = \dot{f}$

QM

wave-particle-duality, uncertainty principle

$\Delta x \Delta p \geq \frac{1}{2}$

canonical quantization recipe

cl. system described by path $q(t)$	$\to$	sys. described by wavefunction $\psi \in \mathcal{H} = L^2(M)$, $\lvert\psi\rvert^2$ prob. density
variables p, q	$\to$	operators $\hat{p}, \hat{q}$ acting on $\mathcal{H}$
PBs $\{f,g\}$	$\to$	$i[\hat{f},\hat{g}]$ commutators
observables A	$\to$	hermitean operators, so $\langle \psi, \hat{A}\psi \rangle \in \mathbb{R}$

Schrödinger's eq.: $i\dot{\psi} = \hat{H}\psi$

time evolution is unitary!

<u>Def</u> the <u>propagator</u> K

$$\psi(x,t) = \int K(x,t,x',t')\,\psi(x',t')\,dx'$$

rewritable using complete basis of eigenfunctions of the Hamiltonian $\hat{H}\varphi_n(x) = E_n\varphi_n(x)$

$$K(x,t|x',t') = \sum_n \varphi_n^*(x)\, e^{-i\hat{H}(t-t')}\varphi_n(x')$$

<u>ex</u>: free particle on the circle of radius R

$S[x(t)] = \frac{1}{2}\int \dot{x}^2 dt$, $x(t) \equiv x(t) + 2\pi R$

$\mathcal{H} = L^2(S^1_R)$; Hamiltonian $\frac{1}{2}\hat{p}^2$

$\hat{p} = -i\frac{\partial}{\partial x}$, $\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} = \frac{1}{2}\Delta$

basis $\exp(ikx/R)\cdot\frac{1}{(2\pi R)^{1/2}}$, eigenvalues $\frac{1}{2}\left(\frac{k}{R}\right)^2$

compute the <u>euclidean partition function</u>

$$Z_E(T) = \mathrm{tr}\, e^{-\hat{H}T}$$

$$= \sum_{k\in\mathbb{Z}} \exp\left(-\frac{1}{2}\left(\frac{k}{R}\right)^2 T\right)$$

Feynman path integral formula:

$$K(x', t', x'', t'') = \int_{\substack{x(t')=x' \\ x(t'')=x''}} \exp i S[x(t)] \, [dx(t)]$$

philosophy: all paths contribute in QM, reflected by integration over path space

pass to imaginary time and use stochastic analysis to define the measure (Wiener measure / Brownian motion)

<u>Fermions</u>

observation: electrons have internal degree of freedom ($\mathbb{Z}/2$-valued) (spin $\frac{1}{2}$)

<u>Fermions</u> are particles of half-integer spin, opposed to <u>bosons</u>

exclusion principle:
2 identical fermions cannot occupy the same quantum state.

In canonical quantization you'll have to use anticommutators:

$$[\hat{p}, \hat{q}]_+ = \hat{p}\hat{q} + \hat{q}\hat{p} = +i$$

path integral derivation of the partition function (free particle on S^1_R):

$$Z_E = \operatorname{tr} e^{-\hat{H}T} = \int_{S^1_R} K(x, -iT, x, 0)\, dx$$

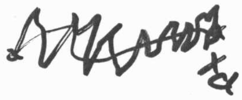
have to consider paths of the form

$$x_{cl} = 2\pi R\, n \cdot \frac{t}{T} \quad , \quad n \in \mathbb{Z} \quad , \quad S[x_{cl}] = \pi n^2 \left(\frac{2\pi R^2}{T}\right)$$

$+ \ x_0 \qquad$ (constant)

$$+ \ x_q = \sum{}' a_k \exp\left(2\pi i k \frac{t}{T}\right) \quad , \quad a_k^* = -a_k \ \text{(reality)} \ ,$$

$$S[x_q] = \frac{1}{2} \sum{}' |a_k|^2 \left(\frac{2\pi k}{T}\right)^2 \cdot T$$



$$\rightsquigarrow Z_E = \int \sum_{n \in \mathbb{Z}} e^{-S_{cl} - S_q} [dx_q]\, dx$$

perform integration over the fourier coefficients (Gaussian integrals) with proper normalization:

$$Z_E = \sum_{n \in \mathbb{Z}} e^{-S_{cl}} \cdot \int_{S^1_R} \underbrace{\left(\prod{}' \left(\frac{2\pi k}{T}\right)^2\right)^{-1/2}}_{= T^{-1} \text{ by } \zeta\text{-reg.}} \cdot \left(\frac{T}{2\pi}\right)^{1/2} dx$$

$$= \left(\frac{2\pi}{T}\right)^{1/2} \cdot R \cdot \sum_{n \in \mathbb{Z}} \exp\left(-\pi n^2 \frac{2\pi R^2}{T}\right)$$

$$= \sum_{n \in \mathbb{Z}} \exp -\pi n^2 \frac{T}{2\pi R^2} \qquad \text{by Poisson resummation}$$

ex.

consider the following action

$$S[\Psi(t)] = \int \psi_\mu^* \, i \, \dot{\psi}^\mu \, dt$$

$$\frac{\partial L}{\partial \dot{\psi}} = i\psi^* \longrightarrow [\hat{\psi}_\mu^*, \hat{\psi}^\nu]_+ = \delta_\mu{}^\nu$$

$$\hat{\psi}_\mu := G_{\mu\nu} \hat{\psi}^\nu$$

$$\Gamma_\mu := \hat{\psi}_\mu - \hat{\psi}_\mu^*$$

$$\Rightarrow [\Gamma_\mu, \Gamma_\nu]_+ = -2 G_{\mu\nu}$$

this realizes Clifford algebra.