Spin structures

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1. Clifford algebras

Let V be a real (or complex) Hilbert space with an isometric involution

$$\alpha : V \to V$$

($\mathbb{C}$-anti linear in the complex case). Write $\bar{v} := \alpha(v)$. Then we get by

$$b(v, w) := \langle \bar{v}, w \rangle$$

a symmetric bilinear form on V.
Write $-V$ for the Hilbert space furnished with the involution $-\alpha$.

Definition 1. The tensor algebra of V is

$$T(V) := \sum_{i=0}^{\infty} \bigotimes^{i} V.$$

Let $I_b(V)$ be the ideal in $T(V)$ generated by all elements of the form

$$v \otimes v + b(v, v) \cdot 1$$

for $v \in V$. We define then

$$Cl(V, b) := T(V)/I_b(V)$$

to be the Clifford algebra associative to V and b.

Remark 2.

1. *There is a natural embedding* $V \hookrightarrow Cl(V,b)$
2. *We have*

$$vw + wv = -2b(v,w) \cdot 1$$

for all $v, w \in V$*.*

Universal property of $Cl(V,b)$:
Let $\tilde{f} : V \to A$ be linear with A an associative algebra with unit such that $\tilde{f}(v)\tilde{f}(v) = -b(v,v) \cdot 1$ for all $v \in V$. Then $\tilde{f}$ extends uniquely to a homomorphism of algebras

$$f : Cl(V,b) \to A.$$

Examples 1.

1. $C_n := Cl(\mathbb{R}^n)$ generated by vectors $v \in \mathbb{R}^n$ subject to the relation $v \cdot v = -|v|^2 \cdot 1$
2. $C_{-n} := Cl(-\mathbb{R}^n)$ generated by vectors $v \in \mathbb{R}^n$ subject to the relation $v \cdot v = |v|^2 \cdot 1$
3. $C_{n,m} := Cl(\mathbb{R}^n \oplus -\mathbb{R}^m)$ generated by vectors $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ subject to the relation $v \cdot v = -|v|^2 \cdot 1$,$w \cdot w = |w|^2 \cdot 1$ and $vw + wv = 0$.

Define now $\tilde{\varepsilon} : V \to Cl(V,b)$ by $\tilde{\varepsilon}(v) := -v$ and extend this map to the involution $\varepsilon : Cl(V,b) \to Cl(V,b)$. Then there is a decomposition

$$Cl(V,b) = Cl^0(V,b) \oplus Cl^1(V,b)$$

into the eigenspaces of ε, which makes $Cl(V,b)$ to a $\mathbb{Z}_2$-graded algebra. $Cl^0(V,b)$ is called the even part and is a subalgebra of $Cl(V,b)$.

Remark 3.

a) $Cl(V \oplus W) \cong Cl(V)\hat{\otimes}Cl(W)$, *where* $\hat{\otimes}$ *is the* $\mathbb{Z}_2$-*graded tensor product:*

$$(v \otimes w) \bullet (v' \otimes w') := (-1)^{|w||v'|} vv' \otimes ww'$$

for $v, v' \in Cl(V)$ *and* $w, w' \in Cl(W)$ *with pure degree.*

b) $Cl(-V) \cong Cl(V)^{op}$, *where* $Cl(V)^{op}$ *is the Clifford algebra with the following new multiplication:*

$$v_1 * v_2 := (-1)^{|v_1||v_2|} v_2 \cdot v_1$$

for $v_1, v_2 \in Cl(V)$ *with pure degree.*

Definition 4. A $\mathbb{Z}_2$-graded module over $Cl(V)$ is a module W with a decomposition $W = W^0 \oplus W^1$ such that

$$Cl^i(V) \cdot W^j \subseteq W^{i+j}$$

for $i, j \in \{0, 1\}$.

Any graded left $Cl(V)\hat{\otimes}Cl(W)$-module M can be interpreted as a $Cl(V) - Cl(W)^{op}$-bimodule via

$$v \cdot m \cdot w := (-1)^{|m||w|}(v \otimes w)m$$

for pure degree elements $v \in V, w \in W$ and $m \in M$ and vice-versa. Together with Remark 3 we can identify left $Cl(V \oplus -W)$-modules with $Cl(V) - Cl(W)$-bimodules.

Definition 5. Let M be a graded $Cl(V) - Cl(W)$-bimodule. We get the opposite $\overline{M}$ of M by changing the grading and keeping the same $Cl(V \oplus -W)$-module structure.

Theorem 6. *Let V be a inner product space of dimension n. Then we can identify orientations on V with isomorphism classes of irreducible graded $Cl(V) - C_n$-bimodules. The opposite bimodule corresponds to the opposite orientation.*

Definition 7.

$$P(V, b) := \{v_1 \cdot \ldots \cdot v_r \in Cl(V, b) \mid b(v_i, v_i) \neq 0, r \in \mathbb{N}\}$$

$$\mathrm{Pin}(V, b) := \{v_1 \cdot \ldots \cdot v_r \in Cl(V, b) \mid b(v_i, v_i) = \pm 1, r \in \mathbb{N}\}$$

$$\mathrm{Spin}(V, b) := \mathrm{Pin}(V, b) \cap Cl^0(V, b)$$

Theorem 8. *Let $V = \mathbb{R}^n \oplus -\mathbb{R}^m$ and $\mathrm{Spin}_{n,m}$ the corresponding spin group.*

$$\mathrm{SO}_{n,m} := \mathrm{SO}(V, b)$$

$$:= \{\lambda \in Gl(V) \mid b(\lambda(v), \lambda(v)) = b(v, v), \det(\lambda) = 1\}$$

Then there is a short exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}_{n,m} \xrightarrow{\zeta_0} \mathrm{SO}_{n,m} \to 1$$

for all (n,m). Furthermore if $(n, m) \neq (1, 1)$ this two-sheeted covering is non-trivial over each component. In particular in the special case

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}_n \xrightarrow{\zeta_0} \mathrm{SO}_n \to 1$$

(where $\mathrm{Spin}_n := \mathrm{Spin}_{n,0}$ and $\mathrm{SO}_n := \mathrm{SO}_{n,0}$) this is the universal covering of SO_n for all $n \geq 3$.

2. Spin structures ans spinor bundle

Let E be an oriented n-dimensional riemannian vector bundle over a manifold X. Let $P_{\mathrm{O}}(E)$ be its bundle of orthonormal frames. This is an principal O_n-bundle.

Choosing an orientation on E is equivalent with choosing a principal SO_n-bundle $P_{\mathrm{SO}}(E) \subset P_{\mathrm{O}}(E)$.

Definition 9. Suppose $n \geq 3$. A spin structure on E is a principal Spin_n-bundle $P_{\mathrm{Spin}}(E)$ together with a two-sheeted covering

$$\zeta : P_{\mathrm{Spin}}(E) \to P_{\mathrm{SO}}(E)$$

such that $\zeta(pg) = \zeta(p)\zeta_0(g)$ for all $p \in P_{\mathrm{Spin}}$ and $g \in \mathrm{Spin}_n$.

When $n = 2$ a spin structure on E is the same with replacing Spin_2 by SO_2 and $\zeta_0 : \mathrm{SO}_2 \to \mathrm{SO}_2$ by the connected 2-fold covering.

When $n = 1$ a spin structure is just a 2-fold covering of X.

Theorem 10. *Let E be an oriented vector bundle over a manifold X. Then there exists a spin structure on E if and only if the second Stiefel-Whitney class of E is zero. Furthermore, if $w_2(E) = 0$, then the distinct spin structures on E are in 1-to-1 correspondence with the elements of $H^1(X, \mathbb{Z}_2)$.*

Definition 11. A spin manifold X is an oriented riemannian manifold with a spin structure on its tangent bundle.

Examples 2. Let $X = S^1$. Then there are two distinct spin structures on X (hence $H^1(S^1, \mathbb{Z}_2) = \mathbb{Z}_2$):

$P_{\mathrm{SO}_1}(S^1) \cong S^1$ and there are two 2-fold coverings of S^1

$$\zeta_1 : S^1 \times \mathbb{Z}_2 \to S^1 \text{ and } \zeta_2 : S^1 \to S^1.$$

These are the two spin structures.

Let $X = T^2$. Then there are four distinct spin structures on X (hence $H^1(T^2, \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$):
$P_{\mathrm{SO}_2}(T^2) \cong T^2 \times S^1$ and there are four 2-fold coverings $\zeta_i : T^2 \times S^1 \to T^2 \times S^1$:

$$\zeta_1(x, y, z) := (x, y, z^2),$$

$$\zeta_2(x, y, z) := (x, y, xz^2),$$

$$\zeta_3(x, y, z) := (x, y, yz^2),$$

$$\zeta_4(x, y, z) := (x, y, xyz^2)$$

These are the four spin structures.

Construction: Let $E \to X$ be a principal G-bundle and let F be another space on which the group G acts. Then G acts on $E \times F$ by

$$g \cdot (e, f) := (eg^{-1}, gf)$$

for $g \in G, e \in E$ and $f \in F$. Define $E \times_G F := E \times F/G$. This is a fibre bundle over X, called the bundle associated to E with fibre F.

Definition 12. The Clifford bundle of the oriented riemannian vector bundle E is the bundle

$$Cl(E) := P_{\mathrm{SO}}(E) \times_{\mathrm{SO}} C_n.$$

Definition 13. Let E be oriented riemannian vector bundle with a spin structure $\zeta : P_{\mathrm{Spin}}(E) \to P_{\mathrm{SO}}(E)$. A real spinor bundle of E is a bundle of the form

$$S(E) = P_{\mathrm{Spin}}(E) \times_{\mathrm{Spin}} M$$

where M is a left module over C_n.

Example 1.

$$Cl_{\mathrm{Spin}}(E) := P_{\mathrm{Spin}}(E) \times_{\mathrm{Spin}} C_n$$

This bundle admits a free action of C_n on the right.

Theorem 14. *Let $S(E)$ be a real spinor bundle of E. Then $S(E)$ is a bundle of modules over the the bundle of algebras $Cl(E)$.*

3. Spin structures a la Stolz/Teichner

Definition 15 (New definition)**.** Let V be a inner product space of dimension n. A spin stucture on V is an irreducible graded $C(V)-C_n$-bimodule equipped with a compatible inner product.

Definition 16 (New definition)**.** Let $E \to X$ be a real riemannian vector bundle of dimension n and let $Cl(E) \to X$ be the Clifford algebra bundle. A spin structure of E is a bundle $S(E) \to X$ of graded irreducible $Cl(E) - C_n$-bimodels.

Remark 17. *Let* $\mathrm{Spin}(E) \to X$ *be a principal* Spin_n*-bundle like in section 2. The spinor bundle*

$$Cl_{\mathrm{Spin}}(E) = \mathrm{Spin}(E) \times_{\mathrm{Spin}_n} C_n,$$

is then a $C(E) - C_n$-bimodule, i.e. a spin structure.

Definition 18. The opposite spin structure of a spin structure $S(E)$ is $\overline{S(E)}$.

By Theorem 6 a spin sturucture $S(E)$ determines an orientation of E. The opposite orientation is induced by $\overline{S(E)}$.

Definition 19. A spin manifold X is a manifold X together with a spin structure on its cotangent bundle T^*X.

Examples 3. Let X be $\mathbb{R}^n$. By identyfying T^*X with $\mathbb{R}^n \times \mathbb{R}^n$ the bundle

$$S := \mathbb{R}^n \times C_n \to \mathbb{R}^n$$

becomes an irreduzible graded $Cl(T^*X) - C_n$-bimodule bundle.

Restricting S on submanifolds of codimension 0 we obtain further spin structures, for example the spin structures on

$$D^n \subset \mathbb{R}^n \quad \text{or} \quad I_t := [0, t] \subset \mathbb{R}.$$

This spin structure S makes sense for $n = 0$: $\mathbb{R}^0$ is just one point pt. Since $C_0 = \mathbb{R}$, $S = \mathbb{R}$ is a graded $\mathbb{R} - \mathbb{R}$-bimodule (even line). The opposite spin structure of pt is $\overline{pt}$ is then an odd line.