Real Johnson-Wilson theories

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Complex Oriented Theories and Formal Group Laws

Suppose E is a complex oriented generalized cohomology theory.

- $E^*(\mathbb{C}P^\infty) \cong E^*[[x]]$ for $x \in E^2(\mathbb{C}P^\infty)$
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- Examples: $H^*(-;R)$, $KU^*(-)$, and $MU^*(-)$

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- The classifying map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\mu} \mathbb{C}P^\infty$ of the tensor product of line bundles gives rise to a power series F_E over E^*

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- F_E is a formal group law over E^*

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Theorem (Quillen)

If F is a formal group law over R, then there is a unique map $MU^* \longrightarrow R$ carrying F_{MU} to F_R.

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Natural Question: When is this a cohomology theory?

Theorem (Landweber)

Let v_i be the coefficient of x^{p^i} in $[p]_F(x)$. If the sequence $(v_0, v_1, v_2, \ldots)$ forms a regular sequence in R for every prime p, then $MU^*(X) \otimes_{MU^*} R$ is a cohomology theory.

Two cohomology theories

Fix prime $p = 2$.

- **Johnson-Wilson theory** $E(n)$: Landweber exact theory with

$$E(n)_* = \mathbb{Z}_{(2)}[v_1, \dots, v_{n-1}, v_n^{\pm}], \qquad |v_i| = 2(2^i - 1)$$

- **Morava** E**-theory** E_n: Landweber exact theory with

$$(E_n)_* = W(\mathbb{F}_{2^n})[[u_1, \dots, u_{n-1}]][u^{\pm}], \qquad |u_i| = 0, |u| = 2$$

- Related by completion and homotopy fixed points:

$$\widehat{E(n)} = L_{K(n)}E(n), \qquad E_n(\mathrm{Gal}) = E_n^{hG}$$

$$\widehat{E(n)} \simeq E_n(\mathrm{Gal})$$

$$\widehat{E(n)}_* = (E(n)_*)^{\wedge}_{I_n} = \widehat{\mathbb{Z}}_2[[v_1, \dots, v_{n-1}]][v_n^{\pm}]$$

A natural question

We have $\widehat{E(n)} \simeq E_n(\mathrm{Gal})$ and...

$\mathbb{Z}/2$ acts on $\widehat{E(n)}$

Complex conjugation action

$\mathbb{Z}/2$ acts on $E_n(\mathrm{Gal})$

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First a little more background...

Real theories

Complex conjugation action on $E(n)$ arises in context of Real theories ($\mathbb{Z}/2$-equivariant $RO(\mathbb{Z}/2)$-graded)

- Atiyah, 1966: Real K-theory $\mathbb{KR}$

$$\mathbb{KR}(X) = G\left\{\text{cplx v.b. } \pi : E \to X \;\middle|\; \begin{array}{c} E, X \ \mathbb{Z}/2\text{-spaces} \\ \text{antilin. on fibers, } \pi \text{ equiv} \end{array}\right\}$$

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- Araki, 1978: Defined $\mathbb{BPR}$ using a Quillen idempotent argument
- Hu & Kriz, 2001: Defined $\mathbb{KR}(n)$ and $\mathbb{ER}(n)$ as $\mathbb{MR}$-modules

Kitchloo and Wilson's real Johnson-Wilson theory

Real theory $\mathbb{E} \rightsquigarrow$ naïve $\mathbb{Z}/2$-equivariant theory

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Moral:

$$\text{complex} \xleftarrow{\text{forget}} \text{Real} \xrightarrow{\text{fixed pts}} \text{real}$$

Kitchloo and Wilson's real Johnson-Wilson theory

The $ER(n)$ are higher real K-theories.

$$E(1) = KU_{(2)} \qquad ER(1) = KO_{(2)}$$

Kitchloo-Wilson: There is a fibration

$$\Sigma^{\lambda(n)} ER(n) \xrightarrow{x(n)} ER(n) \to E(n)$$

that reduces when $n = 1$ to the classical fibration

$$\Sigma KO_{(2)} \xrightarrow{\eta} KO_{(2)} \to KU_{(2)}$$

Makes computations feasible (Bockstein spectral sequence).

$\lambda(n) = 2^{2n+1} - 2^{n+2} + 1$

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The Morava stabilizer group

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Extended Morava stabilizer group: $\mathbb{G}_n := \mathrm{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2) \ltimes S_n$

Hopkins-Miller-Goerss

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- $E_n(\mathrm{Gal})_* = \widehat{\mathbb{Z}}_2[[v_1, \ldots, v_{n-1}]][v_n^\pm]$
- Order 2 subgroup generated by $i(x)$ acts on $E_n(\mathrm{Gal})$

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Answer: Yes

Theorem (A.)

There is an equivalence

$$\widehat{E(n)}^{h\mathbb{Z}/2} \simeq E_n(\mathrm{Gal})^{h\mathbb{Z}/2}$$

and the natural map

$$ER(n) = E(n)^{h\mathbb{Z}/2} \to \widehat{E(n)}^{h\mathbb{Z}/2}$$

induces an algebraic completion on coefficients.

Consequences

Corollary

After completion, $ER(n)$ is an E_∞-ring spectrum.

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$$(E_n(\mathrm{Gal})^{h\mathbb{Z}/2})_* = \widehat{\mathbb{Z}}_2[[\hat{v}_k(l) \mid 0 \leq k < n, l \in \mathbb{Z}]][x, v_n^{\pm 2^{n+1}}]/J$$

J is the ideal generated by the relations

$$\hat{v}_0(0) = 2 \qquad x^{2^{k+1}-1}\hat{v}_k(l) = 0$$

$$\textit{and for } k \leq m, \qquad \hat{v}_m(l)\hat{v}_k(2^{m-k}s) = \hat{v}_m(l+s)\hat{v}_k(0)$$

$$|x| = \lambda(n) = 2^{2n+1} - 2^{n+2} + 1 \qquad |v_n^{2^{n+1}}| = 2^{n+2}(2^n - 1)^2$$

$$|\hat{v}_k(l)| = 2(2^k - 1) + l2^{k+2}(2^n - 1)^2 - 2(2^k - 1)(2^n - 1)^2$$

A bit about the proof

We'd like *equivariant* map $\varphi : \widehat{E(n)} \to E_n(\mathrm{Gal})$ that is also an equivalence.

But we only have a *homotopy equivariant* one.

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- Try replacing $\widehat{E(n)}$ by $\widehat{E(n)} \wedge F(\widehat{E(n)}, E_n(\text{Gal}))_\varphi$
- φ homotopy equivariant $\Rightarrow$ conjugation action on $F(\widehat{E(n)}, E_n(\text{Gal}))_\varphi$
- $ev : \widehat{E(n)} \wedge F(\widehat{E(n)}, E_n(\text{Gal}))_\varphi \to E_n(\text{Gal})$ is honestly equivariant

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- If $F(\widehat{E(n)}, E_n(\mathrm{Gal}))_\varphi \simeq pt$, then $\widehat{E(n)} \wedge F(\widehat{E(n)}, E_n(\mathrm{Gal}))_\varphi \simeq \widehat{E(n)}$.
- Need appropriate category so that $F(\widehat{E(n)}, E_n(\mathrm{Gal}))_\varphi \simeq pt$. Try S-algebra maps.
- Problem: not known if $\mathbb{Z}/2$-action on $E(n)$ is a S-algebra map.

A bit about the proof

- Instead use $F_{S-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))$. New problem: not contractible.
- Dirty trick: create S-algebra T so that $F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))$ is homotopy discrete.

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- Instead use $F_{S-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))$. New problem: not contractible.
- Dirty trick: create S-algebra T so that $F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))$ is homotopy discrete.
  - $T =$ free S-algebra on a bunch of spheres
  - $\pi_*(v_n^{-1}\widehat{MU}) = \pi_*(\widehat{E(n)} \wedge T)$
  - Compute BKSS for $F_{T-\mathrm{alg}}(-, E_n(\mathrm{Gal}))$
  - A map $\widehat{E(n)} \wedge T \to v_n^{-1}\widehat{MU}$ gives a map of spectral sequences that is an iso on E_2
  - Since that for $\widehat{E(n)} \wedge T$ collapses, so does that for $v_n^{-1}\widehat{MU}$

A bit about the proof

- Now

$$v_n^{-1}\widehat{MU} \wedge F_{T-\mathrm{alg}}(v_n^{-1}\widehat{MU}, E_n(\mathrm{Gal}))_\nu \to E_n(\mathrm{Gal})$$

  is equivariant.

- After taking homotopy fixed points, obtain a factorization

$$\begin{array}{ccc} v_n^{-1}\widehat{MU}^{h\widetilde{\mathbb{Z}/2}} & \longrightarrow & E_n(\mathrm{Gal})^{h\widetilde{\mathbb{Z}/2}} \\ \downarrow & \nearrow \simeq & \\ \widehat{E(n)}^{h\widetilde{\mathbb{Z}/2}} & & \end{array}$$

Thank you!