Algebraic models for rational equivariant stable homotopy theory (joint work with John Greenlees)

Conjecture.(Greenlees) For any compact Lie group G there is an abelian category $\mathcal{A}(G)$ such that

$$\mathbb{Q}\text{ - }G\text{-spectra} \simeq_Q \text{d. g. } (\mathcal{A}(G))$$

where $\mathcal{A}(G)$ has injective dimension equal to the rank of G.

Verified for finite groups, SO(2), O(2), SO(3) (G.-May, G., S., Barnes)

Theorem 1.(G.-S., '09) For G connected compact Lie,

$$\mathbb{Q} \text{ free } G\text{-spectra} \simeq_Q \text{tor-}H^*BG\text{-Mod}$$

Theorem 2.(preprint in progress) The conjecture holds for G any torus.

The rest of this talk will outline the five steps of the proof of Theorem 2 for $G = S^1$.

We will concentrate on step one.

Step 1 Variation on fixed point diagram.

Definitions. Let $\mathcal{F} = \{F\}$ be the family of finite subgroups of G. Define

$$(E\mathcal{F})^H = \begin{cases} \text{pt} & H \text{ finite} \\ \emptyset & H \text{ not finite} \end{cases}$$

Define $\widetilde{E}\mathcal{F}$ as the cofiber of the map $E\mathcal{F}_+ \to S^0$.

Define $DE\mathcal{F}_+ = \mathrm{Hom}(E\mathcal{F}_+, S^0)$.

Proposition. For $G = SO(2)$ there is a homotopy pullback of G-equivariant commutative ring spectra.

$$\begin{array}{ccc} S^0 & \longrightarrow & \widetilde{E}\mathcal{F} \\ \downarrow & & \downarrow \\ DE\mathcal{F}_+ & \longrightarrow & DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F} \end{array}$$

Analogues

Proposition. There is a pullback square

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \prod_p \mathbb{Z}_p & \longrightarrow & \prod_p \mathbb{Z}_p \otimes \mathbb{Q} \end{array}$$

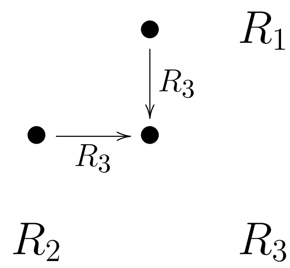
General Case: Assume given a homotopy pullback of rings (ring spectra or DGAs):

$$\begin{array}{ccc} R & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ R_2 & \longrightarrow & R_3 \end{array}$$

Let $R^{\lrcorner}$ denote the diagram of rings above with R deleted.

$$\begin{array}{ccc} & & R_1 \\ & & \downarrow \\ R_2 & \longrightarrow & R_3 \end{array}$$

Definition. $R^{\lrcorner}$-modules is the category of modules over the ring with three objects with $\mathrm{Hom}(1,3) = R_3$ and $\mathrm{Hom}(2,3) = R_3$.



Such a module is a collection $\{M_i\}_{i=1,2,3}$ of $\{R_i\}$-modules with structure maps $R_3 \otimes_{R_1} M_1 \to M_3$ and $R_3 \otimes_{R_2} M_2 \to M_3$. (The adjoints of these structure maps are an R_1-morphism $M_1 \to M_3$ and an R_2-morphism $M_2 \to M_3$.)

Note $R^{\lrcorner}$ determines such a module.

$R^{\lrcorner}$-Mod has three generators R-Mod has only one.

Proposition. The derived category of R-modules is equivalent to the localizing subcategory of $R^{\lrcorner}$-modules generated by $R^{\lrcorner}$. This equivalence is
induced by a Quillen equivalence of model categories.

$$R\text{-Mod} \simeq_Q \text{cell}_{\{R^{\lrcorner}\}}\text{ - } R^{\lrcorner}\text{-Mod}$$

Proof. Consider the adjoint functors on the
generators.

$$M \quad \rightarrow \quad R^{\lrcorner} \otimes_R M$$

$$\text{pullback}(\{M_i\}) \quad \leftarrow \quad \{M_i\}$$

Step 1:
Rational G-spectra are S^0-modules; apply above proposition with above square with $R^{\lrcorner} =$

$$\begin{array}{ccc} & & \widetilde{E}\mathfrak{F} \\ & & \downarrow \\ DE\mathfrak{F}_+ & \longrightarrow & DE\mathfrak{F}_+ \wedge \widetilde{E}\mathfrak{F}. \end{array}$$

Here cellularize with respect to $\{G/H_+ \wedge R^{\lrcorner}\}_H$.

Conclude:

$$\mathbb{Q}\text{ - }G\text{-spectra} = S^0\text{-Mod} \simeq_1 \text{cell}_{\{G/H_+ \wedge R^{\lrcorner}\}}\text{ - }R^{\lrcorner}\text{-Mod}$$

Step 2: Move from G-spectra to spectra.

$$A\text{-Mod}_{(G\text{ - spectra})} \leftrightarrow A^G\text{-Mod}_{(\text{spectra.})}$$

This induces an equivalence on each of the cells $\{G/H_+ \wedge R^{\lrcorner}\}_H$ for each of the relevant rings.

$$S^0\text{-Mod}_G \simeq_1 \text{cell-}R^{\lrcorner}\text{-Mod}_G \simeq_2 \text{cell-}(R^{\lrcorner})^G\text{-Mod}$$

Step 3: Make algebraic:
rational commutative ring spectra are modeled by
rational commutative DGA's

$$\simeq_2 \text{cell-}(R^{\lrcorner})^G\text{-Mod} \simeq_3 \text{cell-d.g.-}(R^{\lrcorner})^G_{DGA}\text{-Mod}$$

Step 4: Rigidity

$(R^{\lrcorner})^G_{DGA}$ is intrinsically formal.

1. $\pi_*(\widetilde{E}\mathcal{F})^G \cong \pi_* S^0 \cong \mathbb{Q}[0]$.

2. Note $E\mathcal{F}_+$ rationally splits as $\vee E\langle F\rangle$.
Since $E\langle 1\rangle = EG$, then

$(DEG_+)^G = \mathrm{Hom}(EG_+, S^0)^G \simeq \mathrm{Hom}(BG_+, S^0)$.

So $\pi_*(DE\mathcal{F}_+)^G \cong \prod_F H^*(BG/F) =: \vartheta_{\mathcal{F}}$.

3. $\pi_*(DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F})^G \cong \mathcal{E}_G^{-1}\vartheta_{\mathcal{F}}$.

Thus $(R^{\lrcorner})^G_{DGA}$ is quasi-isomorphic to $R^{\lrcorner}_{alg} = H_*(R^{\lrcorner})^G_{DGA}$.

$$\begin{array}{ccc} & & \mathbb{Q} \\ & & \downarrow \\ \vartheta_{\mathcal{F}} & \longrightarrow & \mathcal{E}_G^{-1}\vartheta_{\mathcal{F}} \end{array}$$

Summary:

$$S^0\text{-Mod}_G \simeq_1 \text{cell-}R^{\lrcorner}\text{-Mod}_G \simeq_2 \text{cell-}(R^{\lrcorner})^G\text{-Mod}$$
$$\simeq_3 \text{cell-d.g.-}(R^{\lrcorner})^G_{DGA}\text{-Mod} \simeq_4 \text{cell-d.g.-}R^{\lrcorner}_{alg}\text{-Mod}$$

Step 5: Small algebraic model.

For $G = SO(2)$, $\mathcal{A}(G)$ is the category of modules $N \to M \leftarrow V$ over

$$\begin{array}{ccc} & & \mathbb{Q} \\ & & \downarrow \\ \vartheta_{\mathcal{F}} & \longrightarrow & \mathcal{E}_G^{-1}\vartheta_{\mathcal{F}} \end{array}$$

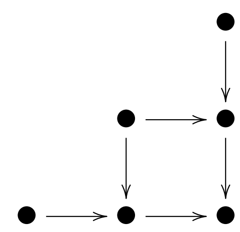
such that both structure maps are isomorphisms.

1. Quasi-coherence: $\mathcal{E}_G^{-1}\vartheta_{\mathcal{F}} \otimes_{\vartheta_{\mathcal{F}}} N \cong \mathcal{E}_G^{-1}N \xrightarrow{\cong} M$.
2. Extended: $\mathcal{E}_G^{-1}\vartheta_{\mathcal{F}} \otimes_{\mathbb{Q}} V \cong M$.

$$\simeq_4 \text{cell-d.g.-}R_{alg}^{\lrcorner}\text{-Mod} \simeq_5 \text{d.g.}\mathcal{A}(G)$$

Theorem 2. For $G = SO(2)$, the homotopy theory of rational G-spectra is modeled by differential graded $\mathcal{A}(G)$-modules. Here $\mathcal{A}(G)$ has injective dimension one.

General outline is the same for all tori, just have larger diagrams. For G a 2-torus, the diagram shape is:



For an n-torus there are n layers.

Can restrict to families of fixed points.

For example, free G-spectra with $G = SO(2)$: have a module N over $H^*(BG)$, with $V = 0, M = 0$. The quasi-coherence condition says $\mathcal{E}_G^{-1} N \cong M = 0$; that is, N is torsion.

Theorem 1. The homotopy theory of free rational G-spectra is modeled by torsion modules over H^*BG.