SEMINAR: BRAUER GROUPS AND THE PERIOD-INDEX PROBLEM, S4A1, SS 2025

The goal of this seminar is to study the Brauer group of general schemes. It consists of three parts:

(1) Introduction to Brauer groups and Brauer–Severi fibrations;
(2) Applications of Brauer groups to rationality questions and the Tate conjecture;
(3) The period-index problem and recent developments.

Organization. This seminar is a combination of a graduate seminar and a research seminar. Credit points can be obtained by following the seminar and either delivering a (joint) talk or submitting an essay on one topic. If you are interested in participating in the seminar, please write an email to *mezzedim@...* before March 14th.

Place and time. Tuesdays 16:00 - 18:00, seminar room 0.011.

1. The Brauer group of a field

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References: [4, Chapter 1.1–1.3], [5, Chapters 1–3]. Briefly recall the language of Galois cohomology. Central simple algebras, Wedderburn theorem. Definition of the Brauer group via central simple algebras and cohomological description of $\mathrm{Br}(k)$ in terms of Galois cohomology. Kummer sequence.

2. The Brauer group of a scheme

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References: [7, 8, 9], [4, Chapter 3]. Azumaya algebras and the Brauer–Azumaya group. Briefly recall the language of étale cohomology. The cohomological Brauer group via étale cohomology and comparison of the two Brauer groups (only the statement, we will see the proof in Talk 6). Kummer sequence. Unramified Brauer classes. Brauer group of smooth curves. Purity for the Brauer group.

3. Brauer–Severi varieties I

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References: [2], [4, Section 6.1], [5, Chapter 5]. Brauer–Severi varieties over a field k as twists of $\mathbb{P}^n_k$. A Brauer–Severi variety is trivial if and only if it has a k-point. Brauer–Severi varieties over general schemes, and conic/quadric bundles. Period and index of a Brauer–Severi variety, and relationship between period and index.

4. Brauer–Severi varieties II

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References: [3]. Explain the (integral and rational) cohomology of a Brauer–Severi variety over a scheme X (Note: These results have not been written up in the literature, so the person in charge would have to dig deep in the literature and probably prove some results themselves. The goal would be to have a description of the cohomology via a twisted version of Leray–Hirsch. For Brauer–Severi varieties over a field, much more is written down). Explain Bernardara's result on the derived category of coherent sheaves on a Brauer–Severi variety.

5. Uni(rationality) and the Artin–Mumford example

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References: [4, Chapters 5, 10, 11]. The Brauer group of a smooth projective variety is a birational invariant. The Artin–Mumford example: a 3-fold that is unirational, but not stably rational. Rationality in families by Hassett–Pirutka–Tschinkel.

6. Gerbes and twisted sheaves

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References: [17, Sections 12.2, 12.3], [6], [4, Section 2.6.4 and 4.2]. Give a self-contained account of gerbes and twisted sheaves, with particular emphasis on the geometric side. Use it to explain de Jong's proof of Gabber's theorem comparing the cohomological Brauer group and the Brauer–Azumaya group.

7. Brauer groups and the Tate conjecture

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References: [4, Chapter 16], [20], [19], [18]. State the Tate conjecture, and explain how it is related to the finiteness of the Brauer group for smooth projective varieties over finite fields. Prove that, for a K3 surface X over a number field k, the group $\mathrm{Br}(\overline{X})^{\mathrm{Gal}(\bar{k}/k)}$ is finite (and its cardinality can be explicitly bounded). If time permits, talk about the Artin–Tate paring on the l-primary part of the quotient $\mathrm{Br}(X)/\mathrm{Br}(X)_{\mathrm{div}}$, where X is a smooth projective surface over a finite field.

8. Period-index problem in dimension 2: characteristic 0

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References: [14], [21]. Introduce the period-index problem, and explain why it is obvious for curves. Explain de Jong's proof of the period-index theorem for surfaces in characteristic 0.

9. Period-index problem in dimension 2: characteristic p

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References: [16]. Explain Lieblich's proof of the period-index theorem for fields of transcendence degree 2 over $\mathbb{F}_p$.

10. Period-index problem for unramified classes

References: [15]. Explain de Jong–Starr's result and its important consequence: if the period–index problem holds for *unramified* classes of an arbitrary variety, then it holds for all Brauer classes.

11. Splitting Brauer classes by abelian varieties

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References: [10, 1, 12, 13] Here we can choose a subset of the referenced works. The first two papers show how to split Brauer classes of degree $\leq$ 7 via genus one curves, while the second two papers prove the splitting of (unramified) Brauer classes via (torsors of) abelian varieties.

12. The period-index problem for abelian threefolds

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References: [11]. Explain the work of Hotchkiss–Perry, proving the period-index conjecture for unramified classes on abelian threefolds.

References

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