ON L-EMBEDDINGS AND DOUBLE COVERS OF TORI OVER LOCAL FIELDS

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Abstract

To a torus T over a local field F and a subset of its character module subject to certain properties, we associate a canonical double cover $T(F)_\pm$ of the topological group $T(F)$ of F-rational points of T. We further associate an L-group ${}^L T_\pm$ to this double cover and establish a natural bijection between L-parameters valued in ${}^L T_\pm$ and genuine characters of $T(F)_\pm$. When T is a maximal torus of a connected reductive group G, we show that there is a canonical L-embedding ${}^L T_\pm \to {}^L G$. This leads to a canonical factorization of Langlands parameters. We associate to a genuine character of $T(F)_\pm$ subject to certain conditions a Harish-Chandra character formula and use it to give a conjectural characterization of the supercuspidal local Langlands correspondence for G, subject to a certain condition on p. This generalizes previous work of Adams-Vogan [AV92] for $F = \mathbb{R}$, and reinterprets computations of Langlands-Shelstad [LS87].

CONTENTS

1   INTRODUCTION

Let G be a connected reductive group defined over a local field F and let $T \subset G$ be a maximal torus. Let $\widehat{T}$ and $\widehat{G}$ be the corresponding dual groups, defined over $\mathbb{C}$, and equipped with algebraic actions of the absolute Galois group Γ of F. There is a canonical $\widehat{G}$-conjugacy class of embeddings $\widehat{T} \to \widehat{G}$. This conjugacy class is Γ-invariant, but need not contain a Γ-invariant member. A natural question is whether a member of this conjugacy class can be extended to an L-embedding $\widehat{T} \rtimes \Gamma \to \widehat{G} \rtimes \Gamma$ between the Galois forms of the corresponding L-groups, and whether a canonical such extension exists. Interest in this question comes from the local Langlands correspondence, which describes representations of $G(F)$ in terms of objects associated to $\widehat{G} \rtimes \Gamma$.

    In some cases the answer to the existence question is negative. For example, when $F = \mathbb{R}$, $G = \mathrm{PGL}_2$, and T is anisotropic, such an embedding would have to take image in the normalizer of a maximal torus of $\mathrm{SL}_2(\mathbb{C})$, which is a non-split extension of $\mathbb{C}^\times$ by $\mathbb{Z}/2\mathbb{Z}$, while $\widehat{T} \rtimes \Gamma$ is the split such extension. In other cases the answer to the existence question is positive, but there is no natural choice.

Langlands observed that if one replaces $\widehat{T} \rtimes \Gamma$ by $\widehat{T} \rtimes W_F$, where W_F is the absolute Weil group of F, then the answer to the existence question is always positive [Lan79, Lemma 4]. This existence result was later strenghtened by Langlands-Shelstad, who gave an explicit construction of an L-embedding $\widehat{T} \rtimes W_F \to \widehat{G} \rtimes W_F$ based on their concept of χ-data [LS87, §2.5,§2.6]. This construction was initially motivated by the study of endoscopic transfer factors [LS87], but has since found other uses as well, for example in explicit constructions of the local Langlands correspondence and in the description of the Harish-Chandra characters of supercuspidal representations [Kal19a]. The group $\widehat{T} \rtimes W_F$ is called the Weil-form of the L-group of T, and the usage of the Weil form of the L-group has now become commonplace due to this phenomenon, since replacing $\widehat{G} \rtimes \Gamma$ by $\widehat{G} \rtimes W_F$ has no deterimental effects on other parts of the theory.

However, the problem that there does not exist a canonical L-embedding $^LT \to {}^LG$ still remains. Since such embeddings embody functorial transfer from $T(F)$ to $G(F)$, and this transfer depends on the particular choice of L-embedding, this makes understanding the nature of such embeddings important. Over the real numbers Shelstad studied in [She81] L-embeddings $^LT \to {}^LG$, and more generally $^LH \to {}^LG$ for endoscopic groups H. She observed that to specify an L-embedding $^LT \to {}^LG$ it is enough to choose a Borel subgroup defined over $\mathbb{C}$ containing T. From the point of view of χ-data this is expressed by the existence of a natural choice, which Shelstad calls "based" in [She08, §9], associated to such a Borel subgroup. Unfortunately, over non-archimedean fields such a simple choice does not exist.

A different point of view emerges from the work of Adams and Vogan [AV92], [Ada91], [AV16], which introduces the concept of the ρ-cover of a maximal torus of a real reductive group. There is an associated L-group, which is a (usually) non-split extension of the dual torus by the Galois group, and therefore not isomorphic to the L-group of the torus itself. Emphasizing this fact, Adams and Vogan call it an "E-group". The ρ-cover and its L-group make the statement of the Harish-Chandara character formula for discrete series representations, as well as the construction of the local Langlands correspondence, more transparent. More recently, a slight variation of the ρ-cover has emerged, the so called ρ_i-cover, which makes constructions even more natural and has become the preferred device [AvLTV]. Its relationship to the ρ-cover is simple and explored in [Ada11, §3].

In this paper we introduce a double cover of the group of rational points of any algebraic torus defined over a local field and equipped with a certain finite subset of its character module (an essential example being the root system of an ambient reductive group). Unlike the ρ_i-cover of Adams-Vogan, which is of algebraic nature, the cover introduced here is of Galois-theoretic nature. Nevertheless, over the real numbers it recovers the ρ_i-cover.

We further define an L-group for our double cover – it is a (usually) non-split extension of the absolute Galois group of F (in fact already of the Galois group of the splitting extension) by the dual torus – and prove a local Lang-

lands correspondence, i.e. a bijection between the set of L-parameters valued in this L-group and the set of genuine characters of the double cover. The computations involved in the proof are all contained in [LS87]. Therefore, this part of the paper can be seen as a reinterpretation of computations made in [LS87]. We hope that the point of view introduced here illuminates the conceptual significance of these computations.

We summarize what was discussed so far in the folllowing theorem, which encompasses the discussion of §§3.1-3.4.

Theorem. *Let F be a local field and let S be an F-torus. Let $R \subset X^*(S)$ be a finite subset that does not contain 0 and is stable under the action of the Galois group and under multiplication by -1. Then there are naturally associated*

1. *extension of topological groups $1 \to \{\pm 1\} \to S(F)_\pm \to S(F) \to 1$,*

2. *extension of topological groups $1 \to \widehat{S} \to {}^L S_\pm \to {}^L S \to 1$,*

3. *bijection between the set of genuine conitnuous characters of $S(F)_\pm \to \mathbb{C}^\times$ and the set of $\widehat{S}$-conjugacy classes of continuous L-homomorphisms $W_F \to {}^L S_\pm$.*

Let us return to the setting of a maximal torus T in a connected reductive group G over F. We can apply the above theorem with R being the absolute root system of G with respect to S and obtain a double cover $T(F)_\pm$ of $T(F)$ with a corresponding L-group ${}^L T_\pm$. It turns out that there is a canonical $\widehat{G}$-conjugacy class of L-embeddings ${}^L T_\pm \to {}^L G$. Here one can take either Galois forms or Weil forms for the L-groups; it makes no difference.

Our main motivation for studying this question comes from the local Langlands correspondence for G. While this conjectural correspondence is expected to satisfy many desiderata, some of which are listed in [Bor79, §10], there does not yet exist a list of conditions that fully characterize it. For certain kinds of groups, a characterization was proposed in [MY20]. In this paper we will propose a characterization for all discrete parameters with trivial monodromy (which we call supercuspidal), provided G is a connected reductive group over a non-archimedean local field of characteristic zero, that splits over a tame extension of F, and the residual characteristic of F is not too small. This characterization is of spectral nature: we give an explicit formula for the stable character of the L-packet associated to a given supercuspidal L-parameter. It is based on the fact that all discrete series representations of real groups, as well as all supercuspidal parameters of p-adic groups (provided the group splits over a tame extension and p does not divide the order of the Weyl group) factor canonically through the L-group ${}^L T_\pm$ of an elliptic maximal torus $T \subset G$. Via the above theorem, such a parameter leads naturally to a genuine character of $T(F)_\pm$. Motivated by the character formula due to Adler, DeBacker, and Spice, and its reinterpretation in [Kal19a], we give a conjectural formula for the stable character associated to the given Langlands parameter, namely

$$S\Theta_\varphi^G(\gamma) = e(J)\epsilon(T_G - T_J) \sum_{j:S \to J} [a_S \cdot \theta](\gamma_0) \cdot \widehat{SO}_{j\,X^*}^J(\log(\gamma_{>0})),$$

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where we refer to Conjecture 4.3 for the relevant notation. This formula is stated in the p-adic case, but the special case $\gamma = \gamma_0$ recovers the known formula for the stable character of a discrete series L-parameter of a real reductive group on its elliptic maximal torus, cf. [AV16]. This conjecture uniquely characterizes the L-packet associated to φ (together with the expected atomic stability property of that L-packet), and thus gives a characterization of the local Langlands conjecture for supercuspidal parameters when G splits over a tame extension and p is large. This provides a different point of view on the explicit constructions of the local Langlands correspondence of [Kal19a] and [Kal19b].

We now return to the discussion of the double covers of tori and their L-groups. The Langlands–Shelstad construction of an L-embedding ${}^L T \to {}^L G$ in terms of χ-data fits into this point of view as follows. First, χ-data is used as a technical device in the proof of the local Langlands correspondence for the double cover $T(F)_\pm$. Second, while the Galois form of ${}^L T_\pm$ can be a non-split extension of Γ by $\widehat{T}$, its pull-back along $W_F \to \Gamma$, i.e. the Weil form of ${}^L T_\pm$, is always split, albeit non-canonically. A choice of χ-data provides a splitting, and hence an L-isomorphism ${}^L T \to {}^L T_\pm$. Composed with the canonical L-embedding ${}^L T_\pm \to {}^L G$ this recovers the Langlands-Shelstad L-embedding associated to this χ-data.

We generalize our investigation from maximal tori to twisted Levi subgroups. These are connected reductive subgroups of G defined over F that are Levi components of parabolic subgroups defined over some field extension of F. In particular, any maximal torus is a twisted Levi subgroup. We show that the Langlands-Shelstad construction can be used to obtain an L-embedding ${}^L M \to {}^L G$ for any twisted Levi subgroup in terms of χ-data for a suitable finite set derived from the root system of G. We then show that $M(F)$ possesses a canonical double cover $M(F)_\pm$, construct an L-group ${}^L M_\pm$ that is again a (usually) non-split extension of Γ by $\widehat{M}$, show how an assumed local Langlands correspondence for $M(F)$ easily implies such a correspondence for genuine representations of $M(F)_\pm$, and show that ${}^L M_\pm$ has a canonical embedding into ${}^L G$. These considerations are motivated by the study of supercuspidal representations of p-adic groups, which naturally arise in an inductive fashion from twisted Levi subgroups [Yu01]. We hope they lay the groundwork for the study of general functorial transfer from twisted Levi subgroups, as well as a characterization of the local Langlands correspondence for reductive groups beyond the case discussed here.

Since the local Langlands correspondence for tori is functorial, it is natural to ask if the correspondence for our double covers is also functorial. Even formulating this question appropriately is not completely straightforward, since the double cover depends on a subset of the character module that satisfies certain properties. In this paper we formulate and prove one instance of such functoriality. It is needed in the study of how an L-embedding ${}^L T \to {}^L G$ factors through the L-group of a twisted Levi subgroup.

In 2008 Benedict Gross circulated privately a set of hand-written notes titled "Groups of type L", in which he made some observations about covers of tori

over local fields and their L-groups. These observations work well for tori S for which the norm map $S(E) \to S(F)$ is surjective, where E/F is the splitting extension of S. While this condition is quite restrictive, it is fulfilled in a very important special case – one-dimensional anisotropic tori. In Appendix B we have included some material from these notes and some further discussion, in the hope of shedding new light on some of the formulas appearing in [LS87] and on our definition of the L-group of the double cover and the corresponding Langlands parameters.

Acknowledgements: The author thanks Jeff Adams for drawing attention to the problem of embedding the L-group of a maximal torus into the L-group of the ambient reductive group and to its elegant solution in the real case via the ρ-cover.

2 NOTATION

Throughout the paper F will denote a local field of arbitrary characteristic, F^s a separable closure, Γ or Γ_F the absolute Galois group of F^s/F, W or W_F the Weil group, and $\Sigma = \Gamma \times \{\pm 1\}$. We let Σ act on F^s with $\{\pm 1\}$ acting trivially.

Consider a set R endowed with an action of Σ, which we will call a Σ-set for short. Following [LS87], a Σ-orbit $O \subset R$ is called symmetric if it is a single Γ-orbit, and asymmetric if it is the disjoint union of two Γ-orbits. For any $\alpha \in R$ we define $\Gamma_\alpha = \mathrm{Stab}(\alpha, \Gamma)$, $\Gamma_{\pm\alpha} = \mathrm{Stab}(\{\pm\alpha\}, \Gamma)$, $F_\alpha = (F^s)^{\Gamma_\alpha}$, $F_{\pm\alpha} = (F^s)^{\Gamma_{\pm\alpha}}$. We introduce Σ_α and $\Sigma_{\pm\alpha}$ analogously. Thus O is symmetric if and only if for one, hence any, $\alpha \in O$, one has $[F_\alpha : F_{\pm\alpha}] = 2$. We may then call each $\alpha \in O$ symmetric. When F is archimedean a symmetric α is traditionally called "imaginary". When F is non-archimedean a symmetric α is called ramified or unramified depending on whether the quadratic extension $F_\alpha/F_{\pm\alpha}$ is ramified or unramified. We have the Σ-invariant decomposition $R = R_{\mathrm{sym}} \cup R_{\mathrm{asym}}$ into the subsets of symmetric and asymmetric elements, respectively. In the non-archimedean case we further have the Σ-invariant decomposition $R_{\mathrm{sym}} = R_{\mathrm{sym,ram}} \cup R_{\mathrm{sym,unram}}$.

For $\alpha \in R_{\mathrm{sym}}$ we denote by κ_α the non-trivial homomorphisms $\Gamma_{\pm\alpha}/\Gamma_\alpha \to \{\pm 1\}$ and $F_{\pm\alpha}^\times/N_{F_\alpha/F_{\pm\alpha}}(F_\alpha^\times) \to \{\pm 1\}$. We denote by τ_α the non-trivial element of $\Gamma_{\pm\alpha}/\Gamma_\alpha$. We denote by $\mathbb{Z}_{(-1)}$ the abelian group $\mathbb{Z}$ on which $\Gamma_{\pm\alpha}/\Gamma_\alpha$ acts by multiplication by κ_α. We denote by $\mathbb{C}_{(-1)}^\times$ the abelian group $\mathbb{C}^\times$ with $\Gamma_{\pm\alpha}/\Gamma_\alpha$-action where τ_α acts by inversion. We denote by F_α^1 the kernel of the norm homomorphism $N_{F_\alpha/F_{\pm\alpha}} : F_\alpha^\times \to F_{\pm\alpha}^\times$.

We will say that R is *admissible* if the action of $-1 \in \Sigma$ does not fix any element of R. Given an algebraic torus S defined over F we shall denote by $X^*(S)$ and $X_*(S)$ its character and co-character modules. These are Σ-modules, with Γ acting in accordance with the F-structure of S, and $\{\pm 1\}$ acting by multiplication. A Σ-invariant subset $R \subset X^*(S)$ is admissible if and only if it does not contain 0.

If R is a Σ-set equipped with a Σ-invariant map $R \to X^*(S)$, the image of an element $\alpha \in R$ in $X^*(S)$ will be denoted by $\bar{\alpha}$.

3   THE DOUBLE COVER

3.1   Construction

Let S be an algebraic torus defined over F. Assume given a finite admissible Σ-set R and a Σ-equivariant map $R \to X^*(S)$. The most important case is when R is a Σ-invariant subset of $X^*(S)$ and the map is just the inclusion, but the slightly more general setting will become useful in §5.4 and §6, where it will give us more flexibility and will lead to a more natural statement of functoriality. Given this data we will define a double cover

$$1 \to \{\pm 1\} \to S(F)_\pm \to S(F) \to 1. \tag{3.1}$$

When we want to emphasize the datum R we shall write $S(F)_R$ in place of $S(F)_\pm$.

We begin by defining for any Σ-orbit $O \subset R$ an algebraic torus J_O and a morphism $S \to J_O$, both defined over F. This homomorphism will factor through $\bar{S}$, where $S \to \bar{S}$ is the quotient such that $X^*(\bar{S}) \subset X^*(S)$ is the submodule generated by the image of R. In fact, we are simply recalling a construction present in the proofs of Lemmas 3.2.D and 3.3.C of [LS87].

First, let O be asymmetric. Choose $\alpha \in O$ and let $J_\alpha = \mathrm{Res}_{F_\alpha/F}\mathbb{G}_m$, so that $X^*(J_\alpha) = \mathrm{Ind}_{\Gamma_\alpha}^\Gamma \mathbb{Z}$. We have $J_\alpha(F) = F_\alpha^\times$. The Γ_α-equivariant morphism $\mathbb{Z} \to X^*(S)$ given by $n \mapsto n\bar{\alpha}$ specifies, by Frobenius reciprocity as reviewed in Appendix A, a Γ-equivariant morphism $X^*(J_\alpha) \to X^*(S)$, given explicitly by $f \mapsto \sum_{\tau \in \Gamma_\alpha \backslash \Gamma} f(\tau)\tau^{-1}\bar{\alpha}$. Another $\beta \in O$ is of the form $\beta = \epsilon\tau\alpha$ for some $\tau \in \Gamma$, whose coset in Γ/Γ_α is unique, and a unique $\epsilon \in \{\pm 1\}$. The map $f_\alpha \mapsto f_\beta$ given by $f_\beta(x) = \epsilon f_\alpha(\tau^{-1}x)$ is an isomorphism $X^*(J_\alpha) \to X^*(J_\beta)$, depending only on α and β, and identifying the two morphisms to $X^*(S)$. Taking the inverse limit over all $\alpha \in O$ we obtain the torus J_O and the morphism $S \to J_O$. An element of $J_O(F)$ is a collection $(\gamma_\alpha)_{\alpha \in O}$, with $\gamma_\alpha \in F_\alpha^\times$, subject to $\gamma_{\epsilon\tau\alpha} = \tau(\gamma_\alpha)^\epsilon$.

Now let O by symmetric. Choose $\alpha \in O$. Let $\mathrm{Res}_{F_\alpha/F_{\pm\alpha}}^1 \mathbb{G}_m$ be the unique one-dimensional anisotropic torus defined over $F_{\pm\alpha}$ and split over F_α. Its character module is the $\Gamma_{\pm\alpha}$-module $\mathbb{Z}_{(-1)}$. Let $J_\alpha = \mathrm{Res}_{F_{\pm\alpha}/F}\mathrm{Res}_{F_\alpha/F_{\pm\alpha}}^1 \mathbb{G}_m$, so that

$$X^*(J_\alpha) = \mathrm{Ind}_{\Gamma_{\pm\alpha}}^\Gamma \mathbb{Z}_{(-1)} = \{f : \Gamma \to \mathbb{Z} | \forall \sigma \in \Gamma_{\pm\alpha} : f(\sigma\tau) = \kappa_\alpha(\sigma)f(\tau)\}.$$

In particular, an element f is constant on Γ_α-cosets. We have $J_\alpha(F) = F_\alpha^1$.

The $\Gamma_{\pm\alpha}$-equivariant morphism $\mathbb{Z}_{(-1)} \to X^*(S)$ given by $n \mapsto n\bar{\alpha}$ specifies, again by Frobenius reciprocity, a Γ-equivariant morphism $X^*(J_\alpha) \to X^*(S)$, given again explicitly by $f \mapsto \sum_{\tau \in \Gamma_{\pm\alpha} \backslash \Gamma} f(\tau)\tau^{-1}\bar{\alpha}$. Another $\beta \in O$ is of the form $\beta = \tau\alpha$ for $\tau \in \Gamma$ whose coset in Γ/Γ_α is unique. Then $\Gamma_{\pm\beta} = \tau\Gamma_{\pm\alpha}\tau^{-1}$

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and $\Gamma_\beta = \tau\Gamma_\alpha\tau^{-1}$, hence $\kappa_\beta = \kappa_\alpha \circ \mathrm{Ad}(\tau)$ and thus $f_\alpha \mapsto f_\beta$ with $f_\beta(x) = f_\alpha(\tau^{-1}x)$ provides again an isomorphism $X^*(J_\alpha) \to X^*(J_\beta)$ that depends only on α and β and identifies the two morphisms to $X^*(S)$. Taking the inverse limit over all $\alpha \in O$ we again obtain a torus J_O together with a morphism $S \to J_O$ defined over F. An element of $J_O(F)$ is again a collection $(\gamma_\alpha)_{\alpha \in O}$ of $\gamma_\alpha \in F_\alpha^1$ satisfying $\gamma_{\sigma\alpha} = \sigma(\gamma_\alpha)$. Note that this is equivalent to the description given in the case of asymmetric O, because of the existence of an element $\tau_\alpha \in \Gamma_{\pm\alpha} \smallsetminus \Gamma_\alpha$.

We have thus defined, for each Σ-orbit $O \subset R$, a torus J_O and a morphism $S \to J_O$, both defined over F. Next we define a double cover $J_O(F)_\pm$ of the topological group $J_O(F)$. When O is asymmetric we take the canonical split cover $J_O(F)_\pm = J_O(F) \times \{\pm 1\}$. When O is symmetric we choose again $\alpha \in O$. The map $F_\alpha^\times \to F_\alpha^1$ sending x to $x/\tau_\alpha(x)$ leads to the double cover

$$1 \to \frac{F_{\pm\alpha}^\times}{N_{F_\alpha^\times/F_{\pm\alpha}^\times}(F_\alpha^\times)} \to \frac{F_\alpha^\times}{N_{F_\alpha^\times/F_{\pm\alpha}^\times}(F_\alpha^\times)} \to F_\alpha^1 \to 1.$$

Using the identification $J_\alpha(F) = F_\alpha^1$ and pusing out by the sign character $\kappa_\alpha : F_{\pm\alpha}^\times/N_{F_\alpha^\times/F_{\pm\alpha}^\times}(F_\alpha^\times) \to \{\pm 1\}$ we obtain a double cover

$$1 \to \{\pm 1\} \to J_\alpha(F)_\pm \to J_\alpha(F) \to 1.$$

It is easy to see that the transition isomorphism $J_\alpha \to J_\beta$ for any pair $\alpha, \beta \in O$ lifts naturally to a transition isomorphism $J_\alpha(F)_\pm \to J_\beta(F)_\pm$; it is simply given by $\tau : F_\alpha^\times \to F_\beta^\times$ for the unique $\tau \in \Gamma/\Gamma_\alpha$ satisfying $\tau\alpha = \beta$. Taking the limit we therefore obtain a double cover

$$1 \to \{\pm 1\} \to J_O(F)_\pm \to J_O(F) \to 1.$$

Definition 3.1.

{dfn:cover}

1. The *big cover* of $S(F)$ is the pull-back of $S(F) \to \prod J_O(F) \leftarrow \prod J_O(F)_\pm$, where the products run over the set of Σ-orbits in R. We denote it by $S(F)_{\pm\pm}$. It is an extension of $S(F)$ by $\prod\{\pm 1\}$.

2. The double cover (3.1) is the pushout of $S(F)_{\pm\pm}$ along the multiplication morphism $\prod\{\pm 1\} \to \{\pm 1\}$.

$\square$

Remark 3.2. The double cover of $J_O(F)$ for a symmetric orbit O came from the exact sequence

$$1 \to F_{\pm\alpha}^\times \to F_\alpha^\times \to F_\alpha^1 \to 1, \quad \alpha \in O.$$

When O is asymmetric there is an analogous exact sequence, namely

$$1 \to F_\alpha^\times \to F_\alpha^\times \times F_{-\alpha}^\times \to F_\alpha^\times \to 1, \quad \alpha \in O,$$

where the first map is the diagonal embedding and the second map sends (a, b) to a/b, and we have used the identity $F_{-\alpha} = F_\alpha$. However, since there is no natural subgroup of $F_\alpha^\times$ of index 2, this exact sequence does not lead to an interesting double cover. $\square$

Remark 3.3. Explicating the definition, an element of $S(F)_{\pm\pm}$ can be represented non-uniquely by a tuple $(\gamma, (\delta_\alpha)_{\alpha \in R})$, where $\gamma \in S(F)$, $\delta_\alpha \in F_\alpha^\times$, $\delta_{\sigma\alpha} = \sigma(\delta_\alpha)$, and $\delta_\alpha/\delta_{-\alpha} = \bar{\alpha}(\gamma)$, for all $\sigma \in \Gamma$. Another collection representing the same element of $S(F)_{\pm\pm}$ is of the form $(\gamma, (\eta_\alpha \cdot \delta_\alpha))$, where $\eta_\alpha \in F_{\pm\alpha}^\times$, $\eta_{\sigma\alpha} = \sigma(\eta_\alpha)$ for $\sigma \in \Gamma$, $\eta_{-\alpha} = \eta_\alpha$, and $\kappa_\alpha(\eta_\alpha) = 1$ for all $\alpha \in R_{\mathrm{sym}}$. In order to represent the same element of $S(F)_\pm$, the last condition should be replaced by $\prod_{\alpha \in R_{\mathrm{sym}}/\Gamma} \kappa_\alpha(\eta_\alpha) = 1$.

When $R = R_{\mathrm{sym}}$ all elements of $S(F)_{\pm\pm}$ are represented this way. When $R_{\mathrm{sym}} \neq \emptyset$, all elements of $S(F)_\pm$ are represented this way.

If we want to represent all elements of $S(F)_\pm$, we can use tuples of the form $(\gamma, \epsilon, (\delta_\alpha)_{\alpha \in R_{\mathrm{sym}}})$, where $\gamma \in S(F)$, $\epsilon \in \{\pm 1\}$, and $\delta_\alpha \in F_\alpha^\times$ satisfy $\delta_{\sigma\alpha} = \sigma(\delta_\alpha)$ and $\delta_\alpha/\delta_{-\alpha} = \bar{\alpha}(\gamma)$ for all $\sigma \in \Gamma$ and $\alpha \in R_{\mathrm{sym}}$. Note that $\delta_{-\alpha} = \tau_\alpha(\delta_\alpha)$. Another such collection representing the same element is of the form $(\gamma, \eta \cdot \epsilon, (\eta_\alpha \cdot \delta_\alpha)_{\alpha \in R_{\mathrm{sym}}})$, for $\eta_\alpha \in F_{\pm\alpha}^\times$ with $\eta_{\sigma\alpha} = \sigma(\eta_\alpha)$, and $\eta = \prod_{\alpha \in R_{\mathrm{sym}}/\Gamma} \kappa_\alpha(\eta_\alpha)$. The unique non-trivial element $\epsilon = -1$ in the kernel of $S(F)_\pm \to S(F)$ is represented by $(1, -1, (1))$. $\qquad\square$

Remark 3.4. In what follows, we will quote some arguments and computations from [LS87]. Strictly speaking, the arguments of [LS87] apply to a finite Σ-invariant subset $0 \notin R \subset X^*(S)$, which appears slightly less general than having a Σ-equivariant map $R \to X^*(S)$ from a finite admissible Σ-set R. But for all arguments that we will quote, one can reduce to the more special case by the following simple construction. Let $\tilde{M}$ be the finitely generated free abelian group generated by the set R. Thus $\tilde{M} = \bigoplus_{\alpha \in R} \mathbb{Z}$. The group Σ operates on $\tilde{M}$ linearly. On the other hand, $\mathbb{Z}$ operates on $\tilde{M}$ via the abelian group structure. We let M be the quotient of $\tilde{M}$ under the action of the group $\{\pm 1\}$ embedded diagonally in $\Sigma \times \mathbb{Z}$. The admissibility of Σ implies that M is a finitely generated free abelian group. By construction it carries a Σ-action where $-1 \in \Sigma$ acts by multiplication by $-1 \in \mathbb{Z}$. The natural map $R \to \tilde{M} \to M$ is injective. The Σ-equivariant map $R \to X^*(S)$ extends canonically to a Σ-equivariant map $M \to X^*(S)$. If T is the F-torus with $X^*(T) = M$, we obtain a natural map $S \to T$. The arguments of [LS87] can be applied to the torus T and then deduced for the torus S. $\qquad\square$

3.2 χ-data and genuine characters

We continue with a torus S defined over F, a finite admissible Σ-set R, and a Σ-equivariant map $R \to X^*(S)$. We recall here the notions of χ-data and ζ-data introduced in [LS87]; see [LS87, §2.5] and [Kal19a, Definition 4.6.4]. A set of χ-data consists of a character $\chi_\alpha : F_\alpha^\times \to \mathbb{C}^\times$ for each $\alpha \in R$. This set satisfies

1. $\chi_{\sigma\alpha} = \chi_\alpha \circ \sigma^{-1}$ for all $\sigma \in \Gamma$;

2. $\chi_{-\alpha} = \chi_\alpha^{-1}$;

3. $\chi_\alpha|_{F_{\pm\alpha}^\times} = \kappa_\alpha$ when α is symmetric.

A set of ζ-data consists of a character $\zeta_\alpha : F_\alpha^\times \to \mathbb{C}^\times$ for each α in R and this set satisfies conditions 1 and 2 above, but in place of 3 it satisfieds $\zeta_\alpha|_{F_{\pm\alpha}^\times} = 1$ when α is symmetric.

We now recall from the proof of [LS87, Lemma 3.5.A] how a set of ζ-data leads to a character $\zeta_S : S(F) \to \mathbb{C}^\times$ and use this as a motivation to show how a set of χ-data leads to a genuine character of the double cover $S(F)_\pm$. Let $(\zeta_\alpha)_\alpha$ be a set of ζ-data. For each Σ-orbit O in R define a character $\zeta_O : J_O(F) \to \mathbb{C}^\times$ as follows. Choosing $\alpha \in O$ we obtain an identification $J_O = J_\alpha$. If O is asymmetric let $\zeta_O = \zeta_\alpha$. If O is symmetric ζ_O be the composition of ζ_α with the isomorphism $F_\alpha^\times/F_{\pm\alpha}^\times \cong F_\alpha^1 = J_O(F)$ sending $x \in F_\alpha^\times$ to $x/\tau_\alpha(x)$. In both cases it is straightforward to check that ζ_O depends only on O and not on the choice of α. Define ζ_S to be the pull-back of $\prod_O \zeta_O$ under the map $S(F) \to \prod_O J_O(F)$, where the products run over the set of Σ-orbits in R. This is the character of [Kal19a, Definition 4.6.5], where it was defined in the slightly less general setting $R \subset X^*(S)$.

Let now $(\chi_\alpha)_\alpha$ be a set of χ-data. We construct a genuine character $\chi_S : S(F)_\pm \to \mathbb{C}^\times$ as follows. Let $O \subset R$ be a Σ-orbit. Choose $\alpha \in O$ and obtain an identification $J_O = J_\alpha$. If O is asymmetric then define $\chi_O = \chi_\alpha$ as a character of $J_O(F) = F_\alpha^\times$ and extend it to a genuine character of $J_O(F)_\pm$ in the unique way. If O is symmetric then we have the commutative diagram

in which the columns are exact and the horizontal maps are isomorphisms. We let χ_O be the character of $J_O(F)_\pm$ that is the composition of χ_α with the middle horizontal isomorphism. It is a genuine character of the double cover $J_O(F)_\pm$ of $J_O(F)$. The product $\prod_O \chi_O$ is a character of $\prod_O J_O(F)_\pm$ whose restriction to $\prod_O \{\pm 1\}$ kills the kernel of the multiplication map $\prod_O \{\pm 1\} \to \{\pm 1\}$ and therefore pulls back to a genunine character of $S(F)_\pm$.

Definition 3.5. Let $\chi_S : S(F)_\pm \to \mathbb{C}^\times$ denote the character just constructed. $\square$

{dfn:chis}

Fact 3.6. If (χ_α) and (ζ_α) are sets of χ-data and ζ-data for R, respectively, and $\chi_S : S(F)_\pm \to \mathbb{C}^\times$ and $\zeta_S : S(F) \to \mathbb{C}^\times$ are the corresponding characters, then $(\chi_\alpha \cdot \zeta_\alpha)$ is another set of χ-data and we have

$$(\chi \cdot \zeta)_S = \chi_S \cdot \zeta_S.$$

$\square$

Proof. Left to the reader. ∎

3.3 The L-group

We continue with a torus S defined over F, a finite admissible Σ-set, and a Σ-equivariant map $R \to X^*(S)$. Let E/F be a finite Galois extension such that Γ_E acts trivially on R and $X^*(S)$. In the case when the map $R \to X^*(S)$ is injective, we can take E/F to be the splitting extension of S. We shall define the L-group of the double cover $S(F)_\pm$, first in its Galois form. This will be an extension of $\Gamma_{E/F}$ by $\widehat{S}$, usually not split, and unique up to isomorphism that is unique up to conjugation by $\widehat{S}$.

Recall the concept of a gauge from [LS87]. It is a function $p : R \to \{\pm 1\}$ having the property $p(-\alpha) = -p(\alpha)$.

Remark 3.7. A gauge need not be Γ-invariant. In fact, a Γ-invariant gauge exists if and only if all elements of R are asymmetric. $\square$

Definition 3.8. The *Tits cocycle* corresponding to p is $t_p \in Z^2(\Gamma_{E/F}, \widehat{S})$ defined by $t_p(\sigma, \tau) = (-1)^{\lambda_p(\sigma,\tau)}$, where $\lambda_p(\sigma, \tau) \in X^*(S) = X_*(\widehat{S})$ is the sum of $\bar{\alpha}$ for $\alpha \in \Lambda_p(\sigma, \tau) \subset R$ defined as

$$\{\alpha \in R | p(\alpha) = +1, p(\sigma^{-1}\alpha) = -1, p((\sigma\tau)^{-1}\alpha) = +1\}.$$

$\square$

It is shown in [LS87, Lemmas 2.1.B] that t_p is indeed a 2-cocycle. According to [LS87, Lemma 2.1.C] the cohomology class of t_p is independent of the choice of gauge p. In fact, according to [LS87, Lemma 2.4.A] there is a canonical co-chain $s_{p/q} \in C^1(\Gamma_{E/F}, \widehat{S})$ for two gauges $p, q : R \to \{\pm 1\}$ with the property $t_p/t_q = \partial s_{p/q}$. We recall its definition: $s_{p/q}(\sigma) = (-1)^{\mu_{p/q}(\sigma)}$, where $\mu_{p/q}(\sigma) \in X^*(S)$ is the sum of $\bar{\alpha}$ for $\alpha \in M_{p/q}(\sigma) \subset R$ defined as the disjoint union of

$$\{\alpha \in R | p(\alpha) = +1, p(\sigma^{-1}\alpha) = -1, q(\alpha) = q(\sigma^{-1}\alpha) = +1\}$$

and

$$\{\alpha \in R | p(\alpha) = p(\sigma^{-1}\alpha) = +1, q(\alpha) = -1, q(\sigma^{-1}\alpha) = +1\}.$$

Finally, [LS87, Lemma 2.4.B] shows that for three gauges p, q, r the 1-cochains $s_{p/q}$ and $s_{q/p}$ are cohomologous, and the 1-cochains $s_{p/q} \cdot s_{q/r}$ and $s_{p/r}$ are cohomologous.

After these preparations we now define the L-group of $S(F)_\pm$ as the twisted product $\widehat{S} \boxtimes_{t_p} \Gamma_{E/F}$ with respect to an arbitrary choice of gauge $p : R \to \{\pm 1\}$ and note that modulo conjugation by $\widehat{S}$ this is independent of the choice of p. More formally we note that to specify an extension of $\Gamma_{E/F}$ by $\widehat{S}$ that is unique up to isomorphism that in turn is unique up to conjugation by $\widehat{S}$ is the same as to specify an object, unique up to unique isomorphism, in the category whose objects are extensions of $\Gamma_{E/F}$ by complex tori, and the set of morphisms from the object $1 \to \widehat{T}_1 \to \mathcal{T}_1 \to \Gamma_{E/F} \to 1$ to the object $1 \to \widehat{T}_2 \to \mathcal{T}_2 \to \Gamma_{E/F} \to 1$ is the set of $\widehat{T}_2$-conjugacy classes of morphisms of extensions.

Definition 3.9. The *Galois form of the L-group* $^L S_\pm$ is the inverse limit, in the above category, of the system of extensions $^L S_\pm^{(p)} = \widehat{S} \boxtimes_{t_p} \Gamma_{E/F}$, one for each gauge $p : R \to \{\pm 1\}$, with transition isomorphisms defined for any pair of gauges p, q by

$$\xi_{p,q} : {}^L S_\pm^{(p)} \to {}^L S_\pm^{(q)}, \quad s \boxtimes \sigma \mapsto s \cdot s_{p/q} \boxtimes \sigma.$$

The *Weil form of the L-group* is the pull-back along $W_{E/F} \to \Gamma_{E/F}$ of the Galois form. $\square$

Remark 3.10. We can also consider the absolute Galois and Weil forms of the L-group, by pulling back along the projections $\Gamma \to \Gamma_{E/F}$ and $W \to W_{E/F}$.

The Galois form of the L-group can often be a non-split extension of Γ by $\widehat{S}$. The Weil form, on the other hand, is always split, but not canonically. In the non-archimedean case, the absolute Galois form is also split, but in general not canonically. We will discuss this in the next subsection. $\square$

Remark 3.11. The formula for $s_{p/q}$, and the definition of $^L S_\pm$, may seem a bit ad-hoc. We provide more motivation in Appendix B, especially Remark B.8 and §B.4. $\square$

Remark 3.12. Assume that all elements of R are asymmetric, so that a Γ-invariant gauge $p : R \to \{\pm 1\}$ exists. Then $t_p = 1$ and hence we have the canonical identification $^L S_\pm = {}^L S$. More generally, the same argument works when all symmetric elements of R are divisible by 2 in $X^*(S)$. In that case we choose the gauge p to have Γ-invariant restriction to all asymmetric orbits, so that the only elements of $\Lambda_p(\sigma, \tau)$ are symmetric, and again see that $t_p = 1$. This is the dual incarnation of the canonical isomorphism $S(F)_\pm = S(F) \times \{\pm 1\}$ of Remark 3.29.

When F is non-archimedean and all symmetric elements are unramified, the situation is slightly more subtle. While the cover $S(F)_\pm$ is still canonically split, its dual group $^L S_\pm$ need not be a split extension of $\Gamma_{E/F}$ by $\widehat{S}$. However, its inflation to Γ does split canonically, see Remark 3.22 below. $\square$

3.4 The local Langlands correspondence

We continue with a torus S defined over F a finite admissible Σ-set, and a Σ-equivariant map $R \to X^*(S)$. We can apply the usual definition of a Langlands

parameter to the L-group $^L S_\pm$:

Definition 3.13. A *Langlands parameter* for $S(F)_\pm$ (equivalently for $^L S_\pm$) is a continuous homomorphism $W_F \to {}^L S_\pm$ that commutes with the maps of both sides to Γ. Two such parameters are considered *equivalent* if they are conjugate under $\widehat{S}$. $\square$

Remark 3.14. Since $^L S$ is a split extension endowed with a splitting, a Langlands parameter for S is the same as a continuous 1-cocycle $W_F \to \widehat{S}$, and two parameters are equivalent if the corresponding 1-cocycles are cohomologous.

We can describe Langlands parameters for $^L S_\pm$ in a similar manner. If we fix a gauge $p : R \to \{\pm 1\}$ then a Langlands parameter for $^L S_\pm$ is the same as a 1-cochain $W_F \to \widehat{S}$ whose differential equals the Tits cocycle t_p. If we prefer not to fix a gauge, then we can describe an equivalence class of Langlands parameters for $^L S_\pm$ as a collection of cohomology classes of 1-cochains u_p, one for each gauge p, s.t. $u_p \cdot s_{q/p} = u_q$ as cohomology classes, and the differential of u_p equals t_p for each p.

Note that if (u_p) is such a collection and $z : W_F \to \widehat{S}$ is a 1-cocycle, then $(z \cdot u_p)$ is another such collection. Conversely if (u_p) and (v_p) are two such collections, then $u_p \cdot v_p$ is a 1-cocycle and independent of p. $\square$

Theorem 3.15.   1. There is a natural bijection between the set of equivalence classes of Langlands parameters for $S(F)_\pm$ and the set of continuous genuine characters of $S(F)_\pm$.

2. Given a genuine character of $S(F)_\pm$ and a character of $S(F)$, the parameter of the product of these characters is the product of their parameters.

3. Given two genuine characters of $S(F)_\pm$, the product of their parameters is the 1-cocycle $W_F \to \widehat{S}$ that is the parameter of their product.

$\square$

The word "natural" in the first part of this theorem has the naive meaning of naturally occurring and independent of choices, and does not (yet) signify that it is functorial. We do prove certain cases of functoriality in §5.

The proof of this theorem is implicit in the computations of [LS87, §2.5]. Before we explain that we offer some motivation for the resulting formulas. In §3.2 we recalled the construction of a character $\zeta_S : S(F) \to \mathbb{C}^\times$ associated to a set (ζ_α) of ζ-data.

Lemma 3.16. The element of $H^1(W_F, \widehat{S})$ corresponding to the character ζ_S can be represented by the 1-cocycle

$$z(w) = z_\zeta(w) = \prod_{\alpha \in R/\Sigma} \prod_{i=1}^{n} \zeta_\alpha(v_0(u_i(w)))^{w_i^{-1}\bar{\alpha}}.$$

Here for each $\alpha \in R$ we have fixed representatives $w_1, \ldots, w_n \in W$ for $W_{\pm\alpha} \backslash W$, $v_0 \in W_\alpha$ is arbitrary, and $v_1 \in W_{\pm\alpha} \smallsetminus W_\alpha$ is arbitrary when α is symmetric. For

$w \in W$ and $1 \le i \le n$ the element $u_i(w) \in W_{\pm\alpha}$ is defined by $w_i \cdot w = u_i(w) \cdot w_j$, where $1 \le j \le n$ is also uniquely determined. For $u \in W_{\pm\alpha}$ and $0 \le i \le 1$ the element $v_i(u) \in W_\alpha$ is defined by $v_i \cdot u = v_i(u) \cdot v_j$, where $0 \le j \le 1$ is also uniquely determined. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We have used here the notation W_α and $W_{\pm\alpha}$ for the preimages in $W = W_F$ of Γ_α and $\Gamma_{\pm\alpha}$.

Proof. The arguments for this are contained in the proof of [LS87, Lemma 3.5.A]. We repeat them here, in a slightly different form, for the sake of illustration.

Since ζ_S was the pull-back to $S(F)$ of the product of characters $\zeta_O : J_O(F) \to \mathbb{C}^\times$ over the set of Σ-orbits O in R, it will be enough to compute the corresponding 1-cocycle for each O. Choose $\alpha \in O$ and obtain identification $J_O = J_\alpha$. When O is asymmetric the character ζ_O is the composition of $\zeta_\alpha : F_\alpha^\times \to \mathbb{C}^\times$ with the identification $J_\alpha(F) = F_\alpha^\times$. Dually we have $\widehat{J_O} = X^*(J_O) \otimes_{\mathbb{Z}} \mathbb{C}^\times = \mathrm{Ind}_{\Gamma_\alpha}^\Gamma \mathbb{C}^\times$ and the parameter of ζ_O is the image of $\zeta_\alpha \in Z^1(W_{F_\alpha}, \mathbb{C}^\times)$ under the Shapiro map $Z^1(W_{F_\alpha}, \mathbb{C}^\times) \to Z^1(W_F, \mathrm{Ind}_{\Gamma_\alpha}^\Gamma \mathbb{C}^\times)$. Tracing through the computation we see that the value at $w \in W$ of this cocycle as an element of $\mathrm{Ind}_{W_\alpha}^W \mathbb{C}^\times$ whose value at w_i is $\zeta_\alpha(u_i(w))$. For the sake of uniformity with the symmetric case we note that this equals $\zeta_\alpha(v_0(u_i(w)))$, since $v_0(u_i(w)) = v_0 \cdot u_i(w) \cdot v_0^{-1}$.

When O is symmetric the character ζ_O is the composition of $\zeta_\alpha : F_\alpha^\times / F_{\pm\alpha}^\times \to \mathbb{C}^\times$, the isomorphism $F_\alpha^\times / F_{\pm\alpha}^\times \cong F_\alpha^1$, and the identification $J_\alpha(F) = F_\alpha^1$. When dualizing this procedure we interpret F_α^1 as the $F_{\pm\alpha}$-points of the torus $\mathrm{Res}_{F_\alpha / F_{\pm\alpha}}^1 \mathbb{G}_m$, and $F_\alpha^\times / F_{\pm\alpha}^\times$ as the $F_{\pm\alpha}$-points of the torus $(\mathrm{Res}_{F_\alpha / F_{\pm\alpha}} \mathbb{G}_m)/\mathbb{G}_m$. The dual torus of $(\mathrm{Res}_{F_\alpha / F_{\pm\alpha}} \mathbb{G}_m)/\mathbb{G}_m$ is the kernel of the norm homomorphism $\mathrm{Ind}_{\Gamma_\alpha}^{\Gamma_{\pm\alpha}} \mathbb{C}^\times \to \mathbb{C}^\times$. The dual torus of $\mathrm{Res}_{F_\alpha / F_{\pm\alpha}}^1 \mathbb{G}_m$ is $\mathbb{C}_{(-1)}^\times$. We have the isomorphism of algebraic tori $(\mathrm{id} - \tau_\alpha) : (\mathrm{Res}_{F_\alpha / F_{\pm\alpha}} \mathbb{G}_m)/\mathbb{G}_m \to \mathrm{Res}_{F_\alpha / F_{\pm\alpha}}^1 \mathbb{G}_m$ which on rational points recovers the isomorphism $F_\alpha^\times / F_{\pm\alpha}^\times \to F_\alpha^1$. Its dual is given by the anti-diagonal embedding of $\mathbb{C}_{(-1)}^\times$ into $\mathrm{Ind}_{\Gamma_\alpha}^{\Gamma_{\pm\alpha}} \mathbb{C}^\times$.

Under the Shapiro map $Z^1(W_\alpha, \mathbb{C}^\times) \to Z^1(W_{\pm\alpha}, \mathrm{Ind}_{W_\alpha}^{W_{\pm\alpha}} \mathbb{C}^\times)$ the character ζ_α maps to the cocycle whose value at $u \in W_{\pm\alpha}$ is the element of $\mathrm{Ind}_{W_\alpha}^{W_{\pm\alpha}} \mathbb{C}^\times$ whose value at v_i is $\zeta_\alpha(v_i(u))$. The assumption that $\zeta_\alpha|_{F_{\pm\alpha}^\times} = 1$ is equivalent to the claim that this element of $Z^1(W_{\pm\alpha}, \mathrm{Ind}_{W_\alpha}^{W_{\pm\alpha}} \mathbb{C}^\times)$ maps trivially to $Z^1(W_{\pm\alpha}, \mathbb{C}^\times)$ under the norm map. In other words, the value of this cocycle at each $u \in W_{\pm\alpha}$ lies in the kernel of the norm map $\mathrm{Ind}_{\Gamma_\alpha}^{\Gamma_{\pm\alpha}} \mathbb{C}^\times \to \mathbb{C}^\times$, hence in the image of the anti-diagonal embedding $\mathbb{C}_{(-1)}^\times \to \mathrm{Ind}_{\Gamma_\alpha}^{\Gamma_{\pm\alpha}} \mathbb{C}^\times$. This gives the element of $Z^1(W_{\pm\alpha}, \mathbb{C}_{(-1)}^\times)$ whose value at $u \in W_{\pm\alpha}$ is $\zeta_\alpha(v_0(u))$. Now $\widehat{J_O} = \mathrm{Ind}_{\Gamma_{\pm\alpha}}^\Gamma \mathbb{C}_{(-1)}^\times$. Under the Shapiro map $Z^1(W_{\pm\alpha}, \mathbb{C}_{(-1)}^\times) \to Z^1(W_F, \widehat{J_O})$ we obtain the 1-cocycle whose value at $w \in W$ is the element of $\mathrm{Ind}_{\Gamma_{\pm\alpha}}^\Gamma \mathbb{C}_{(-1)}^\times$ whose value at w_i is $\zeta_\alpha(v_0(u_i(w)))$.

It remains to see that, in both cases, the image of the parameter for ζ_O under the map $\widehat{J_O} \to \widehat{S}$ is given by the product over i in the statement of the lemma.

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This is immediate from the description of the map $S \to J_O$ given in §3.1, which implies that its dual, $\mathrm{Ind}_{\Gamma_\alpha}^{\Gamma} \mathbb{C}^\times \to \widehat{S}$ when α is asymmetric and $\mathrm{Ind}_{\Gamma_{\pm\alpha}}^{\Gamma} \mathbb{C}^\times_{(-1)} \to \widehat{S}$ when α is symmetric, is given by the formula $f \mapsto \prod_{\tau \in \Gamma_{\pm\alpha} \backslash \Gamma} f(\tau)^{\tau^{-1}\bar\alpha}$. ∎

This Lemma motivates the following definition. Let (χ_α) be a set of χ-data for R and let $\chi_S : S(F)_\pm \to \mathbb{C}^\times$ be the character of Definition 3.5.

Definition 3.17. The *Langlands parameter of* χ_S is the 1-cochain

$$r_p(w) = r_{p,\chi}(w) = \prod_{\alpha \in R/\Sigma} \prod_{i=1}^{n} \chi_\alpha(v_0(u_i(w)))^{w_i^{-1}\bar\alpha},$$

where the notation is as in Lemma 3.16 and $p : R \to \{\pm 1\}$ is the gauge specified by $p(w_i^{-1}\alpha) = +1$. For an arbitrary gauge q the 1-cochain r_q is defined by $s_{q/p} r_p$. □

Remark 3.18. This is the formula occurring before the statement of Lemma 2.5.A in [LS87] and we have kept the name r_p of loc. cit. to emphasize that. For a different expression, in terms of a section $s : W_{\pm\alpha} \backslash W \to W$, see (5.3). For a further motivation about this formula, see Appendix B, especially §B.4. □

Lemma 3.19. The collection (r_p) is indeed a valid Langlands parameter for $S(F)_\pm$. □

Proof. Following the discussion in Remark 3.14 we must show the following:

1. Up to equivalence the collection (r_p) does not depend on the choices of w_i and v_i made in Lemma 3.16;

2. The differential of r_p is t_p.

The second point is [LS87, Lemma 2.5.A] and the first point is discussed right after the proof of that lemma in loc. cit. ∎

Lemma 3.20. Let (ζ_α) be a set of ζ-data for R and let $\zeta_S : S(F) \to \mathbb{C}^\times$ be the associated character. The Langlands parameter for $(\chi\zeta)_S$ is the product of the Langlands parameters for χ_S and ζ_S. □

Proof. This is stated without proof in [LS87, (2.6.3)] and here we also leave the proof to the reader. ∎

Proof of Theorem 3.15. Choose a set of χ-data (χ_α). Let χ_S be the corresponding genuine character of Definition 3.5 and let (r_p) be its parameter as in Definition 3.17. Every other genuine character is of the form $\theta \cdot \chi_S$ for a character $\theta : S(F) \to \mathbb{C}^\times$. Letting $t : W_F \to \widehat{S}$ be the parameter for θ we assign to $\theta \cdot \chi_S$ the parameter $(t \cdot r_p)$. This gives the desired bijection.

If (χ'_α) is another set of χ-data with corresponding character χ'_S and parameter r'_p we define ζ-data (ζ_α) by $\zeta_\alpha = \chi'_\alpha \chi_\alpha^{-1}$. Let $z : W \to \widehat{S}$ be the parameter of ζ_S as in Lemma 3.16. Then Fact 3.6 and Lemma 3.20 show that $\chi'_S = \chi_S \cdot \zeta_S$ and $r'_p = r_p \cdot z$, proving that the bijection is independent of the choice of (χ_α).

The multiplicativity statements are immediate. ∎

We can reinterpet Theorem 3.15 by considering the Weil form of $^{L}S_{\pm}$. As we remarked earlier, [Lan79, Lemma 4] implies that this is a split extension of W by $\widehat{S}$ but without a distinguished splitting. That is, the Weil forms of $^{L}S_{\pm}$ and ^{L}S are non-canonically isomorphic.

Proposition 3.21. Let (χ_α) be a set of χ-data. The map

$$\widehat{S} \rtimes W \to \widehat{S} \boxtimes_{t_p} W, \qquad s \rtimes w \mapsto s \cdot r_{p,\chi}(w) \boxtimes w$$

is an isomorphism of extensions whose $\widehat{S}$-conjugacy class depends only on (χ_α). If $\theta : S(F) \to \mathbb{C}^\times$ is a character with parameter t, the composition of t with this isomorphism is the parameter for $\theta \cdot \chi_S$. $\qquad\square$

Proof. This is just a reformulation of Theorem 3.15. $\qquad\blacksquare$

Remark 3.22. When F is non-archimedean and all symmetric elements of R are unramified, the cover $S(F)_\pm$ splits canonically. This is not true for the L-group when viewed as an extension of $\Gamma_{E/F}$ by $\widehat{S}$. Indeed, in the most basic case where E/F is an unramified quadratic extension and S is the 1-dimensional anisotropic torus defined over F and split over E and $R \subset X^*(S)$ consists of the two generators of this free $\mathbb{Z}$-module of rank 1, then choosing a gauge on R is equivalent to choosing a generator, and then t_p is the unique non-trivial element of $H^2(\Gamma_{E/F}, \mathbb{C}^\times_{(-1)})$, and so $^{L}S_\pm$ is not a split extension.

However, there is a canonical choice of χ-data, namely χ_α is the unramified quadratic character for every symmetric α. The corresponding Langlands parameter $r_{p,\chi}$ of Definition 3.17 extends to Γ and provides a 1-cochain splitting t_p. Thus $^{L}S_\pm$, when viewed as an extension of Γ by $\widehat{S}$, splits canonically.

More generally, when F is non-archimedean, the absolute Galois form of the L-group $^{L}S_\pm$ always splits, but not canonically in general. The reason for the splitting is that there always exist χ-data of finite order, and then $r_{p,\chi}$ extends to Γ. $\qquad\square$

3.5 a-data as a genuine sign function on the double cover

We continue with a torus S defined over F, a finite admissible Σ-set R, and a Σ-equivariant map $R \to X^*(S)$. Recall from [LS87, §2.2] the notion of a-data: A set of a-data consists of an element $a_\alpha \in F_\alpha^\times$ for each $\alpha \in R$ s.t. $a_{\sigma\alpha} = \sigma(a_\alpha)$ and $a_{-\alpha} = -a_\alpha$ for all $\alpha \in R$ and $\sigma \in \Gamma$. Given such a set we obtain the function

$$a_S : S(F)_\pm \to \{\pm 1\} \qquad (\gamma, (\delta_\alpha)_\alpha) \mapsto \prod_{\substack{\alpha \in R_{\mathrm{sym}}/\Gamma \\ \bar{\alpha}(\gamma) \neq 1}} \kappa_\alpha \left(\frac{\delta_\alpha - \delta_{-\alpha}}{a_\alpha} \right) \cdot \prod_{\substack{\alpha \in R_{\mathrm{sym}}/\Gamma \\ \bar{\alpha}(\gamma) = 1}} \kappa_\alpha(\delta_\alpha).$$

(3.2)

Here elements of $S(F)_\pm$ are represented as in Remark 3.3. Both the numerator and the denominator of the argument of κ_α are non-zero elements of F_α of trace zero, so their quotient is a non-zero element of $F_{\pm\alpha}$. The values a_α for $\alpha \in R_{\mathrm{asym}}$ are irrelevant to this function.

Fact 3.23. The function a_S is genuine: For $\tilde{\gamma} \in S(F)_\pm$ and $\epsilon = -1 \in S(F)_\pm$ one has $a_S(\epsilon\tilde{\gamma}) = -a_S(\tilde{\gamma})$. $\qquad\square$

Remark 3.24. Let F be non-archimedean. Recall from [Kal19a, Definition 4.6.8] the concept of mod-a-data: It consists of a real number r_α and a non-zero element $\bar{a}_\alpha \in [F_\alpha]_{r_\alpha}/[F_\alpha]_{r_\alpha+}$ for each $\alpha \in R$ s.t. $r_{\sigma\alpha} = r_\alpha = r_{-\alpha}$, $\bar{a}_{\sigma\alpha} = \sigma(\bar{a}_\alpha)$, and $\bar{a}_{-\alpha} = -\bar{a}_\alpha$, for all $\alpha \in R$ and $\sigma \in \Gamma$.

If $p \neq 2$, the function a_S can be defined by the same formula with mod-a-data in place of a-data. Indeed, note first that mod-a-data can be lifted to a-data. For this it is enough to lift $\bar{a}_\alpha$ to an element $a_\alpha \in F_\alpha$ of trace zero. An arbitrary lift $\dot{a}_\alpha$ satisfies $z_\alpha := \dot{a}_\alpha + \tau_\alpha(\dot{a}_\alpha) \in [F_{\pm\alpha}]_{r_\alpha+}$ and $a_\alpha = \dot{a}_\alpha - z_\alpha/2$ works. Next, two lifts a_α and a'_α of $\bar{a}_\alpha$ to F_α are related by $a'_\alpha = a_\alpha \cdot u_\alpha$ for some $u_\alpha \in [F_{\pm\alpha}^\times]_{0+}$, but κ_α is trivial on $[F_{\pm\alpha}^\times]_{0+}$ when $p \neq 2$. $\qquad\square$

Remark 3.25. The construction of (3.2) can be turned around by saying that a regular element of $S(F)_{\pm\pm}$, i.e. an element whose image $\gamma \in S(F)$ satisfies $\bar{\alpha}(\gamma) \neq 1$ for all $\alpha \in R$, leads to a set of a-data for R, namely $a_\alpha = \delta_\alpha - \delta_{-\alpha}$. $\quad\square$

Remark 3.26. One can define a notion analogous to a-data, which we may call b-data, and show that it leads to elements of $H^1(F, S)$, cf. Appendix C. $\qquad\square$

3.6 Splittings

We now discuss various ways in which the covers $S(F)_{\pm\pm}$ and $S(F)_\pm$ can be split, canonically or not.

Consider an extension of abstract groups

$$1 \to A \xrightarrow{i} B \xrightarrow{p} C \to 1,$$

with A abelian. A *set-theoretic section* $s : C \to B$ is a map satisfying $p \circ s = \mathrm{id}_C$. We call it a *homomorphic section* if s is a group homomorphism.

A *set-theoretic retraction* is a map $r : B \to A$ satisfying $r(i(a)b) = ar(b)$. We call it a *homomorphic retraction* of it is a 1-cocycle for the conjugation action of B on A.

Specifying a (set-theoretic resp. homomorphic) section is equivalent to specifying a (set-theoretic resp. homomorphic) retraction, and vice versa, by the formula

$$b = i(r(b))s(p(b)). \tag{3.3}$$

When the groups in question carry a topology, we require i to be a closed immersion and p to be open. A section s is continuous if and only if the corresponding retraction r is.

Let S be an F-torus and R a finite admissible Σ-set equipped with a Σ-invariant map $\Sigma \to X^*(S)$.

Remark 3.27. Assume that O is symmetric. The double cover $J_O(F)_\pm$ is split if and only if the canonical character $\kappa_\alpha : F_{\pm\alpha}^\times/N_{F_\alpha^\times/F_{\pm\alpha}^\times}(F_\alpha^\times) \to \{\pm 1\}$ can be extended to a character $F_\alpha^\times/N_{F_\alpha^\times/F_{\pm\alpha}^\times}(F_\alpha^\times) \to \{\pm 1\}$, because that latter character

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is then a homomorphic retraction. When F is non-archimedean and $F_\alpha/F_{\pm\alpha}$ is unramified there is a canonical such extension, so the cover $J_O(F)_\pm$ splits canonically. When F is non-archimedean, $F_\alpha/F_{\pm\alpha}$ is ramified, and the size q_α of the residue field of F_α is congruent to 1 modulo 4, then such an extension does exist, but there is no canonical choice, so the cover $J_O(F)_\pm$ splits non-canonically. When F is non-archimedean, $F_\alpha/F_{\pm\alpha}$ is ramified, and q_α is congruent to 3 modulo 4, or when $F = \mathbb{R}$, then there is no such extension, so the cover $J_O(F)_\pm$ does not split. $\qquad\square$

Remark 3.28. The pull-back of the double cover $J_O(F)_\pm \to J_O(F)$ along the squaring map $J_O(F) \to J_O(F)$ is canonically split. Indeed, it is enough to treat the case of symmetric O, in which case we choose $\alpha \in O$ and present this pull-back as

$$\{(x,y)|x \in F_\alpha^1, y \in F_\alpha^\times/N_{F_\alpha/F_{\pm\alpha}}(F_\alpha^\times), x^2 = y/\tau(y)\}.$$

The splitting of the map $(x,y) \mapsto x$ is given by $x \mapsto (x,x)$. $\qquad\square$

Remark 3.29. There are some situations in which the cover $S(F)_\pm$ splits canonically, i.e. there is a canonical identification $S(F)_\pm = S(F) \times \{\pm 1\}$.

1. If all elements of R are asymmetric, then $S(F)_\pm$ splits canonically. This is because each $J_O(F)_\pm$ is by definition the canonical split extension.

2. More generally, if the image in $X^*(S)$ of every symmetric element of R is divisible by 2, then $S(F)_\pm$ splits canonically. This follows from Remark 3.28 and the fact that the morphism $S \to J_O$ factors through the squaring map $J_O \to J_O$ for each symmetric O.

3. If F is non-archimedean and all symmetric elements of R are unramified, then $S(F)_\pm$ splits canonically. This follows from Remark 3.27.

$\qquad\square$

Remark 3.30. Assume that $R_{\text{sym}} \neq \emptyset$. As we saw in §3.5, a set of a-data (a_α) leads to the set-theoretic retraction (3.2), hence to a set-theoretic splitting of $S(F)_\pm$.

This section is continuous on the subgroup of $S(F)$ of regular elements, i.e. those for which $\bar{\alpha}(\gamma) \neq 1$ for all $\alpha \in R$. But it is not in general continuous on all of $S(F)$. Indeed, let us consider the case of the 1-dimensional anisotropic torus $S = \mathbb{R}_{E/F}^1 \mathbb{G}_m$ associated to a separable quadratic extension E/F, and let $R = \{1, -1\} \in X^*(S)$. Thus we are considering the retraction $E^\times/N_{E/F}(E^\times) \to F^\times/N_{E/F}(E^\times) = \{\pm 1\}$ given by $x \mapsto \kappa((x - \bar{x})/a)$ when $x \notin F^\times$. We can decompose $x = t + as$ with $t, s \in F$. Then $\kappa((x - \bar{x})/a) = \kappa(2s)$. Since $0 \in F$ is a limit point for both $N_{E/F}(E^\times)$ and its complement, we see that this retraction does not have a continuous extension to $F^\times \subset E^\times$. $\qquad\square$

Remark 3.31. Consider a set of χ-data. We saw in §3.2 that it leads to a genuine character $\chi : S(F)_\pm \to \mathbb{C}^\times$. This character is a homomorphic retraction if and

only if it is valued in $\{\pm 1\}$, which in general it is not. However, we can form a set-theoretic retraction from it by taking

$$x \mapsto \chi(x)(\sqrt{\chi(x)^2})^{-1},$$

where $\sqrt{\ }$ is a chosen branch of square root on $\mathbb{C}^\times$, provided it can be chosen so that its domain contains all values of $\chi(x)^2$. If χ is valued in $\{\pm 1\}$, this recovers the homomorphic retraction given by χ. $\qquad\square$

3.7 The Lie algebra and the exponential map

In this subsection we assume that F has characteristic zero. Let S be an F-torus. Since $S(F)_{\pm\pm}$ and $S(F)_{\pm}$ are finite covers of (the real or p-adic Lie group) $S(F)$, their Lie algebras are canonically identified with that of $S(F)$, and the exponential map, defined in a neighborhood of 0 in $\mathrm{Lie}(S)(F)$ and mapping to $S(F)$, factors through these covers. In this subsection we will explicate this factoring.

We begin by recalling the exponential map. Assume first that S is split. Its Lie algebra of $S(F)$ is naturally identified with $X_*(S) \otimes_{\mathbb{Z}} F$. Let $U \subset F$ be the open subgroup on which the exponential power series

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges. Recall that $U = F$ when F is archimedean, and $U = \{x \in F | \omega(x) > 1/(p-1)\}$ when F is a finite extension of $\mathbb{Q}_p$ and ω the valuation normalized so that $\omega(p) = 1$. Let $\mathrm{Lie}(S)(U) = X_*(S) \otimes_{\mathbb{Z}} U$, so that $\mathrm{Lie}(S)(U)$ is a lattice in the F-vector space $\mathrm{Lie}(S)(F)$ when F is non-archimedean, and equals that F-vector space when F is archimedean. Then $\mathrm{id} \otimes \exp$ converges on $\mathrm{Lie}(S)(U)$ and provides a continuous (even analytic) group homomorphism

$$\exp : \mathrm{Lie}(S)(U) \to S(F).$$

Note that for every $\alpha \in X_*(S)$ we have $\alpha(\exp(X)) = \exp(d\alpha(X))$ by construction.

Consider now a general F-torus S. Its Lie algebra is naturally identified with $\mathrm{Lie}(S)(F^s)^\Gamma$, which we can also express as $\mathrm{Lie}(S)(E)^{\Gamma_{E/F}}$ for any finite Galois extension E/F splitting S. The exponential map $\exp : \mathrm{Lie}(S)(U_E) \to S(E)$ is $\Gamma_{E/F}$-equivariant, and restricts to an exponential map

$$\exp : \mathrm{Lie}(S)(U) \to S(F),$$

where $U_E \subset E$ is the open subgroup on which the exponential map for E converges, and $\mathrm{Lie}(S)(U) := \mathrm{Lie}(S)(U_E)^{\Gamma_{E/F}}$.

This exponential map factors through the cover $S(F)_{\pm\pm} \to S(F)$ as

$$\exp : \mathrm{Lie}(S)(U) \to S(F)_{\pm\pm}, \qquad X \mapsto (\exp(X), (\exp(\bar\alpha(X)/2)_\alpha),$$

where we are using the presentation of elements of $S(F)_{\pm\pm}$ of Remark 3.3.

4.1   The canonical L-embedding

Let now G be a connected reductive group defined over F with complex dual $\widehat{G}$. Let $S \subset G$ be a maximal torus defined over F. It is well known that there exists a canonical $\widehat{G}$-conjugacy class of embeddings $\widehat{S} \to \widehat{G}$, see e.g. [Kal19a, §5.1]. We will show that this conjugacy class extends to a canonical $\widehat{G}$-conjugacy class of L-embeddings ${}^L S_\pm \to {}^L G$, where $S(F)_\pm$ is constructed with respect to the root system $R \subset X^*(S)$ of G.

Choose a Γ-invariant pinning $(\widehat{T}, \widehat{B}, \{X_\alpha\})$ of $\widehat{G}$ and let $\Omega = N(\widehat{T}, \widehat{G})/\widehat{T}$. Recall the Tits section

$$n : \Omega \to N(\widehat{T}, \widehat{G}).$$

It sends a simple reflection $s_\alpha \in \Omega$ to the image of the standard Weyl element of SL_2 under the unique morphism $\mathrm{SL}_2 \to \widehat{G}$ mapping the standard nilpotent element to the element X_α of the pinning. Explicitly, s_α is sent to $\exp(X_\alpha)\exp(-X_{-\alpha})\exp(X_\alpha)$, where $X_{-\alpha}$ is the unique vector in the root space for the root $-\alpha$ that satisfies $[X_\alpha, X_{-\alpha}] = H_\alpha = d\alpha^\vee(1)$. It sends a general $\omega \in \Omega$ to $n(s_1)\dots n(s_k)$, where $\omega = s_1 \dots s_k$ is any reduced expression. It is easy to see that n is Γ-equivariant, hence extends to a section

$$n : \Omega \rtimes \Gamma \to N(\widehat{T}, \widehat{G}) \rtimes \Gamma, \qquad \omega \rtimes \sigma \mapsto n(\omega) \rtimes \sigma.$$

Choose a member of the conjugacy class of embeddings $\widehat{S} \to \widehat{G}$ with image $\widehat{T}$; it is an isomorphism $\widehat{j} : \widehat{S} \to \widehat{T}$. This isomorphism is usually not Γ-equivariant, but since its $\widehat{G}$-conjugacy class is, the difference between $\widehat{j}$ and $\sigma \circ \widehat{j} \circ \sigma^{-1}$ is realized by an element $\sigma_{S,G} \in \Omega$ for any $\sigma \in \Gamma$. Thus the transport of the Γ-action on $\widehat{S}$ via $\widehat{j}$ is given by the group homomorphism $\Gamma \to \Omega \rtimes \Gamma$ sending σ to $\sigma_{S,G} \rtimes \sigma$. Pulling back along this homomorphism the extension with splitting

$$1 \longrightarrow \widehat{T} \longrightarrow N(\widehat{T}, \widehat{G}) \rtimes \Gamma \overset{n}{\underset{\longrightarrow}{}} \Omega \rtimes \Gamma \longrightarrow 1$$

and using $\widehat{j}$ to identify $\widehat{S}$ with $\widehat{T}$ we obtain an extension of Γ by $\widehat{S}$ with a splitting. According to [LS87, Lemma 2.1.A] the 2-cocycle associated to this splitting is the Tits cocycle t_p of Definition 3.8, where p is the gauge that assigns $+1$ to the $\widehat{B}$-positive roots, and $R \subset X^*(S)$ is the root system of G. In other words, we have just extended $\widehat{j} : \widehat{S} \to \widehat{T}$ to an L-embedding ${}^L S_\pm \to {}^L G$ taking image in $N(\widehat{T}, \widehat{G}) \rtimes \Gamma$. We record explicitly that it is given by

$$\widehat{S} \boxtimes_{t_p} \Gamma \to \widehat{G} \rtimes \Gamma, s \boxtimes \sigma \mapsto \widehat{j}(s) \cdot n(\sigma_{S,G}) \rtimes \sigma. \tag{4.1}$$

The only choice involved in the construction of this L-embedding ${}^L S_\pm \to {}^L G$ was the Γ-invariant pinning. Since any two such are conjugate under $\widehat{G}$, in

fact even under $\widehat{G}^{\Gamma}$ [Kot84, Corollary 1.7], it is immediate to check that the two resulting L-embeddings are also conjugate.

4.2 Factorization of Langlands parameters

Let S be an algebraic torus defined over F and equipped with a stable class of embeddings $S \to G$ defined over F whose images are maximal tori. This endows S with various structures [Kal19a, §5.1], in particular a finite Σ-invariant subset $0 \notin R \subset X^*(S)$. Therefore we have the double cover $S(F)_{\pm}$.

Consider two data consisting of an algebraic torus S_i, a stable class of embeddings $S_i \to G$ as above, and a genuine character θ_i of $S_i(F)_{\pm}$. We say that they are equivalent if there exists an isomorphism $S_1 \to S_2$ that identifies the two stable classes of embeddings and whose natural lift $S_1(F)_{\pm} \to S_2(F)_{\pm}$ identifies the two genuine characters.

Proposition 4.1. There is a canonical bijection between the set of equivalence classes of Langlands parameters $W \to {}^L G$ whose image normalizes a maximal torus, and the set of equivalence classes of data consising of an algebraic torus S defined over F, a stable class of embeddings $S \to G$ defined over F whose images are maximal tori, and a genuine character θ of $S(F)_{\pm}$. Given such a datum, the bijection sends it to the composition of the Langlands parameter of θ with the canonical embedding ${}^L S_{\pm} \to {}^L G$ of §4.1. □

Remark 4.2. It is well known that discrete series parameters when $F = \mathbb{R}$ have the above property, and it is proved in [Kal19a, §5.2] that this is also the case for regular supercuspidal parameters. □

Proof. Such constructions have been explained in detail in various places. In the real case we refer to [She83, §3.4] for the statement without covers and to [AV16, §6] for the statment with covers. In the p-adic case we refer to [Kal19a, §5.2] for the statement without covers. We limit ourselves to a brief sketch here, in order to illustrate the simplicity of the situation once the double cover has been introduced and χ-data is no longer required.

Let $\varphi : W \to {}^L G$ be a Langlands parameter whose image is contained in the normalizer of a maximal torus. Let $\widehat{S}$ denote this torus together with the action of W by conjugation by φ. It is immediate that this action extends to Γ. Let S be the torus over F with dual $\widehat{S}$. Then S comes equipped with a stable class of embeddings $S \to G$ [Kal19a, §5.1]. We have the canonical $\widehat{G}$-conjugacy class of L-embeddings ${}^L S_{\pm} \to {}^L G$ of §4.1. Since $\widehat{S}$ is already (tautologically) embedded into $\widehat{G}$, inside of this $\widehat{G}$-conjugacy class there is a unique $\widehat{S}$-conjugacy class of L-embeddings ${}^L S_{\pm} \to {}^L G$ that extends the tautological embedding $\widehat{S} \to \widehat{G}$. The parameter φ factors through this embedding and leads in this way to a canonical parameter $\varphi_{\pm}$ for $S(F)_{\pm}$, hence to a canonical genuine character $\theta_{\pm}$ of $S(F)_{\pm}$ via Theorem 3.15. ■

4.3 The character formula associated to a genuine character

In this section we assume that in the case of a non-archimedean base field F the residual characteristic is not 2.

Let $S \subset G$ be an elliptic maximal torus and $\theta : S(F)_\pm \to \mathbb{C}^\times$ a genuine character. Fix a non-trivial character $\Lambda : F \to \mathbb{C}^\times$. Then θ and Λ determine a set of a-data in the archimedean case, and a set of mod-a-data in the non-archimedean case, that may happen to be degenerate (i.e. $a_\alpha = 0$), as follows. The character θ^2 descends to $S(F)$. In the archimedean case we define a_α by the equation

$$\theta^2(N_{F_\alpha/F}(\alpha^\vee(\exp(z)))) = \Lambda(\operatorname{tr}_{F_\alpha/F}(2a_\alpha z)),$$

where z is a variable in F_α. In the non-archimedean case we use the equation

$$\theta^2(N_{F_\alpha/F}(\alpha^\vee(z+1))) = \Lambda(\operatorname{tr}_{F_\alpha/F}(2\bar{a}_\alpha z)),$$

where z is a variable in $[F_\alpha]_{r_{\theta,\alpha}}/[F_\alpha]_{r_{\theta,\alpha}+}$ and $r_{\theta,\alpha}$ is the largest real number for which the left hand side is non-zero for such z. Let $r_{\Lambda,\alpha}$ be the depth of $\Lambda \circ \operatorname{tr}_{F_\alpha/F}$ and let $r_\alpha = r_{\Lambda,\alpha} - r_{\theta,\alpha}$.

It is quite obvious that $a_{-\alpha} = -a_\alpha$ and $a_{\sigma\alpha} = \sigma(a_\alpha)$ for $\alpha \in R$ and $\sigma \in \Gamma$. But it may well happen that $a_\alpha = 0$. We'll be particularly interested in characters θ for which this does not happen. For $F = \mathbb{R}$ one checks easily that $a_\alpha = \langle \alpha^\vee, d\theta \rangle/(2\pi i)$, where $d\theta \in [X^*(S) \otimes \mathbb{C}]^\Gamma$ is the differential of θ. Thus $a_\alpha \neq 0$ for all α is equivalent to $d\theta$ being regular. This is the case for θ that come, via Proposition 4.1, from a discrete series parameter. In that case, let (χ_α) be the based χ-data specified by the Weyl chamber determined by $d\theta$. Then $\theta \cdot \chi_S^{-1}$ is the character denoted by θ in [Kal19a, §4.11] and the a-data computed here coincides with the one computed there.

For a non-archimedean F we will only apply this construction when the depth of $\theta^2 \circ N_{F_\alpha/F} \circ \alpha^\vee$ is positive, which guarantees $a_\alpha \neq 0$. For the remaining α we shall take a_α to be units, chosen arbitrarily. When θ comes, via Proposition 4.1, from a regular supercuspidal parameter, let (χ_α) be arbitrary minimally ramified χ-data [Kal19a, Definition 4.6.1]. Then $\theta \cdot \chi_S^{-1}$ satisfies [Kal19a, Definition 3.7.5] and the a-data computed here coincides with the one computed from $\theta \cdot \chi_S^{-1}$ by [Kal19a, (4.10.1)]. Indeed, it is easy to see that the double cover $S(F)_\pm$ splits canonically over $S(F)_{0+}$, and χ_S^{-1} restricts trivially to $S(F)_{0+}$. This relies on our assumption $p \neq 2$.

This set of a-data, respectively mod-a-data, leads to the genuine function a_S of (3.2). The product $a_S \cdot \theta$ is a function on $S(F)_\pm$ that descends to $S(F)$. Let T be the minimal Levi subgroup in the quasi-split inner form of G. Consider the function $S(F) \to \mathbb{C}$ sending $s \in S(F)$ to

$$e(G)\epsilon(X^*(T)_\mathbb{C} - X^*(S)_\mathbb{C}, \Lambda) \sum_{w \in N(S,G)(F)/S(F)} [a_S \cdot \theta](ws). \qquad (4.2) \quad \text{{eq:char}}$$

This function does not depend on the choice of Λ. When $F = \mathbb{R}$ there is a unique irreducible discrete series representation of $G(F)$ whose Harish-Chandra character restricted to $S(F)_{\mathrm{reg}}$ is given by this function [Kal19a, §4.11].

When F is non-archimedean, G splits over a tame extension of F, and p is not a bad prime for G, there exists a natural regular supercuspidal representation whose Harish-Chandra character restricted to the set of regular topologically semi-simple elements of $S(F)$ is given by this function. This representation is obtained by letting (χ_α) be the χ-data obtained from (a_α) by [Kal19a, (4.7.2)] and applying [Kal19a, Proposition 3.7.14] to the character $\theta \cdot \chi_S^{-1} \cdot \epsilon_{f,\text{ram}} \cdot \epsilon^{\text{ram}}$ and the torus S. Here $\epsilon_{f,\text{ram}}$ and ϵ^{ram} are defined in [Kal19a, Definition 4.7.3 and Equation (4.3.3)]. We have had to replace "unique" by "natural" as there may not be sufficiently many such elements in order to determine the representation uniquely.

4.4 The stable character of a supercuspidal Langlands parameter

Let F be non-archimedean. Assume that G splits over a tame extension of F. Assume further that p is sufficiently large so that it does not divide the order of the Weyl group and the exponential function $\text{Lie}(G(F)) \to G(F)$ converges on the set of topologically nilpotent elements. Fix a non-trivial additive character $\Lambda : F \to \mathbb{C}^\times$.

We will now combine the discussions of §§4.1,4.2,4.3 to obtain a spectral characterization of the local Langlands correspondence for supercuspidal parameters, i.e. discrete Langlands parameters $\varphi : W_F \to {}^L G$. For this, we will present a formula for the stable character $S\Theta_\varphi$ of the L-packet associated to φ. The theory of endoscopy produces from this stable character, and the stable characters on all endoscopic groups, the characters of the individual members of the L-packet, and in this way determines the L-packet completely.

It is shown in [Kal19b, Lemma 4.1.3] and [Kal19a, Lemma 5.2.2] that for a supercuspidal parameter φ, $\widehat{S} = \text{Cent}(\text{Cent}(\varphi(I_F), \widehat{G})^\circ, \text{Cent}(\varphi(P_F), \widehat{G})^\circ)$ is a maximal torus of $\widehat{G}$. It is clearly normalized by φ. Proposition 4.1 produces the pair (S, θ) of a torus defined over F and equipped with a stable class of embeddings $S \to G$ whose images are elliptic maximal tori, and a genuine character θ of $S(F)_\pm$.

The prime p is odd, which implies that the double cover $S(F)_\pm$ splits canonically over the pro-p-Sylow subgroup $S(F)_{0+} \subset S(F)$. The restriction of θ to $S(F)_{0+}$ is given by $\Lambda\langle X^*, \log(-)\rangle$ for an element $X^* \in \text{Lie}^*(S)(F)$. We also obtain a-data as in §4.3.

Conjecture 4.3. For any strongly regular semi-simple $\gamma \in G(F)$ with topological Jordan decomposition $\gamma = \gamma_0 \cdot \gamma_{>0}$,

$$S\Theta_\varphi^G(\gamma) = e(J)\epsilon(T_G - T_J) \sum_{j:S \to J} [a_S \cdot \theta](\gamma_0) \cdot \widehat{SO}_{jX^*}^J(\log(\gamma_{>0})).$$

Here J is the connected centralizer of γ_0 in G, $e(J)$ is the Kottwitz sign of J, $\epsilon(T_G - T_J)$ is the root number of the virtual Galois representation $X^*(T_G)_\mathbb{C} - X^*(T_J)_\mathbb{C}$ for T_G and T_J the minimal Levi subgroups in the quasi-split inner forms of G and J, and the sum runs over the set of stable classes of admissible embeddings $S \to J$. $\square$

This conjecture, together with the endoscopic character identities of [Kal16, §5.4], characterize completely the local Langlands correspondence for supercuspidal parameters. Indeed, given a supercuspidal parameter φ, this conjecture characterizes the stable characters of the associated L-packet for G, as well as all endoscopic groups of G through which φ factors. The endoscopic character identities then give a formula for the Harish-Chandra character of each constituent of the L-packet for G.

This conjecture is proved in [FKS21, §4.4] for the construction of [Kal19b]. We have stated it here as a conjecture, because the notion of "the" local Langlands correspondence is somewhat ambiguous.

5   FUNCTORIALITY OF THE LOCAL CORRESPONDENCE

Consider two tori S and T defined over F, two finite admissible Σ-sets R_S and R_T, and two Σ-equivariant maps $R_S \to X^*(S)$ and $R_T \to X^*(T)$. Let $f : S \to T$ be a morphism defined over F and let $f^* : X^*(T) \to X^*(S)$ be the map induced from f. We would like to study what data induces a lift of f to a homomorphism $S(F)_\pm \to T(F)_\pm$ of covers, as well as an extension of $\widehat{f} : \widehat{T} \to \widehat{S}$ to an L-homomorphism ${}^L T_\pm \to {}^L S_\pm$, so that the local correspondences for the two double covers match. The most naive guess would be a map $R_T \to R_S$ commuting with f^*, but it turns out that this is not correct, and the right condition is slightly different. More precisely, we will see in Example 5.6 that, when the map $R_T \to R_S$ is not surjective, functoriality cannot hold in general. It will turn out that requiring $R_T \to R_S$ to be surjective, as well as satisfy a compatibility condition with respect to f^*, which is *not* the commutativity of the obvious square, is enough to produce the desired map between covers and their L-groups. This will be discussed in §5.4.

5.1   Initial observations

Consider given a finite abelian group A and two central extensions $1 \to A \to S(F)_A \to S(F) \to 1$ and $1 \to A \to T(F)_A \to T(F) \to 1$ where S and T are F-tori. The main examples we have in mind are $A = \{\pm 1\}$ or $A = \{\pm 1\}^{R/\Sigma}$ for some finite admissible Σ-set R, and the extensions are those of Definition 3.1.

Definition 5.1. A *lift* of $f : S \to T$ is a continuous homomorphism $S(F)_A \to T(F)_A$ whose restriction to A is the identity, and whose composition with the natural projection $T(F)_A \to T(F)$ equals the composition of the natural projection $S(F)_A \to S(F)$ with f. □

Fact 5.2.   1. For any lift $f_A : S(F)_A \to T(F)_A$ of f, the commutative diagram

$$
\begin{array}{ccc}
S(F)_A & \xrightarrow{\;f_A\;} & T(F)_A \\
\downarrow & & \downarrow \\
S(F) & \xrightarrow{\;f\;} & T(F)
\end{array}
$$

is Cartesian.

2. The set of lifts of f is a torsor for the abelian group of continuous homomorphisms $S(F) \to A$. This torsor is non-empty precisely when the restriction map $H^2(T(F), A) \to H^2(S(F), A)$ along f maps the class of the extension $T(F)_A$ to that of the extension $S(F)_A$.

$\square$

We now return to the setting of the preamble of §5. The simplest case of functoriality is that of isomorphisms. Despite its simplicity, it has some useful applications, so we record it here.

Example 5.3. Assume that $f : S \to T$ is an isomorphism and assume given a Σ-equivariant bijection $f_R : R_T \to R_S$ such that the square

$$
\begin{array}{ccc}
R_T & \xrightarrow{\ f_R\ } & R_S \\
\downarrow & & \downarrow \\
X^*(T) & \xrightarrow{\ f^*\ } & X^*(S)
\end{array}
$$

commutes. This data specifies a lift $f_{\pm\pm} : S(F)_{\pm\pm} \to T(F)_{\pm\pm}$. In terms of the presentations of elements of $S(F)_{\pm\pm}$ discussed in Remark 3.3, $(\gamma, (\delta_\alpha)_\alpha)$ maps to $(f(\gamma), (\delta_{f_R(\alpha)})_\alpha)$ under $f_{\pm\pm}$. Moreover, $f_{\pm\pm}$ factors through a lift $f_\pm : S(F)_\pm \to T(F)_\pm$.

In particular, the inversion automorphism of $S(F)$ lifts canonically to both $S(F)_\pm$ and $S(F)_{\pm\pm}$, and this lift sends $(\gamma, (\delta_\alpha)_\alpha)$ to $(\gamma^{-1}, (\delta_{-\alpha})_\alpha)$. Note that this lift is simply the inversion on the abstract group $S(F)_\pm$, since $\delta_\alpha \delta_{-\alpha} = \delta_\alpha \bar{\delta}_\alpha$.

$\square$

Example 5.4. Consider a connected reductive group G, a maximal torus $S \subset G$, and $R = R(S, G)$ the absolute root system. Applying the above example to each element w of the Weyl group $\Omega(S, G)(F)$ we obtain a lift of the action of $\Omega(S, G)(F)$ on $S(F)$ to $S(F)_{\pm\pm}$ and this lift factors through $S(F)_\pm$. Explictly, an element $w \in \Omega(S, G)(F)$ sends $(\gamma, (\delta_\alpha)_\alpha)$ to $(w\gamma, (\delta_{w^{-1}\alpha})_\alpha)$.

$\square$

We now introduce a basic construction that reduces the question of functoriality, when f is no longer assumed to be an isomorphism, to the case of a single torus.

Construction 5.5. Assume that $R_T = R_S$ and denote this set by R. Assume further that the composition of the given Σ-equivariant map $R \to X^*(T)$ with f^* equals the given Σ-equivariant map $R \to X^*(S)$.

For every Σ-orbit $O \subset R$ we have the torus J_O and the double cover $J_O(F)_\pm$ of $J_O(F)$. The diagram $S(F) \to \prod J_O(F) \leftarrow \prod J_O(F)_\pm$ maps the the diagram $T(F) \to \prod J_O(F) \leftarrow \prod J_O(F)_\pm$, and hence the pull-back of the first diagram maps to the pull-back of the second. In this way we obtain a lift $S(F)_{\pm\pm} \to T(F)_{\pm\pm}$ of $f : S(F) \to T(F)$ and this lift factors through a lift $S(F)_\pm \to T(F)_\pm$

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Dually, the map $\widehat{f} : \widehat{T} \to \widehat{S}$ identifies the two Tits cocycles and hence extends canonically to an L-homomorphism $^L T_{\pm} \to {}^L S_{\pm}$. The fact that the local correspondences for $S(F)_{\pm}$ and $T(F)_{\pm}$ are compatible with these maps is immediate and left to the reader. $\qquad\square$

Note that the flexibility of considering a map $R \to X^*(S)$ rather than a subset $R \subset X^*(S)$ is useful here: even if R is a subset of $X^*(T)$, the restriction of f^* to R may not be injective.

This construction reduces the question of functoriality to the case when one considers a single torus S and two finite admissible Σ-sets R and R' with Σ-equivariant maps $R \to X^*(S)$ and $R' \to X^*(S)$. We will now give two examples showing that having a map $R \to R'$ that makes the obvious triangle commute does not imply that the identity of $S(F)$ lifts to a homomorphism $S(F)_R \to S(F)_{R'}$.

Example 5.6. Let $S = \mathrm{Res}^1_{E/F}\mathbb{G}_m$ for a separable quadratic extension E/F. Then $X^*(S) = \mathbb{Z}_{(-1)}$. Let $R = \varnothing \subset X^*(S)$ and $R' = \{1, -1\} \subset X^*(S)$. The tautological injective map $R \to R'$ commutes with $R \to X^*(S)$ and $R' \to X^*(S)$. But $S(F)_R$ is the split extension $S(F) \times \{\pm 1\}$, while $S(F)_{R'}$ may be non-split, cf. Remark 3.27. Therefore, there is no lift $S(F)_R \to S(F)_{R'}$ of the identity of $S(F)$.

The reader may object that $R = \varnothing$ is not a valid choice of a finite admissible Σ-set. One can easily take a product of the above example with, say $\mathbb{G}_m$, and add to both R and R' an element of $\mathbb{Z} = X^*(\mathbb{G}_m)$ and its negative. $\qquad\square$

Example 5.7. Let $S = \mathrm{Res}^1_{E/F}\mathbb{G}_m$, $R' = \{1, -1\} \subset \mathbb{Z}_{(-1)} = X^*(S)$, $R = \{1_a, 1_b, -1_a, -1_b\}$ and $R \to R'$ is the surjective map that sends $1_a, 1_b$ to 1 and $-1_a, -1_b$ to -1. The action of Σ on R descends to $\Gamma_{E/F} \times \{\pm 1\}$ and both the non-trivial element of $\Gamma_{E/F}$ and $-1 \in \{\pm 1\}$ send $1_a \mapsto -1_a$ and $1_b \mapsto -1_b$. As in the previous example, we know that $S(F)_{R'}$ may be non-split. On the other hand, we claim that $S(F)_R$ is canonically split. Indeed, this extension of $S(F) = E^1$ is obtained by taking the product of two copies of the extension

$$1 \to F^{\times}/N_{E/F}(E^{\times}) \to E^{\times}/N_{E/F}(E^{\times}) \to E^1 \to 1$$

pushing out via the product map $(F^{\times}/N_{E/F}(E^{\times}))^2 = (\{\pm 1\})^2 \to \{\pm 1\}$, and then pulling back along the diagonal inclusion $E^1 \to (E^1)^2$. Therefore, there is no lift $S(F)_R \to S(F)_{R'}$ of the identity of $S(F)$.

The reader may now object that this pathological behavior comes from us allowing maps $R \to X^*(S)$ that are not injective. This is not the case. Indeed, we can consider a second torus $T = S \times S$ and let $f : S \to T$ be the the diagonal embedding. Thus $f^* : X^*(T) = \mathbb{Z}_{(-1)} \oplus \mathbb{Z}_{(-1)} \to \mathbb{Z}_{(-1)} = X^*(S)$ is the addition map. If we let $R_T = \{(1, 0), (0, 1), (-1, 0), (0, -1)\} \subset X^*(T)$, then we now have a version of this example where R_T is a subset of $X^*(T)$ and $R_S = R'$ is a subset of $X^*(S)$. This version reduces to the previous version by Construction 5.5. $\qquad\square$

We now consider a second special case of functoriality. It will be used in §5.2 to compare our double cover construction with that of Adams-Vogan. It will be then substantially generalized in §5.4.

Construction 5.8. Consider a single torus S and a finite Σ-invariant subset $0 \notin R \subset X^*(S)$. Let $\alpha, \beta \in R_{\text{sym}}$ be such that $\Sigma \cdot \alpha \neq \Sigma \cdot \beta$, $\Gamma_\alpha = \Gamma_\beta$, and $\Gamma_{\pm\alpha} = \Gamma_{\pm\beta}$. Set $\gamma = \alpha + \beta$. Let $R' = \Sigma \cdot \gamma \cup R \smallsetminus (\Sigma \cdot \alpha \cup \Sigma \cdot \beta)$. We will construct canonical isomorphisms $S(F)_R \to S(F)_{R'}$ and $^L S_{R'} \to {}^L S_R$.

We have $J_\alpha = J_\beta = J_\gamma$ canonically. The multiplication map $J_\alpha \times J_\beta \to J_\gamma$ fits into the commutative diagram

$$
\begin{array}{ccc}
[J_\alpha(F)_\pm \times J_\beta(F)_\pm]/\{\pm 1\} & \longrightarrow & J_\gamma(F)_\pm \\
\downarrow & & \downarrow \\
J_\alpha(F) \times J_\beta(F) & \longrightarrow & J_\gamma(F)
\end{array}
$$

and this diagram is cartesian. The transitivity of pull-back now gives the isomorphism $S(F)_R \to S(F)_{R'}$.

On the dual side, let p be a gauge on R s.t. $p(\sigma\alpha) = p(\sigma\beta)$ for all $\sigma \in \Sigma$. Define a gauge p' on R' by setting $p'(\sigma\gamma) = p(\sigma\alpha) = p(\sigma\beta)$ for all $\sigma \in \Sigma$ and $p'(\delta) = p(\delta)$ for $\delta \in R' \smallsetminus \Sigma \cdot \gamma$. Then the Tits cocycle t_p with respect to R is equal to the Tits cocycle $t_{p'}$ with respect to R', so the L-groups $^L S_R$ and $^L S_{R'}$ are canonically identified.

Finally we observe that the identifications $S(F)_R = S(F)_{R'}$ and $^L S_R = {}^L S_{R'}$ are compatible with the Langlands correspondence. For this, take a set of χ-data for R'. Define $\chi_\alpha = \chi_\beta = \chi_\gamma$ and obtain in this way a set of χ-data for R. The characters of $S(F)_R$ and $S(F)_{R'}$ obtained from these sets of χ-data are readily seen to be equal under the identification $S(F)_R = S(F)_{R'}$. The same holds for the 1-cochains r_p. $\qquad \square$

5.2 Comparison with the ρ_i-cover of Adams-Vogan

Let $F = \mathbb{R}$, S an algebraic torus over F, and $\gamma \in \frac{1}{2} X^*(S)$. Adams and Vogan [AV16, §3] define the diagonalizable group $S_\gamma = \{(s, z) \in S \times \mathbb{G}_m | [2\gamma](s) = z^2\}$. Thus $X^*(S_\gamma)$ is the push-out of

$$
\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xrightarrow{2\gamma} X^*(S).
$$

The natural map $X^*(S) \to X^*(S_\gamma)$ is injective and the quotient is $\mathbb{Z}/2\mathbb{Z}$, so we have the exact sequence

$$
1 \to \{\pm 1\} \to S_\gamma \to S \to 1.
$$

They define $S(\mathbb{R})_\gamma$ to be the preimage in $S_\gamma(\mathbb{C})$ of $S(\mathbb{R})$. Thus $S(\mathbb{R})_\gamma$ is a double cover of $S(\mathbb{R})$. Let $\gamma \in X^*(S_\gamma)$ by the morphism $S_\gamma \to \mathbb{G}_m$ given by $\gamma(s, z) = z$. Then γ is a geniune character of S_γ whose square is $2\gamma \in X^*(S)$.

The extension S_γ splits if and only if $X^*(S_\gamma)$ has torsion, which is the case if and only if $\gamma \in X^*(S)$. If on the other hand $\gamma \in \frac{1}{2}X^*(S) \setminus X^*(S)$, then S_γ is a torus, its character module is the submodule of $\frac{1}{2}X^*(S)$ generated by $X^*(S)$ and γ, and the map $S_\gamma \to S$ is a degree 2 isogeny.

Adams and Vogan further define an L-group for $S(\mathbb{R})_\gamma$. It is the group generated by $\widehat{S}$ and an element $\widehat{\delta}$ acting on $\widehat{S}$ as the non-trivial element of $\Gamma_{\mathbb{C}/\mathbb{R}}$ and satisfying $\widehat{\delta}^2 = \exp(2\pi i\gamma)$, where $\exp : X_*(\widehat{S}) \otimes \mathbb{C} \to \widehat{S}$. They prove a local Langlands correspondence for their double cover [AV16, Lemma 3.3]. In fact, these results were first proved in [AV92], but [AV16] is sometimes a more convenient reference.

Lemma 5.9. Assume γ is symmetric (i.e. imaginary) and let $R = \{2\gamma, -2\gamma\}$. There are natural isomorphisms $S(\mathbb{R})_\gamma \to S(\mathbb{R})_\pm$ and $\langle \widehat{S}, \widehat{\delta} \rangle \to {}^L S_\pm$. These isomorphisms identify the local correspondence of Theorem 3.15 with the local correspondence of Adams-Vogan. $\square$

Proof. The construction of $S(\mathbb{R})_\gamma$ can be rephrased as the pull-back diagram

$$
\begin{array}{ccc}
S(\mathbb{R})_\gamma & \longrightarrow & S(\mathbb{R}) \\
\downarrow & & \downarrow{\scriptstyle 2\gamma} \\
\mathbb{C}^\times & \xrightarrow{\;(-)^2\;} & \mathbb{C}^\times
\end{array}
$$

Since 2γ is imaginary the image of $S(\mathbb{R})$ lies in $\mathbb{S}^1 \subset \mathbb{C}^\times$. The preimage of $\mathbb{S}^1$ under the squaring map is again $\mathbb{S}^1$. Therefore we may replace the bottom row in the above diagram by the squaring map $\mathbb{S}^1 \to \mathbb{S}^1$. The inclusion $\mathbb{S}^1 \to \mathbb{C}^\times$ induces an isomorphism $\mathbb{S}^1 \to \mathbb{C}^\times/\mathbb{R}_{>0}$. Composing its inverse with the squaring map on $\mathbb{S}^1$ gives the map $\mathbb{C}^\times/\mathbb{R}_{>0} \to \mathbb{S}^1$ sending z to $z/\bar{z}$. Again we replace the bottom row of above diagram, now with the map $\mathbb{C}^\times/\mathbb{R}_{>0} \to \mathbb{S}^1$ sending z to $z/\bar{z}$. The corresponding pull-back diagram is the definition of $S(\mathbb{R})_\pm$.

Turning to the dual side we compute the Tits cocycle for R. Define a gauge by $p(2\gamma) = +1$. The Tits cocycle t_p sends (σ, σ) to $(-1)^{2\gamma} = \exp(2\pi i\gamma)$. Therefore the map $\langle \widehat{S}, \widehat{\delta} \rangle \to \widehat{S} \boxtimes_{t_p} \Gamma_{E/F}$ that is the identity on $\widehat{S}$ and sends $\widehat{\delta}$ to $1 \boxtimes \sigma$ is an isomorphism.

It is enough to compare the two versions of the local correspondence on one genuine character of $S(\mathbb{R})_\pm$ since both are compatible with multiplication by non-genuine characters. We take the one sending (s, δ) with $\delta/\bar{\delta} = [2\gamma](s)$ to $\mathrm{sgn}(\delta)$, where $\mathrm{sgn} : \mathbb{C} \to \mathbb{S}^1$ is the argument function. In terms of the Adams-Vogan cover $S(\mathbb{R})_\gamma$ this is the canonical genuine character γ. Present the Weil group $W_{\mathbb{C}/\mathbb{R}}$ as the group generated by $\mathbb{C}^\times$ and j with $jzj^{-1} = \bar{z}$ and $j^2 = -1$. The L-parameter of γ associated by [AV16, Lemma 3.3] sends z to $(z/\bar{z})^\gamma$ and j to $\widehat{\delta}$. The L-parameter associated by Theorem 3.15 is $r_p(z) = [2\gamma](\mathrm{sgn}(z))$ and $r_p(zj) = 2[\gamma](\mathrm{sgn}(z))$. These are manifestly the same parameter. $\blacksquare$

The construction of Adams-Vogan can be applied in particular when S is a maximal torus of a connected reductive group G and γ is either half the sum

of all positive roots (with respect to some fixed order), or half the sum of the positive imaginary roots, denoted by ρ and ρ_i, respectively.

Corollary 5.10. Let $R \subset X^*(S)$ be the root system of G. There is are natural isomorphisms $S(\mathbb{R})_\pm \to S(\mathbb{R})_{\rho_i}$ between the two covers, as well as between their L-groups, that identify the two versions of the local correspondence. $\square$

Proof. We may discard all asymmetric elements from R, since they do not influence the cover $S(\mathbb{R})_\pm$. All symmetric elements of $R \subset X^*(S)$ share the same $F_\alpha = \mathbb{C}$ and $F_{\pm\alpha} = \mathbb{R}$, which allows us to apply Construction 5.8 inductively and replace R with the sum of representatives for its Γ-orbits. We choose these representatives to be the positive (imaginary) roots with respect to a given order. Their sum is then $2\rho_i$. The claim follows from Lemma 5.9. $\blacksquare$

Remark 5.11. For the comparison of the Adams-Vogan cover and the cover defined in this paper the two essential points were the fact that the inclusion $\mathbb{C}^1 = \mathbb{S}^1 \to \mathbb{C}^\times/\mathbb{R}_{>0} = \mathbb{C}^\times/N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times)$ is an isomorphism and that all symmetric elements $\alpha \in X^*(S)$ share the same $F_\alpha = \mathbb{C}$ and $F_{\pm\alpha} = \mathbb{R}$. Neither of these statements is true in the p-adic case, which explains why a direct translation of their construction to the p-adic case was not possible. $\square$

5.3 Transport of χ-data

We return to the case of an arbitrary local field F. In the next subsection we will discuss a more general case of functoriality for the double cover $S(F)_\pm$ of a torus determined by a Σ-stable finite set $R \subset X^*(S)$. As a preparation, we will discuss in this section some basic results about χ-data. For this, we consider two admissible sets R and R' with Σ-action and a surjective Σ-equivariant map $f : R \to R'$.

Fact 5.12. Let $\alpha' \in R'$. If $\alpha_1, \ldots, \alpha_k$ is a set of representatives for the $\Gamma_{\alpha'}$-orbits in the preimage of α', then it is also a set of representatives for the Γ-orbits in the preimage of $\Gamma \cdot \alpha'$. $\square$

Lemma 5.13. Let $\alpha \in R_{\mathrm{sym}}$ and let $\alpha' = f(\alpha)$.

1. We have $\alpha' \in R'_{\mathrm{sym}}$. The diagram

$$
\begin{array}{ccc}
\Gamma_\alpha & \hookrightarrow & \Gamma_{\alpha'} \\
\cup & & \cup \\
\downarrow & & \downarrow \\
\Gamma_{\pm\alpha} & \hookrightarrow & \Gamma_{\pm\alpha'}
\end{array}
$$

   is both cartesian and cocartesian. In other words, $\Gamma_\alpha = \Gamma_{\pm\alpha} \cap \Gamma_{\alpha'}$ and $\Gamma_{\pm\alpha'} = \Gamma_{\pm\alpha} \cdot \Gamma_{\alpha'}$.

2. Restricting the quadratic character $\Gamma_{\pm\alpha'}/\Gamma_{\alpha'} \to \{\pm 1\}$ to $\Gamma_{\pm\alpha}$ gives the quadratic character $\Gamma_{\pm\alpha}/\Gamma_\alpha \to \{\pm 1\}$.

3. The following diagram commutes

$$\begin{array}{ccc} \Gamma_\alpha^{\mathrm{ab}} & \longrightarrow & \Gamma_{\alpha'}^{\mathrm{ab}} \\ \uparrow & & \uparrow \\ \Gamma_{\pm\alpha}^{\mathrm{ab}} & \longrightarrow & \Gamma_{\pm\alpha'}^{\mathrm{ab}} \end{array}$$

where the horizontal maps are induced by the inclusions, and the vertical maps are the transfer homomorphisms.

$\square$

Proof. That $\alpha' \in R'_{\mathrm{sym}}$ is immediate. Consider the equalities $\Gamma_\alpha = \Gamma_{\pm\alpha} \cap \Gamma_{\alpha'}$ and $\Gamma_{\pm\alpha'} = \Gamma_{\pm\alpha} \cdot \Gamma_{\alpha'}$. For the second, notice that since $\Gamma_{\alpha'}$ is normal in $\Gamma_{\pm\alpha'}$ and $\Gamma_{\pm\alpha} \subset \Gamma_{\pm\alpha'}$, the product is a subgroup. The two equations are equivalent to the injectivity and surjectivity, respectively, of the natural map $\Gamma_{\pm\alpha}/\Gamma_\alpha \to \Gamma_{\pm\alpha'}/\Gamma_{\alpha'}$. These two quotients being of order two, the two statements are equivalent. For the first equality, the inclusion $\Gamma_\alpha \subset \Gamma_{\pm\alpha} \cap \Gamma_{\alpha'}$ is trivial, while the opposite inclusion follows from the admissibility of R'.

The second point is now immediate.

For the second diagram we choose $\tau \in \Gamma_{\pm\alpha} \smallsetminus \Gamma_\alpha$ and recall that the map $t : \Gamma_{\pm\alpha} \to \Gamma_\alpha$ defined by $t(\gamma) = \gamma \cdot \tau\gamma\tau^{-1}$ and $t(\gamma\tau) = \gamma \cdot \tau\gamma\tau^{-1} \cdot \tau^2$ for $\gamma \in \Gamma_\alpha$ describes the transfer homomorphism. The commutativity follows from the fact that the image of τ in $\Gamma_{\pm\alpha'}$ avoids $\Gamma_{\alpha'}$, which is implied by the first part of this lemma. $\blacksquare$

Corollary 5.14. Let $\alpha \in R_{\mathrm{sym}}$ and let $\alpha' = f(\alpha)$. Then

1. $F_\alpha = F_{\pm\alpha} \cdot F_{\alpha'}$ and $F_{\pm\alpha'} = F_{\pm\alpha} \cap F_{\alpha'}$.

2. Restriction to $F_{\alpha'}$ provides an isomorphism

$$\mathrm{Gal}(F_\alpha/F_{\pm\alpha}) \to \mathrm{Gal}(F_{\alpha'}/F_{\pm\alpha'}).$$

3. The restriction of $N_{F_\alpha/F_{\alpha'}}$ to $F_{\pm\alpha}$ equals $N_{F_{\pm\alpha}/F_{\pm\alpha'}}$.

4. $\kappa_{\alpha'} \circ N_{F_{\pm\alpha}/F_{\pm\alpha'}} = \kappa_\alpha$.

$\square$

Proof. The composite $F_{\pm\alpha} \cdot F_{\alpha'}$ is the fixed field of $\Gamma_{\pm\alpha} \cap \Gamma_{\alpha'}$, which by Lemma 5.13 equals Γ_α, hence the first equality. The second follows from the fact that $F_{\pm\alpha} \cap F_{\alpha'}$ is the subfield of $F_{\alpha'}$ fixed by the group $\Gamma_{\pm\alpha}$, which according to Lemma 5.13 generates $\Gamma_{\pm\alpha'}$ together with $\Gamma_{\alpha'}$. The second and third points are immediate. The fourth point follows again from Lemma 5.13 and the fact that the Artin reciprocity map translates inclusion on the Galois side to the norm on the local field side. $\blacksquare$

Corollary 5.15. Let $\chi' = (\chi_{\alpha'})_{\alpha'}$ be a set of χ-data for R'. For each $\alpha \in R$ define $\chi_\alpha = \chi_{\alpha'} \circ N_{F_\alpha/F_{\alpha'}}$, where $\alpha' = f(\alpha)$. Then $\chi = (\chi_\alpha)_\alpha$ is a set of χ-data for R. The analogous statement holds for ζ-data. $\square$

Proof. The Σ-equivariance of f implies $\chi_{-\alpha} = \chi_\alpha^{-1}$ and $\chi_{\sigma\alpha} = \chi_\alpha \circ \sigma^{-1}$. Corollary 5.14 implies

$$\chi_\alpha|_{F_{\pm\alpha}^\times} = \chi_{\alpha'} \circ N_{F_\alpha/F_{\alpha'}}|_{F_{\pm\alpha}^\times} = \chi_{\alpha'}|_{F_{\pm\alpha'}^\times} \circ N_{F_{\pm\alpha}/F_{\pm\alpha'}} = \kappa_{\alpha'} \circ N_{F_{\pm\alpha}/F_{\pm\alpha'}} = \kappa_\alpha.$$

$\blacksquare$

Definition 5.16. Given χ-data $\chi' = (\chi_{\alpha'})$ for R', we denote by $\inf \chi'$ the χ-data $(\chi_{f(\alpha)} \circ N_{F_\alpha/F_{f(\alpha)}})_\alpha$ for R. Analogously we define $\inf \zeta'$ for a collection of ζ-data ζ' for R'. $\square$

5.4 A general case of functoriality

We continue with an arbitrary local field F. Consider two tori S and T defined over F, two finite admissible Σ-sets R_S and R_T, and two Σ-equivariant maps $R_S \to X^*(S)$ and $R_T \to X^*(S)$. Let $f : S \to T$ be a map of tori and $f^* : X^*(T) \to X^*(S)$ its dual. Let $f : R_T \to R_S$ be a Σ-equivariant surjective map. We do *not* assume that the obvious square commutes! Rather, we make the following assumption: for any $\alpha_S \in R_S$

$$\bar\alpha_S = \sum_{\substack{\alpha_T \in R_T \\ f(\alpha_T) = \alpha_S}} f^*(\bar\alpha_T). \tag{5.1}$$

In this setting we are going to construct a natural homomorphism $S(F)_\pm \to T(F)_\pm$ of extensions lifting f, as well as a natural homomorphism ${}^L T_\pm \to {}^L S_\pm$, and then prove that the Langlands correspondence for the two double covers $S(F)_\pm$ and $T(F)_\pm$ is compatible with these two maps.

Using Construction 5.5 we immediately reduce to the case $S = T$. So from now on we focus on a single torus S and a surjective Σ-equivariant map $f : R \to R'$ between to finite admissible Σ-sets equipped with maps to $X^*(S)$. The above assumption takes the form: for any $\alpha' \in R'$

$$\bar\alpha' = \sum_{\substack{\alpha \in R \\ f(\alpha) = \alpha'}} \bar\alpha. \tag{5.2}$$

Remark 5.17. Note that Construction 5.8 is a special case of this. In that case, the map $f : R \to R'$ is the identity on all Σ-orbits in R except those of α and β, and sends both α and β to γ. $\square$

Remark 5.18. The situation discussed here may arise in practice as follows. Consider a torus S defined over F, a finite admissible Σ-set R and a Σ-equivariant map $R \to X^*(S)$. Consider further a Σ-equivariant equivalence relation

$\sim$ on R. That is $\alpha \sim \beta$ implies $\sigma\alpha \sim \sigma\beta$ for all $\sigma \in \Sigma$. Assume that there does not exist $\alpha \in R$ such that $-\alpha \sim \alpha$. Then the quotient $R' = R/\sim$ is again a finite admissible Σ-set. We equip it with the map $R' \to X^*(S)$ that sends $\alpha' \in R'$ to the element $\bar{\alpha}'$ that is the sum of $\bar{\alpha}$ for all α in the equivalence class α'. We will see an important instance of this in the discussion of twisted Levi subgroups in §6. $\qquad \square$

Remark 5.19. The flexibility of allowing a map $R \to X^*(S)$ rather than a subset $R \subset X^*(S)$ is essential here. Indeed, even if R were a subset of $X^*(S)$, the map $R' \to X^*(S)$ that we just constructed may fail to be injective, because two different equivalence classes may happen to yield the same sum of elements. This does happen in practice when dealing with Levi subgroups, as we will show in §6.4. $\qquad \square$

Construction 5.20. We now construct the homomorphism $S(F)_R \to S(F)_{R'}$ lifting the identity of $S(F)$. Let $\tilde{\gamma} = [\gamma, \epsilon, (\delta_\alpha)_{\alpha \in R_{\mathrm{sym}}}] \in S(F)_R$ as in Remark 3.3. We map it to the element $\tilde{\gamma}' = [\gamma, \epsilon, (\delta_{\alpha'})_{\alpha' \in R'_{\mathrm{sym}}}] \in S(F)_{R'}$, where $\delta_{\alpha'}$ is defined as follows. Let $\alpha_1, \ldots, \alpha_k$ be the representatives for the $\Gamma_{\alpha'}$-orbits in the preimage of α'. A given α_i may be symmetric or asymmetric. Assume first that it is symmetric. Write Γ_i instead of Γ_α. Then any $\tau_i \in \Gamma_{\pm i} \smallsetminus \Gamma_i$ lies in $\Gamma_{\pm\alpha'} \smallsetminus \Gamma_{\alpha'}$ by Lemma 5.13. It normalizes both Γ_i and $\Gamma_{\alpha'}$ and hence commutes with $N_i := N_{F_{\alpha_i}/F_{\alpha'}}$. Define $\delta_i' = N_i(\delta_i)$. Then $\delta_i'/\tau_i(\delta_i') = N_i(\delta_i/\tau_i(\delta_i)) = N_i(\bar{\alpha}_i(\gamma))$. Assume now that α_i is asymmetric. Thus $-\Gamma\alpha_i \neq \Gamma\alpha_i$ so there exists $\tau_i \in \Gamma_{\pm\alpha'} \smallsetminus \Gamma_{\alpha'}$ s.t. $-\tau_i\alpha_i = \alpha_j$ for some $j \neq i$. Define $\delta_i' = N_i(\bar{\alpha}_i(\gamma))$. Then $\delta_i'/\tau_i(\delta_i') = N_i(\bar{\alpha}_i(\gamma))N_j(\bar{\alpha}_j(\gamma))$. Now define $\delta_{\alpha'}$ to be the product of δ_i', where i runs over those i for which α_i is symmetric, as well as one representative for each pair of indices i, j s.t. α_i, α_j are asymmetric and $-\Gamma\alpha_i = \Gamma\alpha_j$. Then $\delta_{\alpha'}/\tau_{\pm\alpha'}(\delta_{\alpha'}) = \prod_{i=1}^k N_i(\bar{\alpha}_i(\gamma)) = (\sum_{i=1}^k \sum_{\sigma \in \Gamma_{\alpha'}/\Gamma_i} \sigma(\bar{\alpha}_i))(\gamma) = \bar{\alpha}'(\gamma)$, the last equality due to (5.2). This completes the construction of $[\gamma, \epsilon, (\delta_{\alpha'})_{\alpha' \in R'_{\mathrm{sym}}}]$ and shows that it represents an element $\tilde{\gamma}' \in S(F)_{R'}$. $\qquad \square$

It is immediately checked that $\tilde{\gamma} \mapsto \tilde{\gamma}'$ is a homomorphism, provided one knows that it is well-defined, which we will verify next.

Lemma 5.21. The element $\tilde{\gamma}'$ just constructed depends only on $\tilde{\gamma}$ and not on the choice of presentation $[\gamma, \epsilon, (\delta_\alpha)_{\alpha \in R_{\mathrm{sym}}}]$. $\qquad \square$

Proof. Let $(\eta_\alpha)_{\alpha \in R_{\mathrm{sym}}}$, define $\eta = \prod_{\alpha \in R_{\mathrm{sym}}/\Gamma} \kappa_\alpha(\eta_\alpha)$, and consider the new presentation $[\gamma, \epsilon\eta, (\delta_\alpha \eta_\alpha)]$ of $\tilde{\gamma}$. This will affect δ_i' for those i for which α_i is symmetric, and replace it by $\delta_i' \cdot N_i(\eta_{\alpha_i})$. From Corollary 5.14 we know $N_i|_{F_{\pm\alpha}^\times} = N_{F_{\pm\alpha}/F_{\pm\alpha'}}$ and so $N_i(\eta_{\alpha_i}) \in F_{\pm\alpha'}^\times$. The element $\delta_{\alpha'}$ is then replaced by $\delta_{\alpha'} \cdot \eta_{\alpha'}$, where $\eta_{\alpha'} = \prod_i N_i(\eta_{\alpha_i})$ and the product running over those i for which α_i is symmetric. Thus we obtain the presentation $[\gamma, \epsilon\eta, (\delta_{\alpha'}\eta_{\alpha'})_{\alpha' \in R'_{\mathrm{sym}}}]$ and need to show that it represents the same element of $S(F)_{R'}$ as the presentation $[\gamma, \epsilon, (\delta_{\alpha'})_{\alpha' \in R'_{\mathrm{sym}}}]$. This amounts to showing

$$\prod_{\alpha' \in R'_{\mathrm{sym}}/\Gamma} \kappa_{\alpha'}(\eta_{\alpha'}) = \prod_{\alpha \in R_{\mathrm{sym}}/\Gamma} \kappa_\alpha(\eta_\alpha).$$

Composing the quadratic character $\kappa_{\alpha'} : F_{\pm\alpha'}^{\times}/N_{F_{\alpha'}/F_{\pm\alpha'}}(F_{\alpha'}^{\times}) \to \{\pm 1\}$ with N_i gives the quadratic character $\kappa_i : F_{\pm i}^{\times}/N_{F_i/F_{\pm i}}(F_i^{\times}) \to \{\pm 1\}$, according to Lemma 5.13. Therefore $\kappa_{\alpha'}(\eta_{\alpha'}) = \prod_i \kappa_i(\eta_{\alpha_i})$. By Fact 5.12 this product runs over representatives for the symmetric Γ-orbits in R mapping to the Γ-orbit of α' in R'. Letting α' run over R'_{sym}/Γ and taking the product of the left-hand side of this equality has the effect of taking the product over all symmetric Γ-orbits α in R, as claimed. ∎

Construction 5.22. Next we construct the isomorphism ${}^L S_{R'} \to {}^L S_R$. Fix a gauge p' on R' and define a gauge p on R by $p(\alpha) = p'(f(\alpha))$. That this is a gauge follows from the equivariance of f with respect to $\{\pm 1\} \subset \Sigma$. The value $t_p(\sigma, \tau)$ of the Tits cocycle $t_p \in Z^2(\Gamma, \widehat{S})$ relative to R and p is given by $\lambda_{p,\sigma,\tau}(-1)$, where $\lambda_{p,\sigma,\tau}$ is the sum of $\bar\alpha$ for α in the following subset of R

$$\Lambda_{p,\sigma,\tau} = \{\alpha > 0, \sigma^{-1}\alpha < 0, (\sigma\tau)^{-1}\alpha > 0\},$$

where $\alpha > 0$ stands for $p(\alpha) > 0$. Equation (5.2), the Σ-equivariance of f, and the definition of p, imply that this sum equals the sum of $\bar\alpha'$ for α' in the following subset of R'

$$\Lambda_{p',\sigma,\tau} = \{\alpha' > 0, \sigma^{-1}\alpha' < 0, (\sigma\tau)^{-1}\alpha' > 0\},$$

where now $\alpha' > 0$ stands for $p'(\alpha') > 0$. We conclude that the Tits cocycle $t_{p'}$ relative to R' and p' equals t_p. A similar argument shows that the 1-cochain $s_{p'/q'}$ equals $s_{p/q}$, so the isomorphisms corresponding to different choices of p' splice together to an isomorphism ${}^L S_{R'} \to {}^L S_R$. □

We now come to the desired compatibility of the Langlands correspondence. Before we state it, we give an alternative expression for the cochain r_p of Definition 3.17. For each $\alpha \in R_{\text{sym}}$ fix a section $s : W_{\pm\alpha}\backslash W \to W$ and let $r : W \to W_{\pm\alpha}$ be defined by $w = r(w)s(w)$. Then

$$r_p(w) = \prod_{\alpha \in R/\Sigma} \prod_{u \in W_{\pm\alpha}\backslash W} \chi_\alpha(v_0(r(u)^{-1}r(uw)))^{s(u)^{-1}\bar\alpha}. \tag{5.3}$$

Proposition 5.23. Let χ' be a set of χ-data for R' and let $\chi = \inf \chi'$ as in Definition 5.16. Then

1. The pull-back along $S(F)_R \to S(F)_{R'}$ of the character χ'_S is equal to the character χ_S.

2. Let q' be a gauge on R' and let $q = q' \circ f$. The 1-cochains $r_{q',\chi'}, r_{q,\chi} \in C^1(W_F, \widehat{S})$ are equal up to 1-coboundary.

□

Proof. To compare the pull-back of the character χ'_S with the character χ_S, consider a Σ-orbit O' in R' and choose $\alpha' \in O'$. Let $\alpha_1, \dots, \alpha_k$ be a set of representatives for the $\Gamma_{\alpha'}$-orbits in the preimage of α'. Let $\tilde\gamma = [\gamma, \epsilon, (\delta_\alpha)_{\alpha \in R_{\text{res}}}]$ be an

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element of $S(F)_{\pm}$ with image $\tilde{\gamma}' = [\gamma, \epsilon, (\delta_{\alpha'})_{\alpha' \in R'_{\text{sym}}}]$ as in Construction 5.20. It is enough to show that the contribution of O' to $\chi'_S(\tilde{\gamma}')$ equals the product of the contributions to $\chi_S(\tilde{\gamma})$ of the Σ-orbits in R lying in the preimage of the Σ-orbit of α'.

Assume first that α' is asymmetric. Then each α_i is also asymmetric. The contribution of O' to $\chi'_S(\tilde{\gamma}')$ is $\chi_{\alpha'}(\bar{\alpha}'(\gamma))$. By (5.2) we have

$$
\begin{aligned}
\chi_{\alpha'}(\bar{\alpha}'(\gamma)) &= \chi_{\alpha'}\left(\Big[\sum_{i=1}^{k} \sum_{\sigma \in \Gamma_{\alpha'}/\Gamma_i} \sigma(\bar{\alpha}_i)\Big](\gamma)\right) \\
&= \chi_{\alpha'}\left(\prod_{i=1}^{k} \prod_{\sigma \in \Gamma_{\alpha'}/\Gamma_i} \sigma(\bar{\alpha}_i(\gamma))\right) \\
&= \prod_{i=1}^{k} \chi_{\alpha'}(N_i(\bar{\alpha}_i(\gamma))).
\end{aligned}
$$

Recalling that $\chi_{\alpha_i} = \chi_{\alpha'} \circ N_i$ and using Fact 5.12 we see that this is indeed the product of the contributions to $\chi_S(\tilde{\gamma})$ of the Σ-orbits in R that are in the preimage of the Σ-orbit of α'.

Assume now that α' is symmetric. Then the contribution of O' to $\chi'_S(\tilde{\gamma}')$ is $\chi_{\alpha'}(\delta_{\alpha'})$. Partition $\{1, \ldots, k\}$ as $I \sqcup J \sqcup J'$, where I is the set of i s.t. α_i is symmetric, and $J \cup J'$ is the set of i s.t. α_i is asymmetric, and for $i \in J$ there exists $j \in J'$ s.t. $-\Gamma\alpha_i = \Gamma\alpha_j$. By Construction 5.20

$$
\delta_{\alpha'} = \prod_{i \in I} N_i(\delta_{\alpha_i}) \cdot \prod_{j \in J} N_j(\bar{\alpha}_j(\gamma)).
$$

For $i \in I$ we have $\chi_{\alpha'}(N_i(\delta_{\alpha_i})) = \chi_{\alpha_i}(\delta_{\alpha_i})$, which is the contribution of the Σ-orbit of α_i to $\chi_S(\tilde{\gamma})$. For $j \in J$ we have $\chi_{\alpha'}(N_j(\bar{\alpha}_j(\gamma))) = \chi_{\alpha_j}(\bar{\alpha}_j(\gamma))$, which is the contribution of the Σ-orbit of α_j ot $\chi_S(\tilde{\gamma})$. By Fact 5.12 the set $\{\alpha_i | i \in I \cup J\}$ represents the Σ-orbits inside the preimage of the Σ-orbit of α'. The comparison of χ'_S and χ_S is thus complete.

We now come to the comparison of $r_{q',\chi'}$ and $r_{q,\chi}$. Again we begin our consideration with an asymmetric Σ-orbit O' in R' and a representative $\alpha' \in O'$, as well as representatives $\alpha_1, \ldots, \alpha_k$ for the $\Gamma_{\alpha'}$-orbits in the preimage of α'. To ease notation, we shall write W_i and W' in place of W_{α_i} and $W_{\alpha'}$, and then later in the symmetric case also $W_{\pm i}$ and $W'_{\pm}$ analogously. The same notational conventions will apply to Γ as well, so that for example $\Gamma' = \Gamma_{\alpha'}$. Consider the subgroups $W_i \subset W' \subset W$. We choose sections $s_1 : W'\backslash W \to W$ and $s_2 : W_i\backslash W' \to W'$. These determine $r_1 : W \to W'$ and $r_2 : W' \to W_i$ as in (5.3). Then $s_3 : W_i\backslash W \to W$ defined by $s_3(w) = s_2(r_1(w))s_1(w)$ is a section and the corresponding $r_3 : W \to W_i$ is defined by $r_3(w) = r_2(r_1(w))$. Of course s_2, s_3, r_2, r_3 depend on i.

As in Definition 3.17 these choices determine a gauge p' on O' and a gauge p on $O_1 \cup \cdots \cup O_k$, where O_i is the Σ-orbit of α_i. Since all these Σ-orbits are

asymmetric these gauges can be described more simply as follows: $p'(\beta') > 0$ if and only if $\beta' \in \Gamma\alpha'$, and $p(\beta) > 0$ if and only if $\beta \in \Gamma\alpha_i$ for some $1 \le i \le k$. In particular, these gauges are related by $p(\beta) = p'(f(\beta))$ and it follows that $s_{q',p'} = s_{q,p}$. It is thus enough to compare the contribution of O' to $r_{p',\chi'}$ with the contribution of $O_1 \cup \cdots \cup O_k$ to $r_{p,\chi}$.

The contribution of O' to $r_{p',\chi'}$ is

$$w \mapsto \prod_{u \in W' \backslash W} u^{-1} \bar{\alpha}'(\chi_{\alpha'}(r_1(u)^{-1} r_1(uw))).$$

According to (5.2) we have $\bar{\alpha}' = \sum_i \sum_{\sigma \in \Gamma_i \backslash \Gamma'} \sigma^{-1} \bar{\alpha}_i$, the above equals

$$w \mapsto \prod_i \prod_{u \in W_i \backslash W} u^{-1} \bar{\alpha}_i(\chi_{\alpha'}(r_1(u)^{-1} r_1(uw))). \tag{5.4} \quad \{\texttt{eq:rpcomp1}\}$$

On the other hand, by Fact 5.12 the set $\alpha_1, \ldots, \alpha_k$ represents the Σ-orbits in R that constitute the preimage of the Σ-orbit of α'. Their combined contribution to $r_{p,\chi}$ is

$$w \mapsto \prod_i \prod_{u \in W_i \backslash W} u^{-1} \bar{\alpha}_i(\chi_{\alpha_i}(r_3(u)^{-1} r_3(uw))). \tag{5.5} \quad \{\texttt{eq:rpcomp2}\}$$

Now χ_{α_i} is simply the restriction of $\chi_{\alpha'}$ to W_i. By construction of r_3 the argument of χ_{α_i} is $r_2(r_1(w))^{-1} r_2(r_1(uw))$. We write

$$
\begin{aligned}
r_1(u)^{-1} r_1(uw) &= (r_2(r_1(u)) s_2(r_1(u)))^{-1} (r_2(r_1(uw)) s_2(r_1(uw))) \\
&= s_2(r_1(u))^{-1} (r_2(r_1(u))^{-1} r_2(r_1(uw))) s_2(r_1(uw)).
\end{aligned}
$$

Comparing the contributions of a fixed i to (5.4) and (5.5) we see that they differ by

$$\prod_u u^{-1} \bar{\alpha}_i \chi_{\alpha_i} \Big(s_2(r_1(u))^{-1} s_2(r_1(uw)) \Big).$$

Splitting the product and reindexing this becomes

$$\prod_u u^{-1} \bar{\alpha}_i \chi_{\alpha_i} \Big(s_2(r_1(u)) \Big)^{-1} \cdot w \prod_u u^{-1} \bar{\alpha}_i \chi_{\alpha_i} \Big(s_2(r_1(u)) \Big),$$

visibly a coboundary. Thus the contribution to $r_{p',\chi'}$ of O' equals up to coboundary the contribution to $r_{p,\chi}$ of the preimage in R of O'.

Consider next a symmetric α'. We have $W' \subset W'_\pm \subset W$. Choose $s_1 : W'_\pm \backslash W \to W$ and obtain $r_1 : W \to W'_\pm$. Choose also $v_0 \in W'$ and $v_1 \in W'_\pm \smallsetminus W'$. The resulting gauge p' on R' declares $s_1(u)^{-1}\alpha'$ positive for all u. The contribution of α' to $r_{p',\chi'}$ is

$$w \mapsto \prod_{u \in W'_\pm \backslash W} s_1(u)^{-1} \bar{\alpha}'(\chi_{\alpha'}(v_0(r_1(u)^{-1} r_1(uw)))).$$

For simplicity we take $v_0 = 1$.

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Again we decompose the preimage of α' into Γ'-orbits. The map $\alpha \mapsto -v_1\alpha$ acts as an involution on the set of these orbits. An orbit is fixed by this involution if and only if it consists of symmetric elements. We choose representatives $\alpha_1, \ldots, \alpha_l$ for the symmetric orbits, as well as representatives $\alpha_{l+1}, \ldots, \alpha_k$, one for each pair of asymmetric orbits. Then we set $\alpha_{k+j+1} = -v_1\alpha_{k-j}$, so that $\alpha_1, \ldots, \alpha_{2k-l}$ are representatives for all orbits. Then (5.2) implies that $\bar\alpha' = \sum_{i=1}^{2k-l} \sum_{\sigma \in \Gamma_i \backslash \Gamma'} \sigma^{-1}\bar\alpha_i$. We plug that into the above formula to get

$$w \mapsto \prod_{i=1}^{2k-l} \prod_{u \in W'_\pm \backslash W} \prod_{\sigma \in \Gamma_i \backslash \Gamma'} s_1(u)^{-1}\sigma^{-1}\bar\alpha_i(\chi_{\alpha'}(v_0(r_1(u)^{-1}r_1(uw)))). \qquad (5.6) \quad \{\texttt{eq:rpcomp10}\}$$

We now consider the contribution of an individual i. We begin with a symmetric α_i. Choose $s_2 : W_i \backslash W' \to W'$ and view it, using the identity $\Gamma'_\pm / \Gamma_{\pm i} = \Gamma' / \Gamma_i$ of Lemma 5.13, as $W_{\pm i} \backslash W'_\pm \to W'_\pm$. Declare as before $s_3 : W_{\pm i} \backslash W \to W$ by $s_3(w) = s_2(r_1(w))s_1(w)$. The i-th factor in the above product equals

$$w \mapsto \prod_{u \in W_{\pm i} \backslash W} s_3(u)^{-1}\bar\alpha_i(\chi_{\alpha'}(v_0(r_1(u)^{-1}r_1(uw)))). \qquad (5.7) \quad \{\texttt{eq:rpcomp11}\}$$

We use again

$$r_1(u)^{-1}r_1(uw) = s_2(r_1(u))^{-1}(r_3(u)^{-1}r_3(uw))s_2(r_1(uw))$$

and write it as

$$s_2(r_1(u))^{-1} \cdot \mathrm{Ad}[r_3(u)^{-1}r_3(uw)](s_2(r_1(uw))) \cdot r_3(u)^{-1}r_3(uw).$$

Since the image of s_2 lies in W' we have that $v_0(r_1(u)^{-1}r_1(uw))$ is equal to

$$s_2(r_1(u))^{-1} \cdot \mathrm{Ad}[r_3(u)^{-1}r_3(uw)](s_2(r_1(uw))) \cdot v_0(r_3(u)^{-1}r_3(uw)).$$

We claim that the first two terms contribute a coboundary. Indeed, the first term contributes

$$w \mapsto \prod_{u \in W_{\pm i} \backslash W} s_3(u)^{-1}\bar\alpha_i(\chi_{\alpha'}(s_2(r_1(u))))^{-1},$$

while the contribution of the second is, upon making the substitution $u \mapsto uw^{-1}$, equal to

$$w \mapsto \prod_{u \in W_{\pm i} \backslash W} s_3(uw^{-1})^{-1}\bar\alpha_i(\chi_{\alpha'}(\mathrm{Ad}[r_3(uw^{-1})^{-1}r_3(u)]s_2(r_1(u)))).$$

Now $s_3(uw^{-1}) = r_3(uw^{-1})^{-1}r_3(u)s_3(u)w^{-1}$, which makes the above equal to

$$w \mapsto w \prod_{u \in W_{\pm i} \backslash W} s_3(u)^{-1}r_3(u)^{-1}r_3(uw^{-1})\bar\alpha_i(\chi_{\alpha'}(\mathrm{Ad}[r_3(uw^{-1})^{-1}r_3(u)]s_2(r_1(u)))).$$

But $(r_3(u)^{-1}r_3(uw^{-1})\bar{\alpha}_i)\circ\chi_{\alpha'}\circ\mathrm{Ad}[r_3(uw^{-1})^{-1}r_3(u)]=\bar{\alpha}_i\circ\chi_{\alpha'}$, and we indeed conclude that the first two terms combine to a coboundary.

We turn to the third term, namely $v_0(r_3(u)^{-1}r_3(uw))$. Its contribution to (5.7) is

$$w\mapsto\prod_{u\in W_{\pm i}\backslash W}s_3(u)^{-1}\bar{\alpha}_i(\chi_{\alpha'}(v_0(r_3(u)^{-1}r_3(uw)))). \qquad (5.8) \quad \{\texttt{eq:rpcomp12}\}$$

Since $\chi_{\alpha'}|_{W_i}=\chi_{\alpha_i}$, this looks exactly like the contribution to $r_{p,\chi}$ of the Σ-orbit of α_i. But we need to be careful here. The chosen v_1 may not lie in $W_{\pm i}$. Choose $v_1'\in W_{\pm i}\smallsetminus W_i$. There exists $a\in W'$ so that $v_1'=av_1$. We denote by $v_0'(-)$ the version of the term $v_0(-)$ with respect to v_1' instead of v_1. The contribution to $r_{p,\chi}$ of the Σ-orbit of α_i is given by

$$w\mapsto\prod_{u\in W_{\pm i}\backslash W}s_3(u)^{-1}\bar{\alpha}_i(\chi_{\alpha'}(v_0'(r_3(u)^{-1}r_3(uw)))). \qquad (5.9) \quad \{\texttt{eq:rpcomp13}\}$$

Now (5.8) is the product of (5.9) and

$$w\mapsto\begin{cases}\prod_{u\in W_{\pm i}\backslash W}s_3(u)^{-1}\bar{\alpha}_i(\chi_{\alpha'}(a^{-1})), & r_3(u)^{-1}r_3(uw)\notin W_i\\1, & r_3(u)^{-1}r_3(uw)\in W_i\end{cases}$$

This is the image under $\mathrm{cor}\circ\bar{\alpha}_i:Z^1(W_{\pm i}/W_i,\mathbb{C}^\times_{(-1)})\to Z^1(W_F,\widehat{S})$ of

$$x\mapsto\begin{cases}\chi_{\alpha'}(a^{-1}) &,x\notin W_i\\1 &,x\in W_i\end{cases}$$

Since $H^1(\Gamma_{\pm i}/\Gamma_i,\mathbb{C}^\times_{(-1)})$ vanishes, this is also a coboundary. We conclude that, up to a coboundary, the 1-cochains (5.7), (5.8), and (5.9), are all equal. Thus, the i-th factor in (5.6) equals up to a coboundary the contribution to $r_{p,\chi}$ of the Σ-orbit O_i of α_i. Here the gauge p on the orbit O_i declares each $s_3(u)^{-1}\alpha_i$ positive. Recalling that $s_3(u)=s_2(r_1(u))s_1(u)$ and that s_2 takes values in W' we see that, as in the case of asymmetric α', we have $p(\beta)=p'(f(\beta))$ for all $\beta\in O_i$. Therefore, as in the asymmetric case, the equality of these parts of $r_{p',\chi'}$ and $r_{p,\chi}$ implies the equality of the corresponding parts of $r_{q',\chi'}$ and $r_{q,\chi}$.

We turn again to the contribution of an individual i to (5.6), but now we consider the case of an asymmetric α_{k-j}. We combine this contribution with that of its partner $\alpha_{k+j+1}=-v_1\alpha_{k-j}$. This joint contribution is

$$\prod_{u\in W_\pm'\backslash W}\prod_{\sigma\in\Gamma_i\backslash\Gamma_{\mathrm{eq}}}s_1(u)^{-1}(\sigma^{-1}\bar{\alpha}_i-v_1\sigma^{-1}\bar{\alpha}_i)(\chi_{\alpha'}(v_0(r_1(u)^{-1}r_1(uw)))),$$

where we have set $i=k-j$.

We now encode $v_0=1$ and v_1 as a section $s^\circ:W'\backslash W_\pm'\to W_\pm'$ and choose further a section $s_2:W_i\backslash W'\to W'$. We can combine these into a section $s_3:W_i\backslash W\to W$ defined by $s_3(a)=s_2r^\circ r_1(a)\cdot s^\circ r_1(a)\cdot s_1(a)$. The corresponding $r_3:W\to W_i$ is defined by $r_3(a)=r_2r^\circ r_1(a)$.

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For any $a \in W$ we can decompose $r_1(a) \in W'_\pm$ as

$$r_1(a) = \underbrace{r_2 r^\circ r_1(a)}_{=r_3(a) \in W_i} \cdot \underbrace{s_2 r^\circ r_1(a)}_{\in W'} \cdot \underbrace{s^\circ r_1(a)}_{\in W'_\pm}.$$

Using this, we compute $v_0(r_1(a)^{-1} r_1(aw))$ for any $a \in W$ by distinguishing four cases, depending on whether $r_1(a)$ and $r_1(aw)$ belong to W' or not, which we record as a sequence of two signs:

$$\begin{cases} [r_3(a)^{-1} r_3(aw)] \cdot [s_2 r^\circ r_1(a)^{-1} s_2 r^\circ r_1(aw)] \cdot [s^\circ r_1(a)^{-1} s^\circ r_1(aw)], & ++ \\ [r_3(a)^{-1} r_3(aw)] \cdot [s_2 r^\circ r_1(a)^{-1} s_2 r^\circ r_1(aw)] \cdot [s^\circ r_1(a)^{-1} s^\circ r_1(aw) v_1^{-1}], & +- \\ [r_3(a)^{-1} r_3(aw)]^c \cdot [s_2 r^\circ r_1(a)^{-1} s_2 r^\circ r_1(aw)]^c \cdot [s^\circ r_1(a)^{-1} s^\circ r_1(aw) v_1^{-1}], & -+ \\ [r_3(a)^{-1} r_3(aw)]^c \cdot [s_2 r^\circ r_1(a)^{-1} s_2 r^\circ r_1(aw)]^c \cdot [s^\circ r_1(a)^{-1} s^\circ r_1(aw)], & -- \end{cases}$$

Each row has three terms isolated by square brackets. Each term belongs to W'. The superscript c denotes the conjugation action of v_1 on W'. We consider first the contributions of the third term in each row. These four terms are equal to 1, 1, v_1^{-2}, and 1, respectively. Since $\chi_{\alpha'}(v_1^2) = -1$, we see that the contributions of the third terms combines to

$$w \mapsto \prod_{\substack{a \in W_i \setminus W \\ r_1(a) \notin W' \\ r_1(aw) \in W'}} s_3(a)^{-1} \bar\alpha_i(-1). \tag{5.10} \quad \texttt{\{eq:rpcomp4\}}$$

Next we consider the contributions of the first term in each row. Since $\chi_{\alpha'}(x^c) = \chi_{-\alpha'}(x) = \chi_{\alpha'}(x)^{-1}$ they combine to

$$\prod_{a \in W_i \setminus W} s_3(a)^{-1} \bar\alpha_i(\chi_{\alpha'}(r_3(a)^{-1} r_3(aw))),$$

which is the contribution of Σ-orbit of the asymmetric α_{k-j} (this orbit contains its partner α_{k+j+1}) to $r_{p,\chi}$, where p is the gauge on the Σ-orbit of α_{k-j} declaring that $s_3(a)^{-1}\alpha_i$ is positive.

Finally we consider the contributions of the second term in each row. They combine to

$$\prod_{a \in W_i \setminus W} s_3(a)^{-1} \bar\alpha_i(\chi_{\alpha'}(s_2 r^\circ r_1(a)^{-1} s_2 r^\circ r_1(aw))).$$

Now both $s_2 r^\circ r_1(a)$ and $s_2 r^\circ r_1(aw)$ belong to W' and can be individually plugged into $\chi_{\alpha'}$. Moreover $s_3(a)^{-1}\alpha_i = a^{-1}\alpha_i$. Substituting $a \mapsto aw^{-1}$ in the second term reveals the above contribution to be a coboundary.

We have concluded that the contribution of the pair of indices $i = k - j$ and $i = k + j + 1$ to (5.6) differs from the contribution of the Σ-orbit of α_{k-j} to $r_{p,\chi}$ by the term (5.10). We claim that this difference is accounted for by the fact that the gauges p and p' are not related by $p(\beta) = p'(f(\beta))$. To study this, let p_0

be the gauge on $\Sigma\alpha_{k-j}$ defined by $p_0(\beta) = p'(f(\beta))$. Reviewing the definitions of p and p' we see that p declares the entire Γ-orbit of the asymmetric α_{k-j} positive and the entire Γ-orbit of α_{k+j+1} negative, while p_0 declares half of the Γ-orbit of both α_{k-j} and α_{k+j+1} positive. To state this more precisely, we recall $s_3(a) = s_2 r^\circ r_1(a) \cdot s^\circ r_1(a) \cdot s_1(a)$. Letting $i = k - j$ we have

$$
\begin{aligned}
p_0(a^{-1}\alpha_i) = \;\; &= \;\; p_0(s_3(a)^{-1}\alpha_i) \\
&= \;\; p_0(s_1(a)^{-1}(s^\circ r_1(a))^{-1}(s_2 r^\circ r_1(a))^{-1}\alpha_i) \\
&= \;\; p'(s_1(a)^{-1}(s^\circ r_1(a))^{-1}\alpha'),
\end{aligned}
$$

where we have used that s_2 takes values in W'. When $r_1(a) \in W'$ then the value $p_0(s_3(a)^{-1}\alpha_i)$ is positive and hence equal to $p(s_3(a)^{-1}\alpha_i)$, and when $r_1(a) \notin W'$ then the value $p_0(s_3(a)^{-1}\alpha_i)$ is negative and hence unequal to $p(s_3(a)^{-1}\alpha_i)$.

Now $r_{p_0,\chi} = s_{p_0/p} r_{p,\chi}$ and up to coboundaries $s_{p_0/p} = s_{p/p_0}$. Reviewing the definition of s_{p/p_0} we see that the product determined by the conditions $p(\lambda) = 1$, $p(\sigma^{-1}\lambda) = -1$, $p_0(\lambda) = 1$, $p_0(\sigma^{-1}\lambda) = 1$ is empty, because p is constant on Γ-orbits, while the product over $p(\lambda) = 1$, $p(\sigma^{-1}\lambda) = 1$ $p_0(\lambda) = -1$, $p_0(\sigma^{-1}\lambda) = 1$ runs over members $a^{-1}\alpha_i$ of the Γ-orbit of α_i satisfying $r_1(a) \notin W'$ and $r_1(aw) \in W'$, where $w \in W$ represents $\sigma \in \Gamma$, and hence equals precisely (5.10). We conclude that $r_{p',\chi'} = r_{p_0,\chi}$, and hence $r_{q',\chi'} = r_{q,\chi}$. ∎

Corollary 5.24. If ζ' is a collection of ζ-data for R' and we define $\zeta = \inf \zeta'$ as in Definition 5.16, then the pull-back to $S(F)$ of the character ζ'_S equals the character ζ_S. □

Proof. Apply Proposition 5.23 to $(\chi_{\alpha'})$ and $(\chi_{\alpha'}\zeta_{\alpha'})$ for an arbitrary choice of χ-data $(\chi_{\alpha'})$. ∎

We now return to the case of two tori S and T and two Σ-equivariant maps $R_S \to X^*(S)$, $R_T \to X^*(T)$, as well as a surjective Σ-equivariant map $f : R_T \to R_S$ satisfying (5.1).

Corollary 5.25. If $\varphi : W_F \to {}^L T_\pm$ is a Langlands parameter and $\chi : T_\pm(F) \to \mathbb{C}^\times$ is the associated genuine character, composing φ with the natural L-homomorphism ${}^L T_\pm \to {}^L S_\pm$ is a parameter whose corresponding character is the pull-back of χ along the natural lift $S_\pm(F) \to T_\pm(F)$ of f. □

Proof. This is immediate from Proposition 5.23 and Construction 5.5. ∎

5.5 A comparison of minimally ramified χ-data for R and R'

Consider a non-archimedean local field F whose residual characteristic p is not equal to 2, a torus S defined over F, two finite admissible Σ-sets R and R', a Σ-equivariant $R \to X^*(S)$, and a surjective Σ-equivariant map $f : R \to R'$. We do not assume the existence of a Σ-equivariant map $R' \to X^*(S)$ or the validity of (5.2).

Consider minimally ramified χ-data for R in the sense of [Kal19a, Definition 4.6.1], which we recall means that, for $\alpha \in R$,

1. $\chi_\alpha = 1$ if α is asymmetric;

2. χ_α is unramified if α is symmetric unramified;

3. χ_α is tamely ramified if α is symmetric ramified.

Given tame χ-data that is not necessarily minimally ramified we can associate canonically minimally ramified χ-data in the following straightforward way:

Definition 5.26. Define $\min \chi$ by

1. $(\min \chi)_\alpha = 1$ if α is asymmetric;

2. $(\min \chi)_\alpha$ is the unique unramified extension of κ_α if α is symmetric unramified;

3. $(\min \chi)_\alpha = \chi_\alpha$ if α is symmetric ramified.

□

If χ' is minimally ramified χ-data for R', then the χ-data $\inf \chi'$ of Definition 5.16 is tame, but need not be minimal. Indeed, this happens when α is asymmetric but α' is symmetric, or when α is symmetric unramified but α' is symmetric ramified. Both of these situations do occur, as we shall see in the examples provided in §6.4.

Consider the torus T whose character module is the $\mathbb{Z}$-span of R in $X^*(S)$. We have the natural map $S \to T$ dual to the inclusion $X^*(T) \to X^*(S)$. Let $S(F)_c$ be the preimage of the bounded subgroup $T(F)_b$ of $T(F)$. If R is elliptic, in the sense that the $\mathbb{Z}$-span of R in $X^*(S)$ does not have non-zero Γ-fixed elements, then T is anisotropic and hence $S(F)_c = S(F)$.

Recall the character $\zeta_S : S(F) \to \mathbb{C}^\times$ associated to a set of ζ-data in §3.2.

Proposition 5.27. Let χ' be minimally ramified and define $\zeta_\alpha = (\min \inf \chi')_\alpha \cdot (\inf \chi')_\alpha^{-1}$. The restriction of $\zeta_S : S(F) \to \mathbb{C}^\times$ to $S(F)_c$ is the following sign character

$$\gamma \mapsto \prod_{\substack{\alpha \in R_{\mathrm{asym}}/\Sigma \\ \alpha' \in R'_{\mathrm{sym,ram}}}} \mathrm{sgn}_{k_\alpha^\times}(\bar{\alpha}(\gamma))^{e(\alpha/\alpha')} \cdot \prod_{\substack{\alpha \in R_{\mathrm{sym,unram}}/\Gamma \\ \alpha' \in R'_{\mathrm{sym,ram}}}} \mathrm{sgn}_{k_\alpha^1}(\bar{\alpha}(\gamma))^{e(\alpha/\alpha')}.$$

In particular, this character is independent of χ'. □

Let us explain the notation. The first product runs over asymmetric Σ-orbits in R whose images in R' are symmetric ramified, while the second product runs over symmetric unramified orbits in R whose images in R' are symmetric ramified. Since γ maps to the bounded subgroup of $T(F)_b$, $\bar{\alpha}(\gamma)$ belongs to $O_{F_\alpha}^\times$ and we can project to $k_\alpha^\times$. When α is symmetric unramified, the image of

$\bar{\alpha}(\gamma)$ in $k_\alpha^\times$ belongs to the subgroup k_α^1 of elements whose norm to $k_{\pm\alpha}^\times$ is 1. Both groups $k_\alpha^\times$ and k_α^1 are cyclic of even order, hence have a canonical sign character, which we denote by sgn. Finally, $e(\alpha/\alpha')$ is the ramification degree of the field extension $F_\alpha/F_{\alpha'}$.

Proof. Write $\chi = \inf \chi'$ and $\chi^m = \min\inf \chi'$. Given $\alpha \in R$, we have the following cases in which ζ_α is automatically trivial, because $\chi_\alpha = \chi_\alpha^m$:

1. α is ramified symmetric (then so is also α').

2. α and α' are both unramified symmetric.

3. α and α' are both asymmetric.

This leaves the following cases, in wich ζ_α is non-trivial:

1. α is unramified symmetric, but α' is ramified symmetric.

2. α is asymmetric, but α' is unramified symmetric.

3. α is asymmetric, but α' is ramified symmetric.

Writing a,u,r, for asymmetric, unramified symmetric, ramified symetric, the three cases we need to consider can be abbreviated as (u,r), (a,u), (a,r).

In the settings (a,u) and (a,r) we have $\zeta_\alpha = \chi_\alpha = \chi_{\alpha'} \circ N_{F_\alpha/F_{\alpha'}}$ and the contribution of the Σ-orbit of α is $\zeta_\alpha \circ \alpha$. In the setting (a,u) the character ζ_α is unramified and $\bar{\alpha}(\gamma) \in O_\alpha^\times$ implies that $\zeta_\alpha \circ \alpha$ is trivial. In the setting (a,r) the restriction of $\chi_{\alpha'}$ to $O_{\alpha'}^\times$ factors to the sign character of $k_{\alpha'}^\times$, and the norm map $N_{F_\alpha/F_{\alpha'}}$ factors as $[N_{k_\alpha/k_{\alpha'}}]^{e(\alpha/\alpha')}$. Composing the sign character of $k_{\alpha'}^\times$ with $N_{k_\alpha/k_{\alpha'}}$ gives the sign character of $k_\alpha^\times$. So the contribution to $\zeta : S(F) \to \mathbb{C}^\times$ of α of type (a,r) is $\operatorname{sgn}_{k_\alpha^\times}(\bar{\alpha}(\gamma))^{e(\alpha/\alpha')}$.

In the setting (u,r) we have $\zeta_\alpha = \chi_\alpha \cdot (\chi_\alpha^m)^{-1}$ and the contribution of α is $\chi_\alpha(\delta_\alpha) \cdot \chi_\alpha^m(\delta_\alpha)^{-1}$, where $\delta_\alpha \in F_\alpha^\times$ is chosen so that $\delta_\alpha/\tau_\alpha(\delta_\alpha) = \bar{\alpha}(\gamma)$. Since $F_\alpha/F_{\pm\alpha}$ is unramified, δ_α can be chosen in $O_\alpha^\times$. Now χ_α^m is the unramified quadratic character of $F_\alpha^\times$, so its restriction to $O_\alpha^\times$ is trivial, while χ_α is as the same as in type (a,r), namely its restriction to $O_\alpha^\times$ factors through the quadratic character of $k_\alpha^\times$ taken to the power $e(\alpha/\alpha')$. Thus the contribution of α to ζ_S becomes $\operatorname{sgn}_{k_\alpha^\times}(\delta_\alpha)^{e(\alpha/\alpha')}$. The quadratic character of $k_\alpha^\times$ kills $k_{\pm\alpha}^\times$ and thus factors through the map $k_\alpha^\times \to k_\alpha^1$ and gives on k_α^1 the quadratic character there. Since the image of δ_α under this map is $\bar{\alpha}(\gamma)$, the contribution to ζ_S of α is $\operatorname{sgn}_{k_\alpha^1}(\bar{\alpha}(\gamma))^{e(\alpha/\alpha')}$. ∎

We now consider the setting of Corollary 5.24. In particular, we now assume given a Σ-equivariant map $R' \to X^*(S)$ and the validity of (5.2).

{cor:llcfunc1b}

Corollary 5.28. Let χ_1' and χ_2' be two minimal χ-data for R' and let $\zeta' = \chi_2' \cdot (\chi_1')^{-1}$. Let $\zeta = \min\inf\chi_2' \cdot (\min\inf\chi_1')^{-1}$. Then ζ_S is the pull-back of ζ_S'. □

Notice that the difference between Corollaries 5.24 and 5.28 is that in the latter case we have passed to minimal χ-data after inflation.

Proof. We have $\zeta = [\min \inf\chi_2' \cdot (\inf\chi_2')^{-1}] \cdot \inf\zeta' \cdot [(\inf\chi_1') \cdot (\min \inf\chi_1')^{-1}]^{-1}$ and hence $\zeta_S = [\min \inf\chi_2' \cdot (\inf\chi_2')^{-1}]_S \cdot [\inf\zeta']_S \cdot [(\inf\chi_1') \cdot (\min \inf\chi_1')^{-1}]_S^{-1}$. By Proposition 5.27 the first and third terms in square brackets are the same and thus cancel out, while Corollary 5.24 tells us that the middle term is the pull-back of ζ_S'. $\blacksquare$

6 TWISTED LEVI SUBGROUPS

Let G be a connected reductive group defined over F. A twisted Levi subgroup of G is a connected reductive subgroup $M \subset G$ defined over F that is the Levi factor of a parabolic subgroup defined over a field extension E/F. When $E = F$ then M is called a (actual) Levi subgroup. In the latter case it is well known that there is a canonical $\widehat{G}$-conjugacy class of L-embeddings $^L M \to {}^L G$. For a twisted Levi subgroup this is no longer the case. Indeed, any maximal torus is a twisted Levi subgroup. We will show presently that the Langlands-Shelstad construction lends itself easily to the construction of L-embeddings $^L M \to {}^L G$ when M is a twisted Levi subgroup. We will then apply the results of §5.4 to discuss compositions of the form $^L T \to {}^L M \to {}^L G$ for a maximal torus $T \subset M \subset G$. We will also reinterpret these matters in terms of double covers, and in particular introduce a double cover $M(F)_{\pm}$ of $M(F)$ and an associate L-group.

6.1 Embedding the L-group of a twisted Levi subgroup

Let G be a connected reductive group defined over F and $M \subset G$ a twisted Levi subgroup. Let $M_{\mathrm{ab}} = M/M_{\mathrm{der}}$ be the maximal abelian quotient of M. It is a torus defined over F. We begin by defining a finite admissible Σ-set $R(M_{\mathrm{ab}}, G)$ with a Σ-equivariant map $R(M_{\mathrm{ab}}, G) \to X^*(M_{\mathrm{ab}})$.

Lemma 6.1. Let $S \subset M$ be a maximal torus. Let R_S be the set of $\Omega(S, M)$-orbits in $R(S, G) \smallsetminus R(S, M)$.

1. The set R_S is a finite admissible Σ-set.

2. The map $R_S \to X^*(S)$ that sends an $\Omega(S, M)$-orbit in $R(S, G) \smallsetminus R(S, M)$ to the sum of its elements takes image in $X^*(M_{\mathrm{ab}})$ and is Σ-equivariant.

3. If $S_1, S_2 \subset M$ are two maximal tori, then any element $m \in M$ such that $S_2 = mS_1m^{-1}$ induces the same bijection $R_{S_2} \to R_{S_1}$. In particular, this bijection is Σ-equivariant. Moreover, it respects the maps $R_{S_1} \to X^*(M_{\mathrm{ab}})$ and $R_{S_2} \to X^*(M_{\mathrm{ab}})$.

4. Let $\Delta(S, G)$ be a set of simple roots for G that contains a set of simple roots for M. The gauge $p : R(S, G) \smallsetminus R(S, M) \to \{\pm 1\}$ that sends a positive root to $+1$ is constant on $\Omega(S, M)$-orbits and hence induces a gauge on R_S.

$\square$

Proof. It is clear that R_S is finite. The action of Σ on $R(S, G)$ preserves $R(S, M)$ and respects the orbits of $\Omega(S, M)$, since M is defined over F. Therefore R_S is a finite Σ-set. Its admissibility will follow from 4.

The map $R(S, G) \smallsetminus R(S, M) \to X^*(S)$ is obviously Σ-equivariant and descends to R_S, and its image lies in $X^*(S)^{\Omega(S,M)} = X^*(M_{\mathrm{ab}})$.

The isomorphism $\mathrm{Ad}(m) : X^*(S_2) \to X^*(S_1)$ identifies $R(S_2, G) \smallsetminus R(S_2, M)$ with $R(S_1, G) \smallsetminus R(S_1, M)$ and is $\Omega(S_2, M) - \Omega(S_1, M)$ equivariant. Moreover, it is the identity on the embedded copy of $X^*(M_{\mathrm{ab}})$ into $X^*(S_2)$ and $X^*(S_1)$. Therefore it induces a bijection $R_{S_2} \to R_{S_1}$ that respects the maps to $X^*(M_{\mathrm{ab}})$. Another choice of m is of the form nm with $n \in N(S_1, M)$, so induces the same bijection $R_{S_2} \to R_{S_1}$.

Consider a set of simple roots $\Delta(S, G)$ for G that contains a set of simple roots for M. Let $\eta \in X_*(S/Z(G)) \otimes \mathbb{R}$ denote a regular element in the corresponding acute Weyl chamber. Then the gauge p is given by $p(\alpha) = \mathrm{sgn}(\langle \eta, \alpha \rangle)$. The action of $\Omega(S, M)$ preserves the set of positive roots in $R(S, G) \smallsetminus R(S, M)$ and therefore for any $\alpha \in R(S, G) \smallsetminus R(S, M)$ the sign of the non-zero real number $\langle \eta, w\alpha \rangle$ is independent of $w \in \Omega(S, M)$. Thus p induces a gauge on R_S. This gauge shows that no element of R_S is fixed by $-1 \in \Sigma$, so R_S is indeed admissible as claimed by 1. ∎

Definition 6.2. We define the set $R(M_{\mathrm{ab}}, G)$ to be the limit over all maximal tori $S \subset M$ of the set R_S of Lemma 6.1, with transition maps given by $\mathrm{Ad}(m)$ for $m \in M$. This set comes equipped with a map to $X^*(M_{\mathrm{ab}})$. □

Remark 6.3. The map $R(M_{\mathrm{ab}}, G) \to X^*(M_{\mathrm{ab}})$ need not be injectve, as we will see in §6.4. □

Lemma 6.4. Let $\xi : G^* \to G$ be an inner twist with G^* quasi-split and let $M \subset G$ be a twisted Levi subgroup. After conjugation by $G^*(F^s)$ we can arrange that the subgroup $M^* = \xi^{-1}(M)$ is defined over F (hence is a twisted Levi subgroup of G^*) and quasi-split. The restriction $\xi : M^* \to M$ is an inner twist. □

Proof. Let $T_M \subset M$ be a maximal torus, and let $Z_M \subset T_M$ be the connected center of M. Then Z_M is a subtorus of T_M defined over F and M is the centralizer of Z_M. The torus T_M transfers to G^*, i.e. after conjugating ξ by an element of $G^*(F^s)$ the preimage $T_M^1 = \xi^{-1}(T_M)$ is defined over F and ξ restricts to an isomorphism $T_M^1 \to T_M$ defined over F. In particular $\xi^{-1}\sigma(\xi)$ is realized by conjugation by an element of T_M^1. Transport Z_M by ξ and take its centralizer in G^*, calling it M^1. Then M^1 is defined over F and $\xi : M^1 \to M$ is an inner twist. Let $t^1 \in Z_{M^1}(F)$ be a regular element, so that M^1 is the centralizer of t^1. According to [Kot82, Lemma 3.3] there exists a stable conjugate t^* of t^1 whose centralizer in G^* is quasi-split. Let M^* be the centralizer of t^* and apply [Kot82, Lemma 3.2]. ∎

Fact 6.5. In the setting of above lemma, ξ induces an isomorphism $M_{\mathrm{ab}}^* \to M_{\mathrm{ab}}$ defined over F that identifies $R(M_{\mathrm{ab}}^*, G^*)$ with $R(M_{\mathrm{ab}}, G)$. □

We now proceed to study L-embeddings ${}^L M \to {}^L G$. Lemma 6.4 allows us to assume that G and M are quasi-split. Let (T_M, B_M) and (T_G, B_G) be Borel pairs in M and G respectively. Choose $g \in G$ s.t. $T_M = gT_G g^{-1}$ and $B_M \subset gB_G g^{-1}$. Then the simple roots $\Delta(T_M, B_M)$ are a subset of $g\Delta(T_G, B_G)g^{-1}$. Let $\sigma_{M,G} \in \Omega(T_G, G)$ be the element $g^{-1}\sigma(g)$.

Consider a pinned dual group $(\widehat{G}, \widehat{T}, \widehat{B}, \{X_{\alpha^\vee}\})$ for G. There exists a unique standard Levi subgroup $\widehat{M}$ of $\widehat{G}$ in the conjugacy class determined by M. Its set of simple roots $\Delta(\widehat{T}, \widehat{M})$ is the subset dual to $g^{-1}\Delta(T_M, B_M)g \subset \Delta(T_G, G) = \Delta(\widehat{T}, \widehat{G})^\vee$. For $\sigma \in \Gamma$ let σ_G denote the pinned action of σ on $\widehat{G}$. The set $\Delta(\widehat{T}, \widehat{M})$, and hence also the group $\widehat{M}$, will in general not be σ_G-stable. Using the identification $\Omega(\widehat{T}, \widehat{G}) = \Omega(T_G, G)$ we obtain the element $\sigma_{M,G} \in \Omega(\widehat{T}, \widehat{G})$. The set of simple roots $\Delta(\widehat{T}, \widehat{G})$ is no longer stable under the twisted action $\sigma_{M,G} \rtimes \sigma_G$, but its subset $\Delta(\widehat{T}, \widehat{M})$ is. Thus $\widehat{M}$ is stable under the twisted action $n(\sigma_{M,G}) \rtimes \sigma_G$, where $n(\dots)$ is the Tits lift relative to the chosen pinning. By [Spr81, 11.2.11] this action preserves the pinning of $\widehat{M}$ inherited from that of $\widehat{G}$ and hence would give an L-action if the map $\sigma \mapsto n(\sigma_{M,G}) \rtimes \sigma_G$ were multiplicative. Thus we are looking for a 1-cochain q of Γ or W, valued in $Z(\widehat{M})$, so that $q(\sigma)n(\sigma_{M,G}) \rtimes \sigma$ is multiplicative. Since q takes values in $Z(\widehat{M})$ it acts trivially on the pinning of $\widehat{M}$, so $q(\sigma)n(\sigma_{M,G}) \rtimes \sigma$ still preserves that pinning. Being in addition multiplicative, we see that

$$\widehat{M} \rtimes W_F \to \widehat{G} \rtimes W_F, \qquad (m, w) \mapsto m \cdot q(w)n(\sigma_{M,G}) \rtimes \sigma_G \qquad (6.1) \quad \{\texttt{eq:tlemb}\}$$

is an L-embedding, where $\sigma \in \Gamma$ is the image of w.

We will now show that q can be constructed explicitly by choosing χ-data for $R(M_{\mathrm{ab}}, G)$. The torus dual to M_{ab} is the complex torus $Z(\widehat{M})^\circ$ equipped with the action of Γ where $\sigma \in \Gamma$ acts as $\sigma_{M,G} \rtimes \sigma_G$. Via the isomorphism dual to $\mathrm{Ad}(g) : T_G \to T_M$ and the identification $\widehat{T}_G = \widehat{T}$, the set $R(M_{\mathrm{ab}}, G)$ is identified with the set of $\Omega(\widehat{T}, \widehat{M})$-orbits in $R^\vee(\widehat{T}, \widehat{G}) \smallsetminus R^\vee(\widehat{T}, \widehat{M})$.

The failure of multiplicativity of $\sigma \mapsto n(\sigma_{M,G}) \rtimes \sigma_G$ is measured by the Tits cocycle $t_p(\sigma, \tau) \in \widehat{T}$:

$$n(\sigma_{M,G}) \rtimes \sigma_G \cdot n(\tau_{M,G}) \rtimes \tau_G = t_p(\sigma, \tau) \cdot n((\sigma\tau)_{M,G}) \rtimes (\sigma\tau)_G,$$

given by $t_p(\sigma, \tau) = \lambda_{\sigma,\tau}(-1)$, where $\lambda_{\sigma,\tau}$ is the sum of elements of $R^\vee(\widehat{T}, \widehat{G}) = R(T_G, G)$ in the set

$$\Lambda_{\sigma,\tau} = \{\alpha > 0, (\sigma_{M,G}\sigma_G)^{-1}\alpha < 0, (\sigma_{M,G}\sigma_G\tau_{M,G}\tau_G)^{-1}\alpha > 0\},$$

where $\alpha > 0$ is taken with respect to the Borel subgroup $\widehat{B}$. Let $p : R^\vee(\widehat{T}, \widehat{G}) \smallsetminus R^\vee(\widehat{T}, \widehat{M}) \to \{\pm 1\}$ denote this gauge, and let $p' : R(M_{\mathrm{ab}}, G) \to \{\pm 1\}$ be the induced gauge as in Lemma 6.1, part 4.

$\{\texttt{lem:lambdalevi}\}$

Lemma 6.6. $\lambda_{\sigma,\tau}$ is equal to the sum of $\bar{\alpha}'$ for all α' belonging to the following subset

$$\Lambda'_{\sigma,\mu} = \{\alpha' > 0, (\sigma_{M,G}\sigma_G)^{-1}\alpha' < 0, (\sigma_{M,G}\sigma_G\tau_{M,G}\tau_G)^{-1}\alpha' > 0\}$$

of $R(M_{\mathrm{ab}}, G)$, where $\alpha' > 0$ means $p'(\alpha') > 0$, and $\bar{\alpha}' \in X^*(M_{\mathrm{ab}}) = X_*(Z(\widehat{M})^\circ)$ denotes the image of α' under the map $R(M_{\mathrm{ab}}, G) \to X^*(M_{\mathrm{ab}})$ of Definition 6.2. $\qquad\square$

Proof. The action of $\sigma_{M,G}\sigma_G$ preserves the root system $R(\widehat{T}, \widehat{M})$ and the set of positive roots in it. Therefore the set $\Lambda_{\sigma,\tau}$ lies in $R(\widehat{T}, \widehat{G}) \smallsetminus R(\widehat{T}, \widehat{M})$ and is invariant under $\Omega(\widehat{T}, \widehat{M})$. This means that the sum of the elements of $\Lambda_{\sigma,\tau}$ can be computed by first summing the elements of each individual $\Omega(\widehat{T}, \widehat{M})$-orbit, and then summing the contributions of those orbits. $\qquad\blacksquare$

{cor:lambdalevi}

Corollary 6.7. The Tits cocycle t_p takes values in $Z(\widehat{M})^\circ$. Given χ-data χ' for $R(M_{\mathrm{ab}}, G)$, the differential of the 1-cochain $r_{p',\chi'}$ equals t_p. Therefore, $q = r_{p',\chi'}$ makes (6.1) into an L-embedding. $\qquad\square$

Proof. Immediate from Lemmas 6.6 and 3.19. $\qquad\blacksquare$

{rem:inf1}

Remark 6.8. Let $R(Z(M)^\circ, G) \subset X^*(Z(M)^\circ)$ be the set of non-zero weights for the action of $Z(M)^\circ$ on the Lie-algebra of G. It is a finite admissible Σ-set and equals, for any maximal torus $S \subset M$, the image of $R(S, G) \smallsetminus R(S, M) \subset X^*(S)$ under the restriction map $X^*(S) \to X^*(Z(M)^\circ)$. This restriction map is $\Omega(S, M)$-invariant and therefore we obtain a surjective Σ-equivariant map $R(M_{\mathrm{ab}}, G) \to R(Z(M)^\circ, G)$. This map need not be injective, as we will see in §6.4. A set of χ-data for $R(Z(M)^\circ, G)$ can be inflated to a set of χ-data for $R(M_{\mathrm{ab}}, G)$ as in Definition 5.16. In this way, a set of χ-data for $R(Z(M)^\circ, G)$ leads, via Corollary 6.7, to an L-embedding $^LM \to {}^LG$. $\qquad\square$

6.2 Factoring the L-embedding of a maximal torus through a twisted Levi subgroup

We continue with a twisted Levi subgroup $M \subset G$ and a maximal torus $S \subset M$. Consider given a set of χ-data χ' for $R(M_{\mathrm{ab}}, G)$. We inflate it, as in Definition 5.16, to a set of χ-data $\chi = \inf \chi'$ for $R(S, G) \smallsetminus R(S, M)$ via the natural surjection $R(S, G) \smallsetminus R(S, M) \to R(M_{\mathrm{ab}}, G)$. Choose a set of χ-data for $R(S, M)$, thereby obtaining a set of χ-data for $R(S, G)$. Let $^Lj_{S,G} : {}^LS \to {}^LG$, $^Lj_{S,M} : {}^LS \to {}^LM$, and $^Lj_{M,G} : {}^LM \to {}^LG$ be the L-embeddings obtained from these sets of χ-data. Each L-embedding is well-defined up to conjugation (by $\widehat{M}$ or $\widehat{G}$ respectively, depending on its target).

{pro:lembfact}

Proposition 6.9. Up to conjugation by $\widehat{G}$ we have

$$^Lj_{M,G} \circ {}^Lj_{S,M} = {}^Lj_{S,G}.$$

$\qquad\square$

Proof. Since S transfers to the quasi-split form of M we reduce by Lemma 6.4 to the case that both G and M are quasi-split. Choose Borel pairs (T_M, B_M) and (T_G, B_G), an element $g \in G$ with $\mathrm{Ad}(g)T_G = T_M$ and $\mathrm{Ad}(g)B_G \supset B_M$, and an element $m \in M$ with $\mathrm{Ad}(m)T_M = S$. This gives elements $\sigma_{S,G} = g^{-1}m^{-1}\sigma(m)\sigma(g) \in \Omega(T_G, G)$, $\sigma_{S,M} = m^{-1}\sigma(m) \in \Omega(T_M, M)$, and $\sigma_{M,G} = g^{-1}\sigma(g) \in \Omega(T_G, G)$. Thus $\sigma_{S,G} = g^{-1}\sigma_{S,M}g \cdot \sigma_{M,G}$. We identify $\sigma_{S,M}$ with $g^{-1}\sigma_{S,M}g \in \Omega(T_G, G)$ and then have $\sigma_{S,G} = \sigma_{S,M} \cdot \sigma_{M,G}$. As in §6.1 we fix a pinning $(\widehat{T}, \widehat{B}, \{X_{\alpha^\vee}\})$ for $\widehat{G}$ and let $\widehat{M}$ be the unique standard Levi subgroup in the $\widehat{G}$-conjugacy class determined by M.

Write $^L j_{M,G}(w) = r_{M,G}(w)n(\sigma_{M,G}) \rtimes \sigma_G$, $^L j_{S,G}(w) = r_{S,G}(w)n(\sigma_{S,G}) \rtimes \sigma_G$, and $^L j_{S,M}(w) = r_{S,M}(w)n(\sigma_{S,M}) \rtimes \sigma_M$. Here $r_{S,G}$ and $r_{M,G}$ are the 1-cochains $r_{p,\chi}$ of Lemma 3.19 for the gauge p given by $\widehat{B}$ and the fixed χ-data for $R(S,G)$ and $R(S,M)$, respectively, while $r_{M,G}$ is given by the same Lemma, but for the χ-data for $R(M_{\mathrm{ab}}, G)$, the map $R(M_{\mathrm{ab}}, G) \to X^*(M_{\mathrm{ab}})$, and the gauge p' on $R(M_{\mathrm{ab}}, G)$ induced by p as in Lemma 6.1.

We have $^L j_{M,G} \circ {}^L j_{S,M}(w) = r_{S,M}(w)n(\sigma_{S,M})r_{M,G}(w)n(\sigma_{M,G}) \rtimes \sigma_G$. Recalling that $r_{M,G}$ takes values in $Z(\widehat{M})^\circ$ we can move it past $n(\sigma_{S,M})$. Furthermore, since $\sigma_{S,M}$ preserves the set of positive roots in $R(T_G, G) \smallsetminus R(T_G, M^g)$, while $\sigma_{M,G}\sigma_G$ preserves the set of positive roots in $R(T_G, M^g)$, with $M^g = g^{-1}Mg$, [LS87, Lemma 2.1.A] implies that $n(\sigma_{S,M})n(\sigma_{M,G}) = n(\sigma_{S,M}\sigma_{M,G}) = n(\sigma_{S,G})$. We conclude $^L j_{M,G} \circ {}^L j_{S,M}(w) = r_{S,M}(w)r_{M,G}(w)n(\sigma_{S,G}) \rtimes \sigma_G$. The two L-embeddings $^L j_{M,G} \circ {}^L j_{S,M}(w)$ and $^L j_{S,G}$ thus differ by the 1-cocycle $r_{S,M}(w)r_{M,G}(w)r_{S,G}(w)^{-1}$ and our goal is to show that it is cohomologically trivial.

Recall that $r_{S,G}(w)$ is a product taken over the Σ-orbits in $R(S,G)$. The 1-cochain $r_{S,M}(w)$ is a sub-product, taken over those Σ-orbits that lie in $R(S,M)$. Therefore $r_{S,G}(w)r_{S,M}(w)^{-1}$ equals the 1-cochain of Lemma 3.19 for the torus S, the finite admissible Σ-set $R(S,G) \smallsetminus R(S,M) \subset X^*(S)$, and the χ-data that by construction is inflated from the χ-data for $R(M_{\mathrm{ab}}, G)$. On the other hand, $r_{M,G}(w)$ is the analogous 1-cochain for the torus M_{ab}, the finite admissible Σ-set $R(M_{\mathrm{ab}}, G)$, the Σ-equivariant map $R(M_{\mathrm{ab}}, G) \to X^*(M_{\mathrm{ab}})$, and the chosen χ-data for $R(M_{\mathrm{ab}}, G)$. The gauges on $R(M_{\mathrm{ab}}, G)$ and $R(S,G) \smallsetminus R(S,M)$ are compatible with the natural projection map $f : R(S,G) \smallsetminus R(S,M) \to R(M_{\mathrm{ab}}, G)$. This natural projection map satisfies (5.1). Proposition 5.23 applied to the epimorphism $S \to M_{\mathrm{ab}}$ asserts that $r_{M,G}$ is cohomologous to $r_{S,G} \cdot r_{S,M}^{-1}$. $\blacksquare$

6.3 The canonical double cover of a twisted Levi subgroup

Let $M \subset G$ be a twisted Levi subgroup. The finite admissible Σ-set $R(M_{\mathrm{ab}}, G)$ and the Σ-equivariant map $R(M_{\mathrm{ab}}, G) \to X^*(M_{\mathrm{ab}})$ of Definition 6.2 lead to the double cover $M_{\mathrm{ab}}(F)_\pm$ of $M_{\mathrm{ab}}(F)$.

Definition 6.10.   1. The double cover $M(F)_\pm$ is the pull-back of $M(F) \to M_{\mathrm{ab}}(F) \leftarrow M_{\mathrm{ab}}(F)_\pm$.

  2. The L-group $^L M_\pm$ of $M(F)_\pm$ is the push-out of $\widehat{M} \leftarrow Z(\widehat{M})^\circ \to {}^L M_{\mathrm{ab},\pm}$.

$\square$

Lemma 6.11. Let $S \subset M$ be a maximal torus. Let $S(F)_\pm$ be the double cover for the finite set $R(S,G) \smallsetminus R(S,M) \subset X^*(S)$. The inclusion $S(F) \to M(F)$ lifts naturally to a homomorphism $S(F)_\pm \to M(F)_\pm$. $\square$

Proof. By definition the diagram

$$
\begin{array}{ccc}
R(M_{\mathrm{ab}}, G) & \longrightarrow & R(S,G) \smallsetminus R(S,M) \\
\downarrow & & \downarrow \\
X^*(M_{\mathrm{ab}}) & \longrightarrow & X^*(S)
\end{array}
$$

satisfies (5.1). Constructions 5.5 and 5.20 produces a lifting $S(F)_\pm \to M_{\mathrm{ab}}(F)_\pm$ of the natural map $S(F) \to M_{\mathrm{ab}}(F)$. The map $S(F)_\pm \to M_{\mathrm{ab}}(F)_\pm \to M_{\mathrm{ab}}(F)$ is equal to the map $S(F)_\pm \to S(F) \to M(F) \to M_{\mathrm{ab}}(F)$, so these two maps induce the desired map $S(F)_\pm \to M(F)_\pm$. $\blacksquare$

{lem:lembfact}

Lemma 6.12. Let $S \subset M$ be a maximal torus. Let $S(F)_\pm$ be the double cover for the finite set $R(S,G) \subset X^*(S)$.

1. The canonical $\widehat{G}$-conjugacy class of embeddings $\widehat{M} \to \widehat{G}$ extends canonically to $\widehat{G}$-conjugacy class of L-embeddings $^L M_\pm \to {}^L G$.

2. The canonical $\widehat{M}$-conjugacy class of embeddings $\widehat{S} \to \widehat{M}$ extends canonically to a $\widehat{M}$-conjugacy class of L-embeddings $^L S_\pm \to {}^L M_\pm$.

3. The $\widehat{G}$-conjugacy class of L-embeddings $^L S_\pm \to {}^L G$ factors as the composition $^L S_\pm \to {}^L M_\pm \to {}^L G$ of the two L-embeddings described in the preceding two points.

$\square$

Proof. In this proof we will use the same set-up as in the proof of Proposition 6.9, so we refer the reader to the first paragraph of that proof rather than repeating it.

Let $t_{M,G}$, $t_{S,M}$, and $t_{S,G}$ be the Tits cocycles for the maps sending σ to $n(\sigma_{M,G})$, $n(\sigma_{S,M})$, and $n(\sigma_{S,G})$, respectively. The embeddings $^L M_\pm \to {}^L G$, $^L S_\pm \to {}^L M_\pm$, and $^L S_\pm \to {}^L G$ are then given by

$$
\widehat{M} \boxtimes_{t_{M,G}} \Gamma \to \widehat{G} \rtimes \Gamma, m \boxtimes \sigma \mapsto m \cdot n(\sigma_{M,G}) \rtimes \sigma_G,
$$

$$
\widehat{S} \boxtimes_{t_{S,G}} \Gamma \to \widehat{M} \boxtimes_{t_{M,G}} \Gamma, s \boxtimes \sigma \mapsto s \cdot n(\sigma_{S,M}) \boxtimes \sigma_G,
$$

$$
\widehat{S} \boxtimes_{t_{S,G}} \Gamma \to \widehat{G} \rtimes \Gamma, s \boxtimes \sigma \mapsto s \cdot n(\sigma_{S,G}) \rtimes \sigma_G.
$$

The first formula is the same as (6.1), but with the 1-cochain q removed. The other two are as in (4.1).

The fact that the first of these is indeed a group homomorphism follows from the computation $t_{M,G}(\sigma,\tau) = (-1)^{\lambda_{\sigma,\tau}}$ with $\lambda_{\sigma,\tau}$ as in §6.1 and Lemma 6.6. This embedding depends only on the choice of pinning of $\widehat{G}$, and since two pinnings are $\widehat{G}$-conjugate, so are the resulting L-embeddings. The fact that the third is a group homomorphism was already discussed in §4.1. The fact that the second is a group homomorphism follows from the same argument combined with the observation $t_{S,G} = t_{S,M} \cdot t_{M,G}$. This observation follows from the equation $n(\sigma_{S,M}) \cdot n(\sigma_{M,G}) = n(\sigma_{S,G})$ also recorded in the proof of Proposition 6.9. That latter equation also implies that the composition of the first two embeddings equals the third embedding. ∎

Fact 6.13. The Weil forms of $^{L}M_{\pm}$ and ^{L}M are non-canonically isomorphic. More precisely, let $(\chi_{\alpha'})$ be a set of χ-data for $R(M_{\mathrm{ab}}, G)$ and let $r_{M,G}$ be the corresponding 1-cochain as in §6.1. Then

$$\widehat{M} \rtimes W \to \widehat{M} \boxtimes_{t_{M,G}} W, \qquad m \rtimes w \mapsto m \cdot r_{M,G}(w) \boxtimes w$$

is an L-isomorphism. □

Remark 6.14. Assume given a bijection between L-parameters $\varphi : W' \to {}^{L}M$ and L-packets for $M(F)$. Assume further this bijection is compatible with twists by co-central characters. Then one obtains a bijection between L-parameters $\varphi : W' \to {}^{L}M_{\pm}$ and L-packets of genuine representations of $M(F)_{\pm}$ as follows: Choose a set of χ-data for $R(M_{\mathrm{ab}}, G)$. We obtain a genuine character χ_M of $M_{\mathrm{ab}}(F)_{\pm}$ which we can pull back to a genuine character of $M(F)_{\pm}$. Via the isomorphism of Fact 6.13 we obtain from a given $\varphi : W' \to {}^{L}M_{\pm}$ an L-packet Π_{φ} on $M(F)$. Tensoring each member of Π_{φ} with χ_M we obtain a genuine representation of $M(F)_{\pm}$. The set of these forms the L-packet $\Pi_{\varphi}^{\pm}$ on $M(F)_{\pm}$. The compatibility of the assumed correspondence for M with co-central characters implies that $\Pi_{\varphi}^{\pm}$ depends only on φ and not on the chosen χ-data. □

Remark 6.15. Assume that there is a correspondence assigning to a pair (S, θ) of a maximal torus $S \subset G$ and a genuine character $\theta : S(F)_{\pm} \to \mathbb{C}^{\times}$, subject to certain conditions, a representation π of $G(F)$, as suggested in §4.3. Assume further that the same holds for M. The factorization of L-embeddings of Lemma 6.12 suggests that there should be a parallel correspondence that assings to a pair (S, θ), where $S \subset G$ is a maximal torus that lies in M and θ is a genuine character of $S(F)_{\pm}$, a genuine representation of the double dover $M(F)_{\pm}$.

This is easy to see formally. Since we are now dealing with different Σ-invariant finite sets we will use them as subscripts to distinguish the various double covers. Thus we have a maximal torus $S \subset M$ and a genuine character $\theta : S(F)_{R(S,G)} \to \mathbb{C}^{\times}$ and want to assign to them a genuine representation $\pi_{\pm}$ of $M(F)_{R(M_{\mathrm{ab}},G)}$. Such representations are classified as follows. Choose a genuine character $\chi : M_{\mathrm{ab}}(F)_{R(M_{\mathrm{ab}},G)} \to \mathbb{C}^{\times}$ and inflate it to $M(F)_{R(M_{\mathrm{ab}},G)}$. Then $\pi := \pi_{\pm} \otimes \chi^{-1}$ is a representation inflated from $M(F)$. The pair (π, χ)

determines $\pi_{\pm}$, but is not determined by it. Another pair determining the same $\pi_{\pm}$ is of the form $(\pi \otimes \delta, \chi \delta^{-1})$ for a character $\delta : M_{\mathrm{ab}}(F) \to \mathbb{C}^{\times}$. Now say that π corresponds to (S, θ_0) with $\theta_0 : S(F)_{R(S,M)} \to \mathbb{C}^{\times}$ genuine. Then $\pi_{\pm}$ is determined, and conversely determines, the torus S and a geniune character of the pull-back of

$$S(F)_{R(S,M)} \to S(F) \to M_{\mathrm{ab}}(F) \leftarrow M_{\mathrm{ab}}(F)_{R(M_{\mathrm{ab}},G)}.$$

This pull-back is an extension of $S(F)$ by $\{\pm1\}^2$ and a genuine character is one that maps both $(-1, 1)$ and $(1, -1)$ to -1. According to Fact 5.2 this pull-back is canonically isomorphic to the pull-back of

$$S(F)_{R(S,M)} \to S(F) \leftarrow S(F)_{R(S,G) \smallsetminus R(S,M)},$$

whose quotient by the diagonal copy of $\{\pm1\}$ equals $S(F)_{R(S,G)}$. $\qquad\square$

6.4 Examples

We will now show by example that the maps $R(M_{\mathrm{ab}}, G) \to X^*(M_{\mathrm{ab}})$ and $R(M_{\mathrm{ab}}, G) \to R(Z(M)^\circ, G)$ need not be injective.

Consider the group $G = \mathrm{PSp}_4$. Note that $Z(M)$ is connected. The root system $R(S, G) \subset X^*(S) = \mathbb{Z}^2$ consists of $\{e_1, e_2, e_2 - e_1, e_2 + e_1\}$ and their negatives. The simple roots for this choice of positive roots are $\{e_1, e_2 - e_1\}$.

Let first $R(S, M) = \{e_1, -e_1\}$. Then $X^*(Z(M)) = \mathbb{Z}$ and the map $X^*(S) \to X^*(Z(M))$ sends e_1 to 0 and e_2 to 1. In particular, $R(Z(M)) = \{1, -1\}$. The action of $\Omega(S, M)$ on the positive part of $R(S, G) \smallsetminus R(S, M)$ has the orbits $\{e_2\}$ and $\{e_2 - e_1, e_2 + e_1\}$. These orbits and their negatives are the elements of $R(M_{\mathrm{ab}}, G)$ and we see that the map $R(M_{\mathrm{ab}}, G) \to R(Z(M), G)$ sends both positive orbits to 1, hence is not injective. On the other hand, $X^*(M_{\mathrm{ab}}) = \langle e_2 \rangle$ and the map $R(M_{\mathrm{ab}}, G) \to X^*(M_{\mathrm{ab}})$ sends $\{e_2\}$ to e_2 and $\{e_2 - e_1, e_2 + e_1\}$ to $2e_2$, and is thus injective.

Let now $R(S, M) = \{e_2 - e_1, e_1 - e_2\}$. Then $X^*(Z(M)) = \mathbb{Z}$ and the map $X^*(S) \to X^*(Z(M))$ sends both e_1 and e_2 to 1. Now $R(Z(M)) = \{1, 2, -1, -2\}$. The action of $\Omega(S, M)$ on the positive part of $R(S, G) \smallsetminus R(S, M)$ has the orbits $\{e_1, e_2\}$ and $\{e_1 + e_2\}$. These orbits and their negatives are the elements of $R(M_{\mathrm{ab}}, G)$ and we see that the map $R(M_{\mathrm{ab}}, G) \to R(Z(M), G)$ sends $\{e_1, e_2\}$ to 1 and $\{e_1 + e_2\}$ to 2, thus it is injective. On the other hand, $X^*(M_{\mathrm{ab}}) = \langle e_1 + e_2 \rangle$ and the map $R(M_{\mathrm{ab}}, G) \to X^*(M_{\mathrm{ab}})$ sends both $\{e_1, e_2\}$ and $\{e_1 + e_2\}$ to $e_1 + e_2$, and is thus not injective.

Appendix

A   FROBENIUS RECIPROCITY

Let $\Delta \subset \Gamma$ be two finite groups, X a $\mathbb{Z}[\Delta]$-module, and Y and $\mathbb{Z}[\Gamma]$-module. We consider

$$\mathrm{Ind}_{\Delta}^{\Gamma} X = \{f : \Gamma \to X | f(\delta\gamma) = \delta f(\gamma)\}.$$

We have the Δ-equivariant maps

$$X \to \mathrm{Ind}_\Delta^\Gamma X \to X,$$

the first sending $x \in X$ to

$$f_x(\gamma) = \begin{cases} \gamma x & , \gamma \in \Delta \\ 0 & , \gamma \notin \Delta \end{cases}$$

and the second sending f to $f(1)$. Their composition is the identity. In particular, the first is injective and the second surjective.

The first Frobenius reciprocity isomorphism is

$$\mathrm{Hom}_\Gamma(Y, \mathrm{Ind}_\Delta^\Gamma X) \leftrightarrow \mathrm{Hom}_\Delta(Y, X).$$

An element Φ on the left is related to an element φ on the right by $\varphi(y) = \Phi(y)(1)$ and $\Phi(y)(\gamma) = \varphi(\gamma y)$.

The second Frobenius reciprocity isomorphism is

$$\mathrm{Hom}_\Gamma(\mathrm{Ind}_\Delta^\Gamma X, Y) \leftrightarrow \mathrm{Hom}_\Delta(X, Y).$$

An element Φ on the left is related to an element φ on the right by $\varphi(x) = \Phi(f_x)$ and $\Phi(f) = \sum_{\tau \in \Delta \backslash \Gamma} \tau^{-1} \varphi(f(\tau))$. Note that each summand depends only on the coset τ, not an individual representative of it, and is thus well-defined.

In checking that the maps of the second Frobenius reciprocity isomorphism are inverse to each other, the following identity is used: $\gamma^{-1} f_{f(\gamma)} = f \cdot \mathbf{1}_{\Delta\gamma}$.

There is a different presentation for $\mathrm{Ind}_\Delta^\Gamma X$. Choose a set of representatives $(\gamma_1, \ldots, \gamma_k)$ for $\Delta \backslash \Gamma$. Then

$$\mathrm{Ind}_\Delta^\Gamma X \to X^{[\Gamma:\Delta]}, \qquad f \mapsto (f(\gamma_1), \ldots, f(\gamma_k))$$

is an isomorphism of $\mathbb{Z}$-modules. To see what Γ-action on the target makes it equivariant, introduce the function

$$\{1, \ldots, k\} \times \Gamma \to \Delta \times \{1, \ldots, k\}, \qquad (i, \gamma) \mapsto (\gamma_i(\gamma), i_\gamma), \qquad \gamma_i \cdot \gamma = \gamma_i(\gamma) \cdot \gamma_{i_\gamma}$$

and then see that the action of Γ on the right is by

$$\gamma \cdot (x_1, \ldots, x_k) = (\gamma_1(\gamma) \cdot x_{1_\gamma}, \ldots, \gamma_k(\gamma) \cdot x_{k_\gamma}).$$

B Comparison with groups of type L

{app:glt}

In this appendix we review and reinterpret material from a set of informal notes called "groups of type L" by Gross, that were created in 2008 and circulated privately. The aim is to generalize the construction of Adams-Vogan of the "universal extension" of a real torus, its various E-groups and the associated local Langlands correspondences [AV92, §5]. We will see that this works under a rather strict assumption on the torus and that some important special cases which satisfy this assumption can be used to shed new light on the computations of Langlands-Shelstad that we have used in this paper.

B.1 Groups of type L

We begin by reviewing the material from Gross' notes. This material has also been reviewed in [AR16, §5].

Consider an algebraic torus S defined over F with complex dual torus $\widehat{S}$. Let E/F be the splitting extension of S. In this subsection we make the following

Assumption B.1. The norm map $S(E) \to S(F)$ is surjective. $\qquad\square$

This leads to the exact sequence

$$1 \to \widehat{H}^{-1}(\Gamma_{E/F}, S(E)) \to \frac{S(E)}{IS(E)} \to S(F) \to 1, \qquad (\text{B.1})$$

where $IS(E) = \langle x^{-1}\sigma(x) | x \in S(E), \sigma \in \Gamma_{E/F} \rangle$ is the augmentation submodule of $S(E)$ for the action of $\Gamma_{E/F}$.

Gross' observation is that characters of $S(E)/IS(E)$ are parameterized by "Langlands parameters" valued in the various extensions of $\Gamma_{E/F}$ by $\widehat{S}$. More precisely, we have the following

Proposition B.2. 1. There is a natural bijection between $H^2(\Gamma_{E/F}, \widehat{S})$ and the Pontryagin dual of the finite abelian group $\widehat{H}^{-1}(\Gamma_{E/F}, S(E))$.

2. Let $c : \widehat{H}^{-1}(\Gamma_{E/F}, S(E)) \to \mathbb{C}^\times$ be a character corresponding to an element of $H^2(\Gamma_{E/F}, \widehat{S})$ and let

$$1 \to \widehat{S} \to \mathcal{S} \to \Gamma_{E/F} \to 1. \qquad (\text{B.2})$$

be an extension corresponding to that element. There is a natural bijection between the set of $\widehat{S}$-conjugacy classes of continuous L-homomorphisms $W \to \mathcal{S}$ and the set of c-genuine continuous characters of $S(E)/IS(E)$.

$\qquad\square$

Remark B.3. The extension (B.2) is is what Gross calls a *group of type L*, and what Adams-Vogan call an E-group in [AV92, Definition 5.9]. $\qquad\square$

Proof. We first claim that the sequence

$$0 \to H^1(W_F, \widehat{S}) \to H^1(W_E, \widehat{S})^{\Gamma_{E/F}} \to H^2(\Gamma_{E/F}, \widehat{S}) \to 0 \qquad (\text{B.3})$$

is exact, where the first map is restriction, and the second is transgression. Indeed, this is a piece of the inflation-restriction exact sequence for $W_E \subset W_F$, which gives exactness in the middle. Exactness on the right is due to [Lan79, Lemma 4]. Exactness on the left is due restriction being dual to the norm $S(E) \to S(F)$, which is assumed surjective.

Since the first map of (B.3) is dual to the second map in (B.1), we conclude that the third term in (B.3) is dual to the first term in (B.1). This proves the first part of the proposition.

The second part of the proposition follows immediately from the fact that the fiber of the second map in (B.3) over a given class in $H^2(\Gamma_{E/F}, \widehat{S})$ is in natural bijection with the set of L-homomorphisms $W_F \to \mathcal{S}$, where $\mathcal{S}$ is the extension (B.2) corresponding to that class. We can be more explicit: Given a continuous $\Gamma_{E/F}$-equivariant homomorphism $\varphi_E : W_E \to \widehat{S}$ choose arbitrarily an extension of φ_E to a continuous map $\varphi_F : W_F \to \widehat{S}$ s.t. $\varphi_F(ww') = \varphi_E(w)\varphi_F(w')$ for $w \in W_E$ and $w' \in W_F$. Let $z \in Z^2(\Gamma_{E/F}, \widehat{S})$ be its differential. It represents the image of φ_E under the transgression map. Then $w \mapsto \varphi_F(w) \boxtimes w$ is a homomorphism from W_F to the twisted product $\widehat{S} \boxtimes_z \Gamma_{E/F}$, which is an extension of $\Gamma_{E/F}$ by $\widehat{S}$ in the isomorphism class specified by the cohomology class of z. Regarding well-definedness, please note the following remark. ∎

Remark B.4. A subtle but important point emphasized in [AV92] is that the L-group of a torus S is not simply a split extension of $\widehat{S}$ by Γ, but rather such an extension equipped with an $\widehat{S}$-conjugacy class of splittings. Similarly, an E-group in their terminology is not just an extension of $\widehat{S}$ by Γ, but one equipped with a $\widehat{S}$-conjugacy class of set-theoretic splittings. Our assumption on the surjectivity of the norm $S(E) \to S(F)$ is equivalent to the vanishing of $\widehat{H}^0(\Gamma_{E/F}, S(E))$, which by [Tat66] is isomorphic to $\widehat{H}^{-2}(\Gamma_{E/F}, X_*(S))$, which is dual to $H^2(\Gamma_{E/F}, X_*(\widehat{S}))$, which is isomorphic to $H^1(\Gamma_{E/F}, \widehat{S})$. Therefore, there is a unique $\widehat{S}$-conjugacy class of set-theoretic splittings for any extension of $\Gamma_{E/F}$ by $\widehat{S}$. Equivalently, specifying an isomorphism class of an extension is the same as specifying the extension itself up to $\widehat{S}$-conjugacy. □

Remark B.5. There is an alternative way to specify an isomorphism between $H^2(\Gamma_{E/F}, \widehat{S})$ and the dual of $\widehat{H}^{-1}(\Gamma_{E/F}, S(E))$, namely by composing the isomorphism $H^2(\Gamma_{E/F}, \widehat{S}) \to H^3(\Gamma_{E/F}, X_*(\widehat{S}))$ coming from the exponential sequence with the duality between $H^3(\Gamma_{E/F}, X_*(\widehat{S}))$ and $\widehat{H}^{-3}(\Gamma_{E/F}, X_*(S))$ and the isomorphism $\widehat{H}^{-3}(\Gamma_{E/F}, X_*(S)) \to \widehat{H}^{-1}(\Gamma_{E/F}, S(E))$ due to [Tat66]. We leave it to the reader to compare this identification with the one of Proposition B.2. □

Remark B.6. Consider the case $F = \mathbb{R}$. Given a torus S we have the exponential exact sequence

$$0 \to X_*(S) \to X_*(S) \otimes_{\mathbb{Z}} \mathbb{C} \to X_*(S) \otimes_{\mathbb{Z}} \mathbb{C}^{\times} \to 1$$

which realizes $X_*(S) \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{Lie}(S)$ as the universal cover of $S(\mathbb{C})$ and $X_*(S)$ as the fundamental group. Adams and Vogan consider [AV92, (5.3b)]

$$S(\mathbb{R})^{\sim} = \{x \in X_*(S) \otimes_{\mathbb{Z}} \mathbb{C} \,|\, \exp(2\pi i x) \in S(\mathbb{R})\}/(1-\sigma)X_*(S)$$

which fits into the exact sequence

$$0 \to \frac{X_*(S)}{(1 - \sigma)X_*(S)} \to S(\mathbb{R})^\sim \to S(\mathbb{R}) \to 1$$

that they call "the universal extension" of $S(\mathbb{R})$. Note that we are using the map $\exp(2\pi i x)$, while they are using the map $\exp(x)$, which accounts for the slight disagreement of notation. They show that an extension

$$1 \to \widehat{S} \to \mathcal{S} \to \Gamma \to 1$$

equipped with a set-theoretic splitting s gives a character c_s of $X_*(S)/(1 - \sigma)X_*(S)$, and c_s-genuine characters of $S(\mathbb{R})^\sim$ are in 1-1 correspondence with L-parameters valued in $\mathcal{S}$.

If S is an anisotropic torus then the norm map $S(\mathbb{C}) \to S(\mathbb{R})$ is surjective, so Proposition B.2 can be applied. Furthermore $\widehat{H}^{-1}(\Gamma, S) = H^1(\Gamma, S) = H^{-1}(\Gamma, X_*(S)) = H_0(\Gamma, X_*(S)) = X_*(S)/(1 - \sigma)X_*(S)$, where we have used the periodicity of cohomology for cyclic groups, the Tate isomorphism, and the anisotropicity of S. In this way Proposition B.2 recovers the construction of Adams-Vogan.

If S is split then the norm map $S(\mathbb{C}) \to S(\mathbb{R})$ is no longer surjective, so Proposition B.2 does not apply. We have $H^2(\Gamma, \widehat{S}) = 0$, so up to equivalence there is no non-trivial extension of Γ by $\widehat{S}$. But if we consider an actual such extension equipped with a set-theoretic splitting we obtain not just an element of $H^2(\Gamma, \widehat{S})$, but an element of $Z^2(\Gamma, \widehat{S})$. A very special feature of $F = \mathbb{R}$ is that Γ is finite cyclic with canonical generator, so there is a canonical element of $Z^2(\Gamma, \mathbb{Z})$ and cup product with this element gives an isomorphism $Z^0(\Gamma, \widehat{S}) \to Z^2(\Gamma, \widehat{S})$. Now $Z^0(\Gamma, \widehat{S}) = H^0(\Gamma, \widehat{S})$ is dual to $H_0(\Gamma, X_*(S)) = X_*(S)$ and hence we obtain a character of $X_*(S)$. The universal extension is

$$0 \to X_*(S) \to S(\mathbb{R})^\sim \to S(\mathbb{R}) \to 1.$$

Thus the construction of Adams-Vogan and the construction presented here diverge.

It is not clear if the construction of Adams-Vogan can be performed in the p-adic case. The crucial isomorphism $Z^2(\Gamma, \widehat{S}) \to Z^0(\Gamma, \widehat{S})$, which relied on the fact that Γ is cyclic with canonical generator, is no longer given. $\qquad\square$

{exa:nns}

Example B.7. We give here an example of an anisotropic torus over $F = \mathbb{Q}_p$ for which the norm map $S(E) \to S(F)$ is not surjective, where E/F is the splitting extension.

Consider $F = \mathbb{Q}_p$ with $4|(p - 3)$. The unramified quadratic extension of F is generated by a fourth root of unity ζ_4. Choose a fourth root ϖ of p and let $E_\pm = \mathbb{Q}_p(\sqrt{p})$, $E_+ = \mathbb{Q}_p(\varpi)$ and $E = \mathbb{Q}_p(\varpi, \zeta_4)$. Then $E_\pm/F$ is ramified quadratic, $E_+/E_\pm$ is also ramified quadratic, and E is the normal closure of E_+. Let S be the restriction of scalars from $E_\pm$ to F of the one-dimensional

anisotropic torus defined over $E_\pm$ and split over E_+. Then S is anisotropic and its splitting extension is E/F.

The map $x/\bar{x}$ from E_+ to E_+^1 is surjective and the preimage of -1 consists of elements of E_+ of odd valuation. No such element is in the image of the norm map $E \to E_+$. Since $S(E)$ is a product of copies of $E^\times$ and the norm map $S(E) \to S(F)$ is made of compositions of the norm map $E \to E_+$ and the map $E_+ \to E_+^1$, we see that $-1 \in E_+^1 = S(F)$ is not in the image of the norm map $S(E) \to S(F)$. $\qquad\square$

B.2 One-dimensional anisotropic tori

In this subsection we consider the one-dimensional anisotropic torus S defined over F and split over a separable quadratic extension E/F, i.e. $S = \operatorname{Res}^1_{E/F}\mathbb{G}_m$. We have $\widehat{S} = \mathbb{C}^\times_{(-1)}$. The Tate periodicity isomorphism

$$H^2(\Gamma_{E/F}, \mathbb{C}^\times_{(-1)}) = \widehat{H}^0(\Gamma_{E/F}, \mathbb{C}^\times_{(-1)}) = \{\pm 1\}$$

shows that there are exactly two isomorphism classes of extensions of $\Gamma_{E/F}$ – the split extension, i.e. the L-group of S, and a second extension. In fact, the second extension has a canonical 2-cocycle, namely $z \in Z^2(\Gamma_{E/F}, \mathbb{C}^\times_{(-1)})$ specified by $z(\sigma, \sigma) = -1$, where $\sigma \in \Gamma_{E/F}$ is the non-trivial element. Therefore we have the canonical representative $\mathbb{C}^\times_{(-1)} \boxtimes_z \Gamma_{E/F}$ of that isomorphism class.

The torus S satisfies $\widehat{H}^0(\Gamma_{E/F}, S(E)) = 0$, i.e. Assumption B.1. At the same time we have $\widehat{H}^{-1}(\Gamma_{E/F}, S(E)) = F^\times/N_{E/F}(E^\times)$. Therefore the extension (B.1) becomes, via push-out under $\kappa : F^\times/N_{E/F}(E^\times) \to \{\pm 1\}$, the double cover

$$1 \to \{\pm 1\} \to E^\times/N_{E/F}(E^\times) \to E^1 \to 1, \tag{B.4}$$

where $E^1 = \operatorname{Ker}(N_{E/F})$ and the map $E^\times \to E^1$ is given by $x \mapsto x/\sigma x$. According to Proposition B.2 there is a canonical bijection between L-parameters valued in $\mathbb{C}^\times_{(-1)} \boxtimes_z \Gamma_{E/F}$ and genuine characters of this double cover.

Let us examine this bijection more carefully. A genuine character of this double cover is the same as a character θ of $E^\times$ that extends κ. In order to obtain the corresponding L-parameter we follow the proof of Proposition B.2. Choose an element $v_1 \in W_{E/F} \setminus E^\times$. Note that $v_1^2 \in F^\times \setminus N_{E/F}(E^\times)$. Extend θ to a map $W_{E/F} \to \mathbb{C}^\times$ by $\theta(ev_1) = \theta(e)$ for $e \in E^\times$. The differential of this map sends $(\sigma, \sigma) \in \Gamma_{E/F}$ to $\theta(v_1^2)^{-1} = \kappa(v_1^2)^{-1} = -1$. Therefore the L-parameter for θ is the map

$$W_{E/F} \to \mathbb{C}^\times \boxtimes_z \Gamma_{E/F}, \quad e \mapsto \theta(e) \boxtimes 1, \quad ev_1 \mapsto \theta(e) \boxtimes \sigma. \tag{B.5}$$

B.3 Induced one-dimensional ansitropic tori

In this subsection we consider the restriction of scalars of a one-dimensional anisotropic torus. Thus let $E_\pm/F$ be a finite separable extension and let $E_+/E_\pm$

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be a quadratic separable extension. Let $S = \mathrm{Res}_{E_\pm/F}\mathrm{Res}^1_{E_+/E_\pm}\mathbb{G}_m$. We have $\widehat{S} = \mathrm{Ind}^{\Gamma_F}_{\Gamma_{E_\pm}}\mathbb{C}^\times_{(-1)}$ and $X^*(S) = \mathrm{Ind}^{\Gamma_F}_{\Gamma_{E_\pm}}\mathbb{Z}_{(-1)}$.

The splitting field of S is the normal closure E of E_+. The norm map $S(E) \to S(F)$ need not be surjective any more as we saw in Example B.7 so this situation is not covered by §B.1. But we can extrapolate the results of §B.2. First, we still have the canonical double cover of $S(F) = E^1_+$, namely $E^\times_+/N_{E_+/E_\pm}(E^\times_+)$, but it is not related any more to $S(E)$. Second, we can define a suitable L-group for that double cover and obtain a local correspondence. This is conceptually rather simple – it just involves Shapiro's lemma, but since we will be dealing with co-chains rather than cohomology classes we must be more careful.

We begin by noting that we still have a canonical class in $H^2(\Gamma, \widehat{S})$, becase the latter is canonically isomorphic to $H^2(\Gamma_{E_\pm}, \mathbb{C}^\times_{(-1)})$ which contains the class inflated from the unique non-trivial class in $H^2(\Gamma_{E_\pm/E_+}, \mathbb{C}^\times_{(-1)})$. But there is no canonical 2-cocycle representing it. Indeed, while the Shapiro isomorphism $H^2(\Gamma_{E_\pm}, \mathbb{C}^\times_{(-1)}) \to H^2(\Gamma, \widehat{S})$ is canonical, a presentation for it $Z^2(\Gamma_{E_\pm}, \mathbb{C}^\times_{(-1)}) \to Z^2(\Gamma, \widehat{S})$ on the level of cocycles is not. It depends on a choice, which we now discuss.

There is a canonical element $e \in X^*(S)$: it is the unique element supported on $\Gamma_{E_\pm}$ and sending the identity in the group to $1 \in \mathbb{Z}$. Its Γ-orbit R is symmetric. The following choices are equivalent:

1. A section $s : \Gamma_{E_\pm}\backslash\Gamma_F \to \Gamma_{E_+}\backslash\Gamma_F$ of the natural projection;

2. a gauge $p : R \to \{\pm 1\}$.

The equivalence is by $p(s(\sigma)^{-1}e) = +1$. Once this choice is made, we compose s by an arbitrary section $\Gamma_{E_+}\backslash\Gamma_F \to \Gamma_F$, define $r : \Gamma_F \to \Gamma_{E_\pm}$ by $\sigma = r(\sigma)s(\sigma)$, and obtain according to [Kal18, Lemma B.3] the Shapiro map $Z^2(\Gamma_{E_\pm}, \mathbb{C}^\times_{(-1)}) \to Z^2(\Gamma, \widehat{S})$ (here translated into inhomogenous coordinates) as
$$\mathrm{Sh}\, z(\sigma, \tau)(a) = r(a)z(r(a)^{-1}r(a\sigma), r(a\sigma)^{-1}r(a\sigma\tau)).$$
Its restriction to $Z^2(\Gamma_{E_\pm/E_+}, \mathbb{C}^\times_{(-1)})$ does not depend on the choice of section $\Gamma_{E_+}\backslash\Gamma_F \to \Gamma_F$.

We are particularly interested in the image of the canonical element $z \in Z^2(\Gamma_{E_\pm/E_+}, \mathbb{C}^\times_{(-1)})$. For fixed σ, τ we have the function
$$\Gamma_{E_\pm}\backslash\Gamma_F \to \{\pm 1\}, \qquad a \mapsto \mathrm{Sh}\, z(\sigma, \tau)(a).$$

Such a function is determined by its values at elements $a \in \Gamma_{E_+}\backslash\Gamma_F$ with $r(a) \in \Gamma_{E_+}$, equivalently $p(a^{-1}e) = +1$. Let us adopt the notation $\alpha > 0$ for $p(\alpha) = +1$. From above formula we see that $\mathrm{Sh}\, z(\sigma, \tau)(a) = -1$ if and only if $r(a\sigma) \in \Gamma_{E_\pm} \smallsetminus \Gamma_{E_+}$ and $r(a\sigma\tau) \in \Gamma_{E_+}$, equivalently $(a\sigma)^{-1}e < 0$ and $(a\sigma\tau)^{-1}e > 0$.

We can rewrite $\mathrm{Sh}\, z(\sigma, \tau)$ by using the element $(-1)^e = e\otimes(-1) \in X^*(S)\otimes_\mathbb{Z} \mathbb{C}^\times = \widehat{S}$, i.e. the function $\Gamma_{E_\pm}\backslash\Gamma_F \to \{\pm 1\}$ sending the trivial coset to -1 and

all other cosets to 1. In these terms the above formula evaluates to

$$\mathrm{Sh}\, z(\sigma, \tau) = \prod_{\substack{a \in \Gamma_{E_\pm} \backslash \Gamma_F \\ \mathrm{Sh}\, z(\sigma,\tau)(a)=-1}} (-1)^{\delta_a} = \prod_{\substack{a \in \Gamma_{E_+} \backslash \Gamma_F \\ a^{-1}e>0 \\ (a\sigma)^{-1}e<0 \\ (a\sigma\tau)^{-1}e>0}} (-1)^{a^{-1}e} = \prod_{\substack{\alpha \in R \\ \alpha>0 \\ \sigma^{-1}\alpha<0 \\ (\sigma\tau)^{-1}\alpha>0}} (-1)^{\alpha}.$$

(B.6)   {eq:tpi1d}

We now recognize this as the Tits cocycle of Definition 3.8.

{rem:spq}

Remark B.8. If we change the choice of gauge, the new version of $\mathrm{Sh}\, z(\sigma, \tau)$ will of course be cohomologous to the old version. But in fact a stronger statement is true – there is a natural element of $C^1(\Gamma, \widehat{S})$ whose coboundary relates the two versions, as observed in [Kal18, Lemma B.4]. Specifying the formula of that Lemma to the case at hand we obtain the cochain $s_{p/q}$ of [LS87, §2.4]. □

Keeping a gauge p fixed we define the L-group of the double cover of $S(F)$ to be $^L S_\pm = \widehat{S} \boxtimes_{t_p} \Gamma_{E/F}$, where t_p is the image of z under the Shapiro map determined by the gauge p. Thus $^L S_\pm$ is an extension of $\Gamma_{E/F}$ by $\widehat{S}$ equipped with a set-theoretic splitting, i.e. an E-group for S in the sense of [AV92]. Since the group $H^1(\Gamma_{E/F}, \widehat{S})$ need not vanish any more, specifying this splitting gives additional information. A parameter for $^L S_\pm$ is now the same as an element of $C^1(W_F, \widehat{S})/B^1(W_F, \widehat{S})$ whose differential is t_p. The Shapiro map in degree 1, again given by the gauge p, identifies these with the elements of $C^1(W_{E_\pm}, \mathbb{C}^\times_{(-1)})/B^1(W_{E_\pm}, \mathbb{C}^\times_{(-1)})$ whose differential is the canonical element $z \in Z^2(\Gamma_{E_\pm/E_+}, \mathbb{C}^\times_{(-1)})$. If θ is a genuine character of $E_+^\times/N_{E_+/E_\pm}(E_+^\times)$ then the image of its parameter (B.5) under the Shapiro map in degree 1 is given by

$$\prod_{a \in \Gamma_{E_\pm} \backslash \Gamma_F} \theta_0(r(a)^{-1}r(a\sigma))^{\delta_{s(a)}},$$

where $\theta_0 : W_{E_\pm} \to \mathbb{C}^\times$ is an arbitrary extension of $\theta : W_{E_+} \to \mathbb{C}^\times$ satisfying $\theta_0(ww') = \theta(w)\theta_0(w')$ for $w \in W_{E_+}$ and $w' \in W_{E_\pm}$. Of course we have arranged here that r and s take values in W in place of Γ.

This formula can be rewritten in another form: If $w_1, \dots, w_n$ are the images of the section s, $v_0 = 1$, and $v_1 \in W_{E_\pm} \smallsetminus W_{E_+}$ arbitrary, and we follow the notation of Lemma 3.16, then we obtain

$$\prod_{i=1}^n \theta(v_0(u_i(w)))^{\delta_{w_i}},$$

(B.7)   {eq:par2}

which is precisely Definition 3.17 in this case.

B.4   General tori

{sub:gentori}

Let us now collect the preceding observations and see how they, applied to a general torus S equipped with a finite Σ-invariant subset $0 \notin R \subset X^*(S)$,

motivate the definition of the L-group of its double cover and the L-parameter of the genuine character χ_S corresponding to a set of χ-data.

The double cover $S(F)_\pm$ of $S(F)$ was obtained as the pull-back of $S(F) \to \prod J_O(F) \leftarrow (\prod J_O(F))_\pm$ and it makes sense to define its L-group as the pushout of $\widehat{S} \leftarrow (\prod \widehat{J}_O) \to {}^L(\prod J_O)_\pm$. Each J_O is an induced one-dimensional anisotropic torus. The L-group of its canonical double cover $J_O(F)_\pm$ was discussed in §B.3 – it is the product of $\widehat{J}_O$ and Γ twisted by the 2-cocycle (B.6). Under the map $\prod \widehat{J}_O \to \widehat{S}$ dual to $S \to \prod J_O$ the image of the product of these cocycles is precisely the Tits cocycle of Definition 3.8. Thus Definition 3.9 of the L-group of $S(F)_\pm$ is indeed this pushout.

Consider a set of χ-data (χ_α). Each χ_α is a genuine character of the double cover $J_\alpha(F)_\pm$ and its L-parameter is given by (B.7). We now recognize Definition 3.17 as image under the pushout map ${}^L(\prod J_O)_\pm \to {}^L S_\pm$ of the product over O of the parameters of these characters.

C b-DATA AND $H^1(F, S)$

{app:bdata}

We will introduce the notion of b-data and relate it to elements of $H^1(F, S)$, clarifying the formula of [LS87, Lemma 2.2.B]. We will use arguments and notation established in Appendix B.

Let S be an F-torus and R a finite admissible Σ-set. Contrary to the data of a Σ-invariant map $R \to X^*(S)$ that we have been using so far, we now assume given a Σ-invariant map $R \to X_*(S)$. Then this leads to a homomorphism $J_O \to S$ that lifts uniquely to a homomorphism $J_O \to S'$, where $S' \to S$ is the map of tori induced by the inclusion into $X_*(S)$ of the submodule generated by the image of R. Indeed, the construction is entirely analogous to the construction of the homomorphism $S \to J_O$ of §3.1 associated to a Σ-invariant map $R \to X^*(S)$. Indeed, $X_*(J_O)$ has the same description as $X^*(J_O)$, and the desired homomorphism is given by the same formula as that for $S \to J_O$, but now in the setting of cocharacter modules, namely applying Frobenius reciprocity to the $\Gamma_{\pm\alpha}$ equivariant homomorphism

$$\mathbb{Z}_{(-1)} \to X_*(S), \quad n \mapsto n\alpha'$$

where $\mathbb{Z}_{(-1)}$ is the abelian group $\mathbb{Z}$ on which the quotient $\Gamma_{\pm\alpha}/\Gamma_\alpha$ acts by the non-trivial character when α is symmetric, and otherwise acts (necessarily) trivially, and where $\alpha \in O$ is arbitrary and $\alpha' \in X_*(S)$ is its image.

We note that when we are given Σ-invariant maps $R \to X^*(S)$ and $R \to X_*(S)$, so that both homomorphisms $J_O \to S$ and $S \to J_O$ are present, the composition $J_O \to S \to J_O$ is the homomorphism that raises every element of J_O to its $\langle \bar{\alpha}, \alpha' \rangle$-power.

Definition C.1.   1. A set of b-data is a collection $(b_\alpha)_{\alpha \in R}$ of non-zero elements $b_\alpha \in F_\alpha^\times$ satisfying $b_{\sigma\alpha} = \sigma(b_\alpha)$ and $b_{-\alpha} = b_\alpha$ for $\alpha \in R$, $\sigma \in \Gamma$ (note this implies $b_\alpha \in F_{\pm\alpha}^\times$).

2. Two b-data (b_α) and (b'_α) are equivalent if $b_\alpha/b'_\alpha \in N_{F_\alpha/F_{\pm\alpha}}(F_\alpha^\times)$.

$\square$

We will now construct an element of $H^1(F, S)$ from an equivalence class of b-data. Each b_α provides under the isomorphism $F_{\pm\alpha}^\times/N_{F_\alpha/F_{\pm\alpha}}(F_\alpha^\times) \to H^1(F, J_O)$ an element of the latter, which we call b_O. Assumung given a Σ-invariant map $R \to X_*(S)$, and taking the product over all Σ-orbits O of the images of the elements b_O in $H^1(F, S)$ we obtain an element b_S.

Lemma C.2. Let p be a gauge on R. The class b_S is represented by the 1-cocycle

$$\sigma \mapsto \prod_{\alpha>0, \sigma^{-1}\alpha<0} \alpha'(b_\alpha).$$

$\square$

Note that this is the formula given in [LS87, Lemma 2.2.B]. The proof will be analogous to the argument used in Appendix B to explain Definition 3.17.

Proof. Since both the class b_S and the given 1-cocycle are products over the Σ-orbits in R, we may focus on a single orbit O, thereby replacing R by O. When O is asymmetric then $H^1(F, J_O) = \{0\}$ by Hilbert 90, while the 1-cocycle $\prod_{\alpha>0, \sigma^{-1}\alpha<0} \alpha'(b_\alpha)$ is the coboundary of $\prod_{\alpha>0} \alpha'(c_\alpha)$, where $c_\alpha = b_\alpha$ for all α in one of the two Γ-orbits in O, and $c_\alpha = 1$ for the other orbit.

When O is symmetric, fix $\alpha \in O$ and abbreviate $\Gamma_+ = \Gamma_\alpha$, $\Gamma_\pm = \Gamma_{\pm\alpha}$, $F_+ = F_\alpha$, $F_\pm = F_{\pm\alpha}$. Before we can apply Shapiro's lemma we need to interpret $J_O(F^s)$ as an induced module. By construction we have $J_O(F^s) = \mathrm{Ind}_{\Gamma_\pm}^\Gamma \mathbb{Z}_{(-1)} \otimes F^{s,\times}$ with the diagonal Γ-action. The isomorphism $\mathbb{Z} \otimes_\mathbb{Z} F^{s,\times} \to F^{s,\times}$ alone doesn't identify this with $\mathrm{Ind}_{\Gamma_\pm}^\Gamma F^{s,\times}$. In addition to using this isomorphism, we need to send a function $f \in \mathrm{Ind}_{\Gamma_\pm}^\Gamma \mathbb{Z}_{(-1)} \otimes F^{s,\times}$ to the function $\phi \in \mathrm{Ind}_{\Gamma_\pm}^\Gamma F^{s,\times}$ defined by $\phi(\sigma) = \sigma(f(\sigma))$. Note that in the latter module we have equipped $F^{s,\times}$ with the twisted $\Gamma_\pm$-action, so that $(F^{s,\times})^{\Gamma_+} = F_+^\times$ but $(F^{s,\times})^{\Gamma_\pm} = F_+^1$. Then the homomorphism $J_O \to S$ sends ϕ to the element

$$\prod_{\tau \in \Gamma_\pm \backslash \Gamma} \tau^{-1}(\phi(\tau))^{\tau^{-1}\alpha'}.$$

As discussed in §B.3, the gauge p leads to a section $s : \Gamma_\pm\backslash\Gamma \to \Gamma_+\backslash\Gamma$, which we compose with an arbitrary section $\Gamma_+\backslash\Gamma \to \Gamma$, and denote again by r the corresponding retraction. Applying [Kal18, Lemma B.3] again we obtain the expression

$$\mathrm{Sh}\, z(\sigma)(a) = r(a)z(r(a)^{-1}r(a\sigma))$$

in inhomogeneous coordinates for the Shapiro map

$$H^1(\Gamma_\pm, F_+^\times) \to H^1(\Gamma, \mathrm{Ind}_{\Gamma_\pm}^\Gamma F^{s,\times}).$$

The element $z(\tau) \in F_{\pm}^{\times}$ has the value b_α if $\tau \in \Gamma_{\pm} \smallsetminus \Gamma_{+}$, and the value 1 if $\tau \in \Gamma_{+}$. Thus the element $z(r(a)^{-1}r(a\sigma))$ is non-trivial if and only if $r(a)^{-1}r(a\sigma) \in \Gamma_{\pm} \smallsetminus \Gamma_{+}$. The elements $a \in \Gamma_{+}\backslash\Gamma$ with $r(a) \in \Gamma_{+}$ (equivalently $p(a^{-1}\alpha) > 0$ are a set of representatives for the quotient $\Gamma_{\pm}\backslash\Gamma$, so restricting to those we see that the non-triviality condition becomes $r(a\sigma) \notin \Gamma_{+}$, equivalently $p((a\sigma)^{-1}\alpha) < 0$. Therefore,

$$\mathrm{Sh}\, z(\sigma) = \prod_{\substack{a \in \Gamma_{+}\backslash\Gamma \\ a^{-1}\alpha > 0 \\ (a\sigma)^{-1}\alpha < 0}} a^{-1}(b^{\alpha'}) = \prod_{\substack{\beta \in O \\ \beta > 0 \\ \sigma^{-1}\beta < 0}} b_\beta^{\beta'}.$$

$\blacksquare$

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