Introduction to Conjugate Plateau Constructions

by H. Karcher, Bonn. Version December 2001

Abstract. We explain how geometric transformations of the solutions of carefully designed Plateau problems lead to complete, often embedded, minimal or constant mean curvature surfaces in space forms.

Introduction.

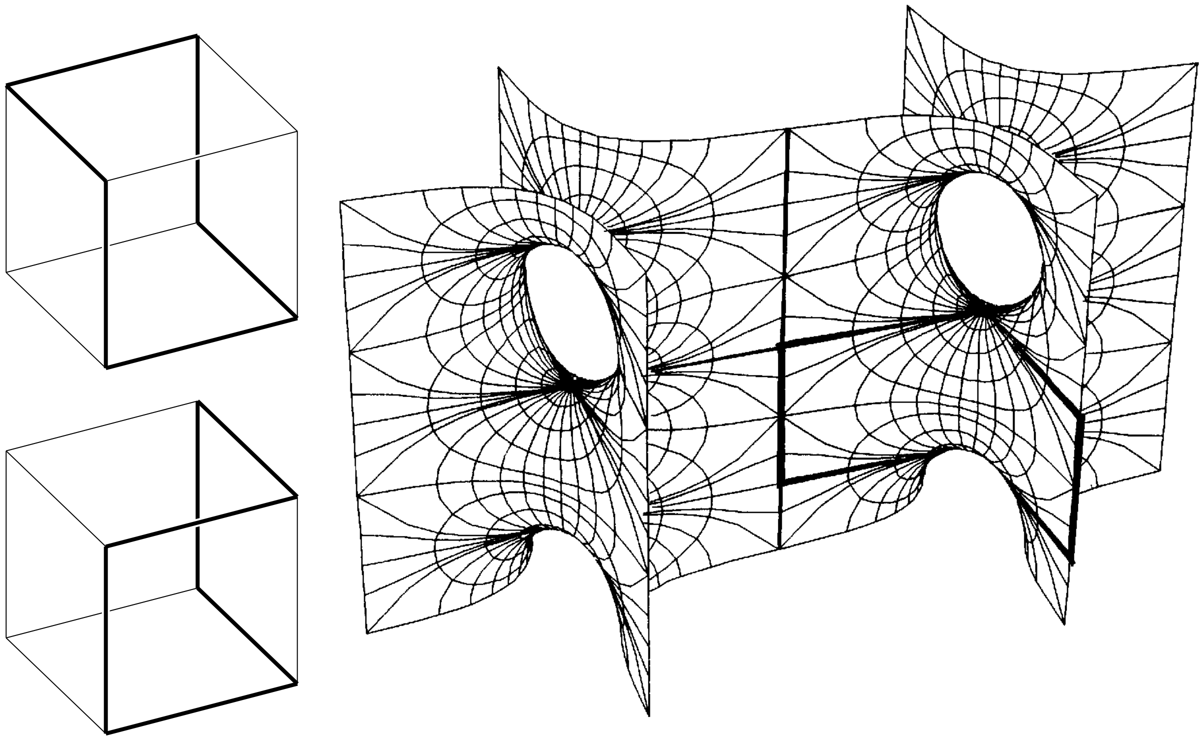
The purpose of this paper is, to explain to a reader who is already familiar with the theory of minimal surfaces another successful method to construct examples. This method is independent of the Weierstraß representation which is the construction method more immediately related to the theory. This second method is called "conjugate Plateau construction" and we summarize it as follows: solve a Plateau problem with polygonal contour, then take its conjugate minimal surface which turns out to be bounded by planar lines of reflectional symmetry (details in section 2), finally use the symmetries to extend the conjugate piece to a complete (and if possible: embedded) minimal surface. For triply periodic minimal surfaces in $\mathbb{R}^3$ this has been by far the simplest and richest method of construction. [KP] is an attempt to explain the method to a broader audience, beyond mathematicians. Large families of doubly and singly periodic minimal surfaces in $\mathbb{R}^3$ have also been obtained. By contrast, for finite total curvature minimal surfaces the Weierstraß representation has been much more successful.
Since *conjugate minimal surfaces* can also be defined in spheres and hyperbolic spaces (section 2) the method has also been used there. However in these applications the contour of the Plateau problem is *not* determined already by the symmetry group with which one wants to work. Even in the simplest cases the correct contour has to be determined by a degree argument from a 2-parameter family of (solved) Plateau problems. For spherical examples see [KPS], for hyperbolic examples see [Po].
There is an even wider range of applications. In [La] two constant mean curvature one surfaces in $\mathbb{R}^3$ were constructed from minimal surfaces in $\mathbb{S}^3$. In [Ka2] it was observed that for large numbers of cases the required spherical Plateau *contours*, surprisingly, can be determined without reference to the Plateau *solutions* by using Hopf vector fields in $\mathbb{S}^3$. Then [Gb] added strings of bubbles to these examples by solving Plateau problems not in $\mathbb{S}^3$ but in mean convex domains such as the universal cover of the *solid* Clifford torus. He also obtained nonperiodic limits.
More generally one can start with minimal surfaces in a spaceform $M^3(k)$ (having constant curvature k). From these it is possible to obtain in a similar way constant mean curvature c surfaces in the space $M^3(k-c^2)$. However, one has the same problem explained for conjugate minimal surfaces: the required Plateau contours have to be chosen from at least 2-parameter families and the choice depends on properties of the solutions. In those cases

where conjugate minimal pieces have been obtained by solving degree arguments, for example in [KPS], [Po], a trivial perturbation argument shows that constant mean curvature deformations of the constructed minimal surfaces also exist.
Constant mean curvature one surfaces in $\mathbb{H}^3$, also called Bryant surfaces, have to be mentioned separately. They are obtained from minimal surfaces in $\mathbb{R}^3$. Again, the contours in $\mathbb{R}^3$ are not determined by the symmetries with which one wants the Bryant surface to be compatible. Nevertheless the situation still is a bit simpler than e.g. the spherical or hyperbolic conjugate minimal surfaces, because one of the parameters of the $\mathbb{R}^3$-contours is just scaling. E.g. to prove existence of Bryant surfaces having the symmetries of a Platonic tessellation of $\mathbb{H}^3$, the scaling parameter allowed to succeed with a once iterated intermediate value argument instead of a general winding number argument [Ka3].
Frequently we will argue: "take the Plateau solution and...". This may suggest difficulties which are not encountered. For our applications it is neccessary to know more about the Plateau solution than just their existence. In particular, all our Plateau contours will be polygons in $\mathbb{R}^3$ or piecewise geodesic in $\mathbb{S}^3$ or $\mathbb{H}^3$. In many cases these solutions can be obtained as graphs over convex polygons with piecewise linear Dirichlet boundary data. It will be convenient to include also projections for which certain edges are projected to points. A reference for such cases is [Ni], where Dirichlet problems for graphs over convex domains are solved even if the boundary values have jump discontinuities. For example over two opposite edges of a square one can prescribe the value $-n$ and over the other pair the value $+n$. The Nitsche graph has then vertical segments over the vertices of the square as boundary, i.e., it is the Plateau solution of the polygon which has two horizontal edges at height $-n$, two horizontal edges at height $+n$ and four vertical edges of length $2n$. The fact that such Dirichlet problems have graph solutions implies that the tangent planes along the vertical edges have to rotate in a monotone way (otherwise the surface could not be a graph over the interior). This implies that the corresponding boundary arcs of the conjugate piece are convex arcs, a very helpful qualitative control of the conjugate piece. – In [JS] this work is extended, now giving sufficient conditions for including *infinite* Dirichlet boundary values. For example on a convex $2n$-gon with *equal* edge lengths (i.e., allowing angles $\leq \pi$) one can alternatingly prescribe the boundary values $+\infty, -\infty$. The conjugate pieces of these Jenkins- Serrin graphs generate a rich family of generalizations of Scherk's singly periodic saddle towers, section 2 and [Ka1], pp.90-93.

This paper has three sections. In the first we do not yet use the conjugate minimal surfaces. We discuss the construction of complete embedded minimal surfaces by extending polygonally bounded minimal surface pieces, not necessarily of disk type. I concentrate on a new family of hyperbolic minimal surfaces in an attempt to show how quantitative facts about hyperbolic geometry are essential for existence and embeddedness. The second section repeats, for the convenience of the reader, those parts of minimal surface theory which are relevant for the conjugation constructions in space forms. Also, I try to indicate, how varied the applications to minimal surfaces in $\mathbb{R}^3$ are. In the third section I concentrate on constant mean curvature one surfaces in $\mathbb{R}^3$ since I feel one has to understand these before one can work on the less explicit problems mentioned in section 2.

1. Extension of minimal surfaces across boundary segments.

Minimal surfaces in $\mathbb{R}^3$. More than 50 years before the general Plateau problem was solved, solutions for certain polygonal contours where found by B. Riemann and by H.A. Schwarz. Their solutions were given as integrals of multivalued functions. In today's terminology they used the Weierstraß representation on nontrivial Riemann surfaces. The following particularly simple examples are due to H.A. Schwarz. Consider the following hexagonal polygons made of edges of a brick with edge lengths a, b, c.
$P_1 : (0,0,0) \to (a,0,0) \to (a,b,0) \to (a,b,c) \to (0,b,c) \to (0,0,c) \to (0,0,0)$,
$P_2 : (0,0,0) \to (a,0,0) \to (a,b,0) \to (a,b,c) \to (a,0,c) \to (0,0,c) \to (0,0,0)$.

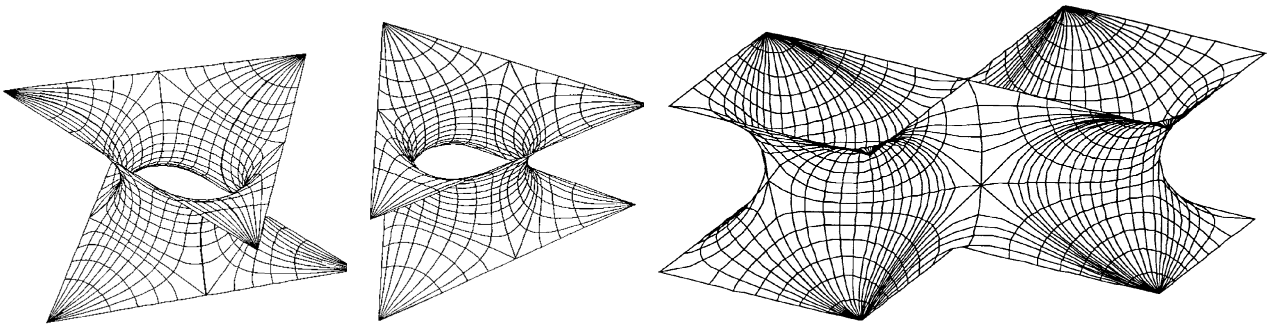


Left: the polygonal contours $P1$, $P2$ on the boundary of a cube (a special brick). Right: the contour $P2$ with its Plateau solution and an extension by 180° rotations around boundary edges. It is named Schwarz' CLP-surface.

In both cases the contour has a 1-1 convex projection (in fact many). The Plateau problem has therefore a unique solution, which is a graph over the interior of the chosen convex projection. Moreover, by the maximum principle, every compact minimal surface lies in the convex hull of its boundary. In particular our Plateau solutions are inside the brick with edge lengths a, b, c. Imagine a black and white ("checkerboard") tessellation of $\mathbb{R}^3$ by these bricks, and imagine our Plateau solution to be in a black brick. The Schwarz reflection theorem says that 180° rotation around any of the boundary edges of the Plateau

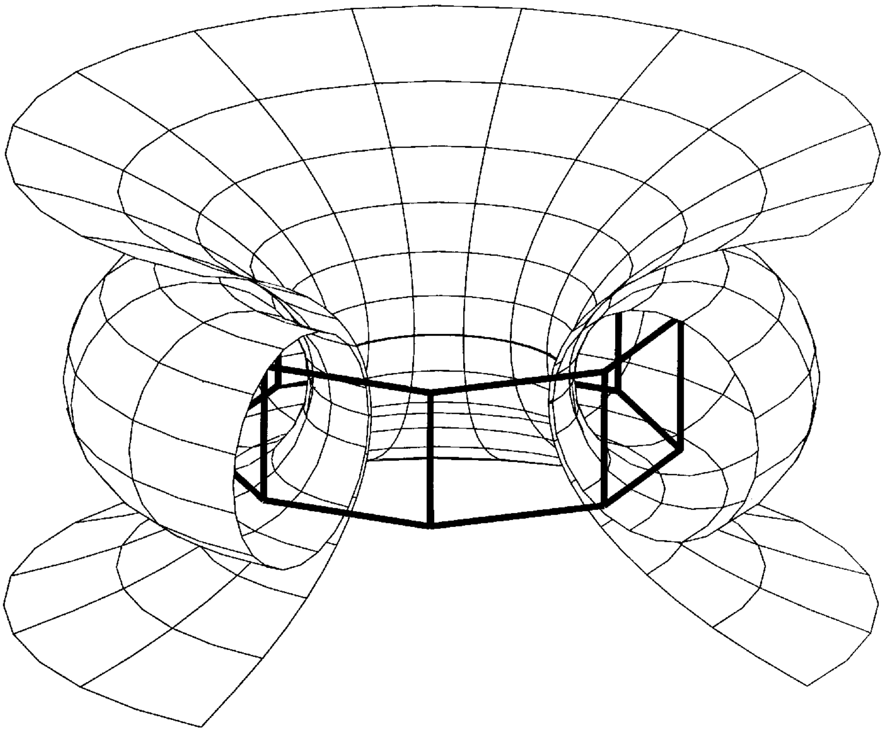
piece gives an analytic continuation of the initial minimal surface piece. Because of the 90° angles of the polygons we can repeatedly extend across the edges which leave from one polygon vertex and thus obtain a smooth extension containing that vertex as an interior point. Notice that all the extensions are inside black bricks, and if the extensions lead us back into the first brick then the whole brick comes back in its original position. This shows that the extensions lead to embedded triply periodic minimal surfaces. The first one was named by A. Schoen *Schwarz D-surface* and the second *Schwarz CLP-surface*, see [DHKW] vol I, plate V, (a)-(c). The names given by A. Schoen are well known in the cristallographic literature.
The cristallographers Fischer and Koch [??] have listed the cristallographic groups which contain enough 180° rotations so that from segments of the rotation axes polygons can be formed with the property that the Plateau solutions of these polygons extend to embedded triply periodic minimal surfaces. Their work includes examples of fairly complicated such polygons.

Minimal surfaces in spheres and hyperbolic spaces. Since the Schwarz reflection theorem extends to minimal surfaces in spheres and hyperbolic spaces one can extend the above idea to space forms. Lawson's minimal surfaces in spheres [La] are constructed in this way from disk type Plateau solutions bounded by great circle quadrilaterals. I will now construct new embedded minimal surfaces in hyperbolic space which have compact *annular* fundamental domains. The purpose of these examples is to illustrate the use of barriers and of basic hyperbolic geometry. - Euclidean analogues of such annuli, where the annulus is bounded by a pair of equilateral triangles in parallel planes, or a pair of squares in parallel planes, were already constructed by H.A. Schwarz, see [DHKW] vol. I fig. 22 (a),(b),(d) and fig. 23 (a),(b). For more complicated minimal annuli see [Ka2], p.342,343. All these extend to embedded triply periodic minimal surfaces.



Three polygonally bounded minimal annuli in $\mathbb{R}^3$, one with noninjective boundary. The middle one is called Schwarz' H-surface.

The existence construction. We have to deal with the following problem: if two circles in parallel planes are too far apart, then there is no catenoid annulus which joins them. The problem is dealt with by barriers. In $\mathbb{R}^3$ for example, assume that two convex polygons in parallel planes are so close together that a catenoid exists which meets only the interior of the two polygons. Then the two polygons are the boundary of a minimal annulus which surrounds the catenoid. In principle this would work also in hyperbolic space $\mathbb{H}^3$, but the meridians of hyperbolic catenoids are given by differential equations, not by formulas in terms of well known functions. Therefore one cannot readily decide whether some prism can surround a catenoid or whether it is too high and cannot. Instead we will rely on tori as (less optimal but more explicit) barriers. A torus of revolution has its mean curvature vector pointing into the solid torus if the radius of the meridian circle is less than half the soul radius. Such a torus is called *mean convex.*



Catenoid, mean-convex torus and polygonally bounded annulus inside.

The minimization technique of the calculus of variation can be applied to the set of surfaces which lie in a domain with mean convex boundary and which have the desired boundary. We can therefore choose any two different latitude circles of the torus and find an *annular minimal surface* (with those two latitudes as boundary) inside the solid torus. This minimal annulus together with the exterior planar domains of the two latitudes bounds another mean convex domain. In this mean convex domain we find a rich collection of minimal annuli: any two simply closed curves which lie in the exterior domains of the two latitude circles and are homotopic to them bound a minimal annulus which can be found by minimization in our mean convex domain. We apply this to the top and bottom polygon of the following regular and tessellating prisms: start with a regular geodesic n-gon, $n \geq 8$, with $90°$ angles, lying in a totally geodesic hyperbolic plane in $\mathbb{H}^3$. (There also exist hyperbolic five-, six- and sevengons with $90°$ angles but for those the torus barriers do not work.) Consider next the infinite prism orthogonal to the n-gon in $\mathbb{H}^3$. Cut this infinite prism above and below its symmetry plane so that the dihedral angles along the top and bottom rim are $60°$. The following hyperbolic computation shows that the two rims of this prism lie in a mean convex torus (see the figure above) and can therefore be joined by a minimal annulus (of course inside the convex prism). And analytic extension of this annulus by repeated $180°$ rotations around boundary edges produces a complete minimal surface which turns out to be embedded.

My reference for *hyperbolic trigonometry* is [Bu, pp.31-42]. It is helpful to know that general trigonometric formulae simplify more than one expects from the general expressions if one specializes to simple figures. To emphasize this I only quote and use two such special cases. First, for right angled triangles with edge lengths a, b, c and angles $\alpha, \beta, \gamma = \pi/2$ we have

$$\cosh c = \cosh a \cosh b = \cot\alpha \cot\beta$$
$$\sinh a = \sin\alpha \sinh c = \cot\beta \tanh b$$
$$\cos\alpha = \cosh a \sin\beta = \tanh b \coth c.$$

The formulas $\cos\alpha = \cosh a \sin\beta,\ \cosh c = \cot\alpha\cot\beta$ imply that a regular n-gon with $90°$ angles has inradius $a = r_i$ and outer radius $c = r_o$ given by

$$\cosh r_i = \frac{\cos(\pi/4)}{\sin(\pi/n)}, \qquad \cosh r_o = \cot(\pi/4)\cot(\pi/n).$$

Our second such simple figure is the trirectangle, a quadrilateral with three angles $\pi/2$, with edge lengths a, b, α, β and with the fourth angle ϕ between edges α, β. Here the general formulas simplify to

$$\cos\phi = \sinh a \sinh b = \tanh\alpha \tanh\beta$$
$$\cosh a = \cosh\alpha \sin\phi = \tanh\beta \coth b$$
$$\sinh\alpha = \sinh a \cosh\beta = \coth b \cot\phi.$$

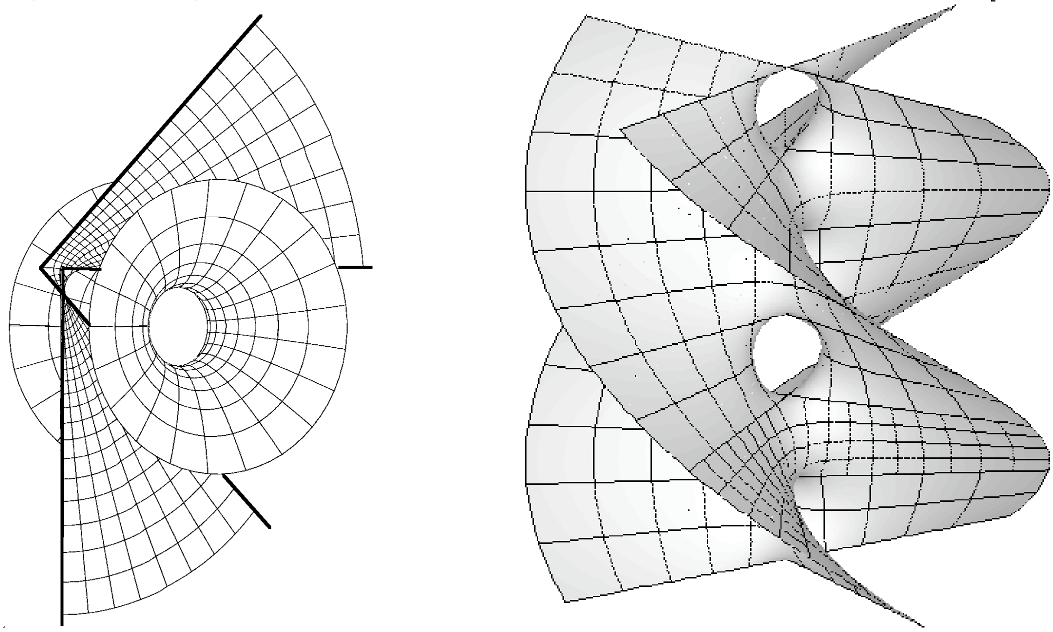
Next consider the prism over the above $90°$-n-gon, which has at its top rim a dihedral angle of $\phi = \pi/3$. We determine its inner height h_i, i.e., the distance between the top and bottom plane, and its outer height h_o, i.e., the distance between the bottom n-gon and the top n-gon (i.e. the distance between the midpoints of the top and bottom edge of a vertical face). Note that a plane through the (vertical) symmetry axis and the midpoint of a (horizontal) edge intersects the prism in a trirectangle with $a = h_i,\ b = r_i,\ \alpha = h_o,\ \phi = \pi/3$. With the formulas $\cos\phi = \sinh a \sinh b,\ \sinh\alpha = \coth b\cot\phi$ we get

$$\sinh h_i = \frac{\cos(\pi/3)}{\sinh r_i}, \qquad \sinh h_o = \frac{\cot(\pi/3)}{\tanh r_i}.$$

Finally we have to check that the two $60°$ rims of the double prism lie in a mean convex domain as above. For this we choose the midpoint M of the meridian circle of the torus on the extension of the edge r_i of the trirectangle and at a distance $r_s = 1.1 \cdot r_i$ (the soul radius) from the symmetry axis. The meridian radius r_m is determined by $\cosh r_m = \cosh h_o \cosh(0.1 r_i)$ The condition for mean convexity of the torus was $r_s \geq 2r_m$ and this is satisfied for $n \geq 8$. Therefore we have the *existence of a minimal annulus* which is bounded by the top and bottom rim of our convex and *tessellating* hyperbolic prism. For fivegons to sevengons either this barrier computation is not good enough or the top and bottom rims are indeed too far apart for the minimal annulus to exist. Note another quantitative aspect of this computation: we do not have other families of such prisms, because the sum of the three dihedral angles at a top vertex of the prism has to be $> \pi$; if the sum of the dihedral angles is $= \pi$ then the vertices of the prism are on the sphere at infinity and if the sum of the dihedral angles is $< \pi$ then then the vertical edges do not meet the top face.

Analytic extension of the annular piece by $180°$ rotations. By repeated $180°$ rotation around boundary edges the annular fundamental piece is analytically extended to a complete minimal surface. The embeddedness proof has two parts: (i) the constructed annulus is embedded and (ii) the continuation does not create selfintersections. We omit the first part because the arguments are disjoint from the topic of this paper. For part (ii) we have to understand the tessellation of hyperbolic space by our prisms; such geometric discussions are always part of a construction of a complete surface from Plateau pieces. The idea is to color the prisms of the tessellation in red, green and blue so that the color changes across a face and $180°$ rotation around an edge of a prism maps that pisma to one of the same color. If that can be achieved then we have the complete surface contained only in the prisms of one color. Since along each edge only two prisms of the same color meet we have avoided selfintersections. We start by making the first prism red, the 2 neighbours above and below we make green and the n neighbours across vertical faces we make blue. Next we describe how the 24 prisms meet which have one vertex in common. Consider how a small sphere around that vertex meets the adjacent prisms: each prism intersects the sphere in a geodesic triangle whose angles are the dihedral angles at the three edges of the prism at that vertex, i.e., the angles are $\pi/2, \pi/3, \pi/3$. Four such triangles around the $\pi/2$-corner fit together to a spherical square with angles $2\pi/3$. Six such squares tessellate the sphere; it is the same tessellation obtained by central projection of a cube to its circumsphere. The colors of the prisms which meet at one vertex are by definition the same as the colors of the 24 triangles of the spherical tessellation and vice versa. Initially we colored four prisms at each vertex of the first prism. Now consider the spherical triangulation which describes the neigbourhood of one vertex of the prism; one can view it on a cube after subdividing each face by its diagonals into four triangles. Our coloring prescription says that opposite triangles on one face have the same colour, since they correspond to two prisms whose position differs by a $180°$ rotation about a $90°$ edge. Also, each triangle has neighbours of both other colors. These two observations imply that the different colours of two neighbouring triangles determine the colours of all the other triangles on the cube uniquely. Therefore we can extend the coloration to all the prisms which meet a vertex of the first prism. We continue the coloration and because of the unique extension of a partial coloration to the full sphere we can assign a unique color to each prism and thus complete the proof.

Further remarks. Minimal annuli in $\mathbb{S}^3$ which are bounded by two geodesic quadrilaterals and which extend to complete embedded minimal surfaces by repeated 180° rotations around boundary edges are described in [Ka2, p.344]. The complete surface is invariant under a rotation of $\mathbb{S}^3$ which has two orthogonal great circles as axes. Near each axis this $\mathbb{S}^3$-rotation looks like a screw motion in $\mathbb{R}^3$ and the constructed surface can therefore be thought of as a generalization to $\mathbb{S}^3$ of the twisted Scherk saddle towers [Ka1, pp.94-99].



Existence proof of the twisted Scherk tower. The surface to the right is generated by a fundamental domain which is a strip (left) between two broken lines (here, each broken line consists of two halflines which meet under 90°). One can make a mean convex domain from the two quarter planes spanned by the broken lines and from two congruent pieces of a helicoid whose axis is the segment between the corners of the quarter planes. The strip between the broken lines is to be constructed as a limit of Plateau solutions which look like the left picture: each pair of halflines is connected by an arc in the mean convex domain, preferably by the helices on the two barrier helicoids. Moving these connecting arcs outwards gives a monotone family of minimal surfaces (the earlier ones are barriers for the later ones). Finally, the shown catenoid is the barrier which prevents this family from converging against the two quarterplanes.

The unbounded Dirichlet problems over convex domains from [JS] yield doubly periodic embedded minimal surfaces which are graphs over the full plane minus parallel lines, see below. (The projection of Scherk's doubly periodic surfaces covers only half the plane.)

A list of the known *disk type* Plateau solutions for polygonal boundaries, which extend to *embedded* minimal surfaces, is rather short. However, we see in the next section that the properties of conjugate minimal surfaces offer very flexible possibilities for the construction of embedded minimal surfaces. We emphasize that the conjugate surface method of the next section requires disk type minimal surfaces; the construction of embedded minimal surfaces from annular (or still higher genus) fundamental domains, which we have seen in this first section, is not compatible with the definition of the conjugate surface.

2. Conjugate minimal surfaces.

Some basic theory. When the theory of minimal surfaces developed in the 19th century it was realized early that the three coordinate functions of a *conformal* parametrization of a minimal surface are the *real parts of holomorphic functions*! Since the existence of conformal parametrizations of surfaces in $\mathbb{R}^3$ is a hard theorem which is rarely completely explained in differential geometry courses we observe that the above discovery can also be stated without a conformal parametrization. If one imagines an atlas of conformal coordinates for the surface, then it makes sense to multiply a tangent vector by the complex number i. But this multiplication by i is, on each tangent space, the positive $90°$ rotation, and this is a geometric description of the multiplication by i which does not use a conformal parametrization. The endomorphism field of tangent space wise $90°$ rotations is therefore also called the *complex structure* and is denoted by J. A differentiable map from the surface to the complex numbers, $f : M^2 \to \mathbb{C}$ is then *holomorphic* if its differential Tf satisfies $Tf(J\cdot X) = i\cdot Tf(X)$ for each tangent vector X. We write f in terms of its real and imaginary part, $f = u+i\cdot v$. Holomorphicity of f then can be expressed as $Tu(J\cdot X) = -Tv(X)$, or $Tv(J\cdot X) = Tu(X)$. These *Cauchy-Riemann equations* say that the differential of the imaginary part can be computed from the differential of the real part and J, namely $Tv = -Tu\circ J$. Moreover, just as in $\mathbb{C}$, a differentiable function u is (locally) the real part of a holomorphic function iff the differential form $\omega := -Tu\circ J$ is closed. Therefore we can now formulate, without reference to a conformal parametrization, the mentioned discovery of the 19th century, namely that the coordinate functions of minimal surfaces are, locally, real parts of holomorphic functions. To see why this fact is true requires the use of the surface equations. Let $F : M^2 \to \mathbb{R}^3$ be a local immersion, $N : M^2 \to \mathbb{S}^2$ the normal Gauß map, S the shape operator, $g(X,Y) = \langle TF(X),\ TF(Y)\rangle_{\mathbb{R}^3}$ the Riemannian metric, Γ the Christoffel map (or symbol), i.e., $T^2F(X,Y)^{tang} = TF\circ\Gamma(X,Y)$, ∇ the covariant derivative of the metric g and K its curvature, then we have the

Surface equations for TF, N with data $\{g, S\}$

Weingarten equation	$TN = TF\circ S,$
Gauß equation	$T^2F(X,Y) = TF\circ\Gamma(X,Y) \ -g(S\cdot X,Y)N,$
Covariant version	$\nabla^2F(X,Y) := T^2F(X,Y) - TF\circ\Gamma(X,Y) = -g(S\cdot X,Y)N.$

Integrability conditions:

Symmetry	$g(S \cdot X, Y) = g(S \cdot Y, X),$
Codazzi equation	$\nabla_X S \cdot Y = \nabla_Y S \cdot X,$
Gauß equation	$\det(S) = K.$

Minimality condition: $\quad \operatorname{trace}(S) = 0.$

The close connection between minimal surfaces and holomorphic maps. The minimal and selfadjoint shape operator S has with respect to an orthonormal basis a trace free and symmetric matrix $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$. This implies $-K = -\det(S) = a^2 + b^2$, hence $S^2 = -K \cdot \mathrm{id}$. This proves *the conformality of the normal Gauß map* since scalar products change only by the scaling factor $-K$:

$$\langle TN(X), TN(Y)\rangle = \langle TF(S \cdot X), TF(S \cdot Y)\rangle = g(S \cdot X, S \cdot Y) = -K \cdot g(X, Y).$$

In particular, we obtain the meromorphic Gauß map G if we compose the normal Gauß map N (which is orientation reversing but angle preserving) with an orientation reversing (and angle preserving) stereographic projection $St : \mathbb{S}^2 \to \mathbb{C}$ as follows $G := St \circ N$.
The complex structure J has with respect to any orthonormal basis the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This and the above matrix of S (which was implied by minimality and symmetry) give $J \circ S = \begin{pmatrix} -b & a \\ a & b \end{pmatrix} = -S \circ J$. This implies first that $S^* := J \circ S$ is again symmetric and has $\operatorname{trace}(S^*) = 0$. Secondly, $\det(J) = 1$ implies that the Gauß equations for S and S^* are equivalent. And finally $\nabla J = 0$, i.e. $\nabla S^* = J \cdot \nabla S$, implies that the Codazzi equations for S and S^* are also equivalent. *Therefore g, S^* satisfy the integrability conditions and define another minimal immersion F^** (of simply connected pieces or coverings). These two pairs of surface data, $\{g, S\}$ and $\{g, S^*\}$, belong to minimal immersions F, F^* which fit together as real and imaginary part of a holomorphic map $F + i \cdot F^*$ for the following reason. We rewrote the second surface equation in its covariant form. This shows immediately that the covariant derivative of the ($\mathbb{R}^3$-valued) 1-form TF, which is equal to $-g(S \cdot X, Y)N$, has trace 0 so that *the three coordinate functions F^j are Riemannian harmonic.* Together with $\nabla J = 0$ the covariant surface equation shows further that the derivative of the ($\mathbb{R}^3$-valued) 1-form $TF \cdot J$, which is equal to $-g(S \cdot J \cdot X, Y)N$, is again symmetric (equal to $-g(X, S \cdot J \cdot Y)N$), i.e. the exterior derivative of $TF \circ J$ is 0. Therefore $TF \circ J$ is (on simply connected domains again) the derivative of some other map. If we define F^* by $TF^* := -TF \circ J$ then we have proved that the surface equations for $\{TF, N\}$ with data $\{g, S\}$ imply immediately that $\{TF^*,\ N^* := N\}$ are solutions for the surface equations with data $\{g, S^*\}$. F^* is called the *conjugate minimal immersion.* Moreover,

$F + iF^* : M^2 \to \mathbb{C}^3$ *is not only differentiable but even holomorphic* because the Cauchy-Riemann equations $T(F+iF^*)\cdot J = i\cdot T(F+iF^*)$ are satisfied. For numerical computations it has been extremely convenient that F^* can be obtained by one integration from *first* derivative data, namely from $TF^* := -TF \circ J$. (The *second* order surface data $\{g, J \circ S\}$, which are numerically more difficult to obtain, determine the surface via an ODE.)

Extension of "conjugate minimal surfaces" to $\mathbb{S}^3$ and $\mathbb{H}^3$. Some of these observations carry over to minimal surfaces in spheres $\mathbb{S}^3(c^2)$ or hyperbolic spaces $\mathbb{H}^3(-c^2)$ of curvature c^2 resp. $-c^2$. One cannot speak of harmonic coordinate functions of minimal surfaces in these spaces. But the surface equations and the integrability conditions are almost the same. One only has to interprete N as a unit normal field along the immersion F, then the surface equations hold. The first two integrability conditions stay the same and the Gauß equation needs only a small adjustment:

Gauß equation in $M^3(k)$: $\qquad k + \det(S) = K.$

Therefore: if $\{g, S\}$ are minimal surface data in a space $M^3(k)$ of constant curvature k, then $\{g, (\mathrm{id}\cos\alpha + J\sin\alpha)\cdot S\}$ is a 1-parameter family of isometric (and in general non-congruent) minimal surface data in $M^3(k)$, the socalled *associate family*.

Extension to constant mean curvature surfaces. In fact, minimal surface data $\{g, S\}$ in one space form $M(k)$ provide constant mean curvature $\pm c$ surface data $\{g, S \pm c\cdot \mathrm{id}\}$ in another space form $M(k - c^2)$. I learnt this from [La], I am told Lawson heard this from Calabi and in any case, it is immediate from the surface equations and their integrability conditions since $\det(S \pm c\cdot\mathrm{id}) = \det(S) + c^2$. We will use this in section 3.

Symmetry lines of minimal surfaces. Before we can exploit this geometric transformation of a simply connected minimal surface to its conjugate minimal surface we need one more piece of geometric information. The usual Frenet theory of curves in 3-dimensional space forms fails at points where the curve has curvature = 0; in particular it cannot handle geodesics. This problem goes away for curves on surfaces. Given a unit speed curve γ in the domain of a parametrized surface $F : D^2 \to M^3$ with normal field $N : D^2 \to TM^3, N(p) \perp \mathrm{image}(TF_p)$. We then choose as frame along the image curve $c := F\circ\gamma$ the tangent field $e_1 := TF(\dot\gamma)$, the conormal field $e_2 := TF(J\cdot\dot\gamma)$, and the surface normal field $e_3 := N\circ\gamma$. We then have with geodesic curvature κ_g, normal curvature $k_n = \langle \dot e_3, e_1\rangle = g(S\dot\gamma, \dot\gamma)$ and normal torsion $\tau_n = \langle \dot e_3, e_2\rangle = g(S\dot\gamma, J\cdot\dot\gamma)$ the following

Frenet equations for curves on surfaces:

$$\dot e_1 = \kappa_g\cdot e_2 - k_n\cdot e_3,\quad \dot e_2 = -\kappa_g\cdot e_1 - \tau_n\cdot e_3,\quad \dot e_3 = k_n\cdot e_1 + \tau_n\cdot e_2.$$

Data on the conjugate surface: $\kappa_g^* = \kappa_g,\ k_n^* = -\tau_n,\ \tau_n^* = k_n.$

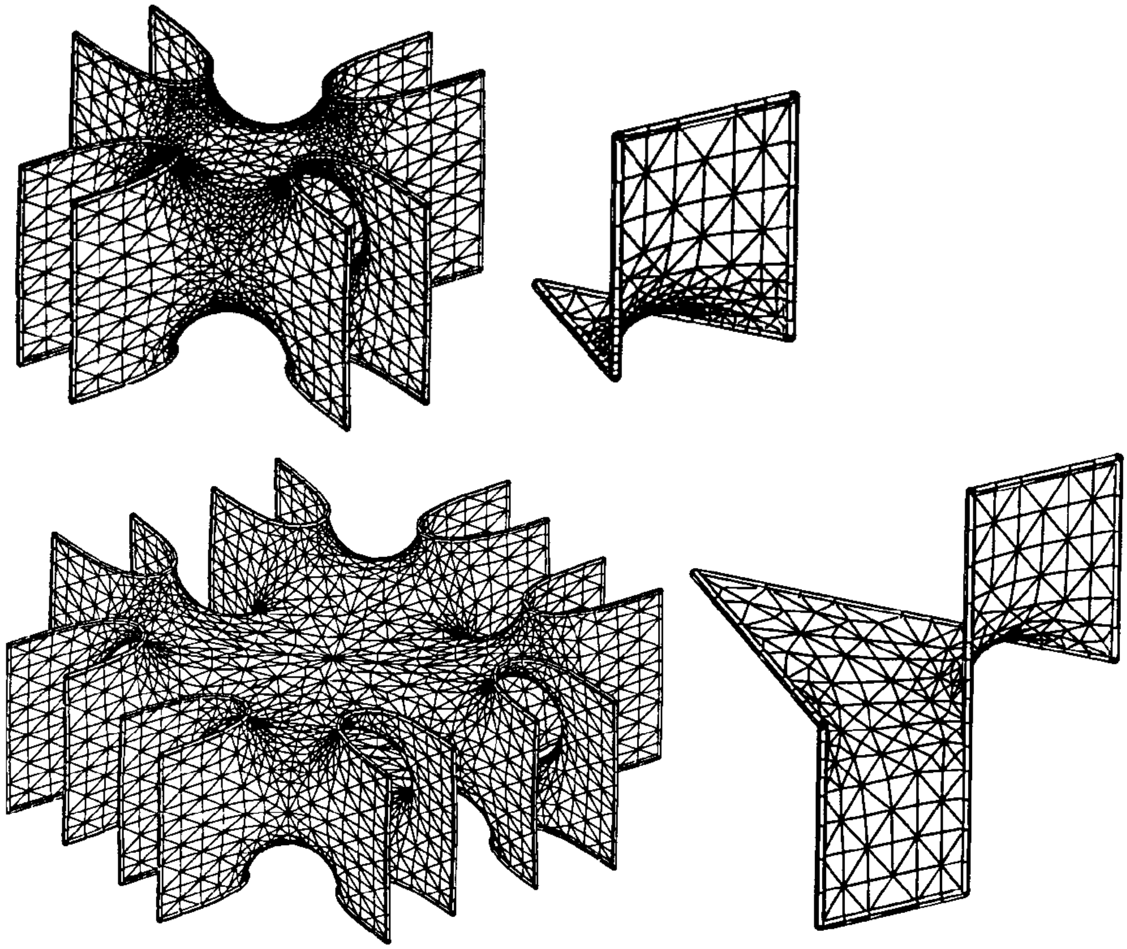
We discuss these equations for geodesics, i.e. $\kappa_g = 0$. Curves are principal curvature lines iff $\tau_n = 0$ and asymptote lines (vanishing normal curvature) iff $k_n = 0$. Observe that a principal curvature line on a minimal immersion is an asymptote line on the conjugate immersion and vice versa. But a geodesic asymptote line has vanishing tangential and normal curvature, hence is even a geodesic in the space form $M(k)$. Consider next a geodesic curvature line; the Frenet equations show (i) that the surface normal $N \circ \gamma$ is the principal curvature normal of the curve and (ii) that its torsion in $M(k)$ is 0, i.e., such a curve is planar (or lies in a 2-dimensional totally geodesic subspace). Finally, for minimal surfaces in any space form we have the

Reflection principle for minimal surfaces in $M^3(k)$:
a) $180°$ *rotation around a geodesic asymptote line (in fact a geodesic in $M^3(k)$) is a congruence of the minimal surface.*
b) Reflection in the plane of a geodesic principle curvature line is a congruence of the minimal surface.
c) Plateau solutions in polygonal contours are sufficiently regular at the boundary so that the symmetry from a) can be used to analytically extend the Plateau piece across each boundary segment. If the conjugate surface is considered then this extension is transformed into the symmetry b).

Embeddedness criterion. The integration of the surface equations usually does not say whether the obtained immersion is in fact an embedding. The following result is an easily applied criterion which covers many interesting cases.
R. Krust's conjugate graph theorem in $\mathbb{R}^3$:
If a minimal surface is a graph over a convex domain then all surfaces of the associate family are graphs (usually not over convex domains) and hence embedded, [DHKW], 118-119.

Some singly periodic examples in $\mathbb{R}^3$. (I give more details for the triply periodic case.) The conjugates of Jenkins-Serrin graphs over *equilateral* convex $2n$-gons were already quoted in the introduction. Being equilateral is a trivial sufficient condition under which the results of [JS] can be applied. And the fact that the strips between the vertical lines of the graph all have the same width implies that, on the conjugate graph, all the pairs of horizontal symmetry lines have the same vertical distance. These conjugate patches are fundamental domains for a rich family of deformations of the singly periodic Scherk saddle towers. Embeddedness follows from Krust's theorem. — Each pair of adjacent saddles of the most symmetric of these saddle towers were connected by a "catenoid like" handle in [Ka1], p. 107-110, giving toroidal saddle towers.
The Jenkins-Serrin criterion allows certain angles of the convex polygon to be equal to π. Corresponding neighboring wings of the saddle tower are then parallel. This suggests to "cut one pair of such parallel wings off, modify the cuts to be planar symmetry lines and reflect to get other toroidal saddle towers". Technically this is done by replacing the boundary values $\pm\infty$ of the pair of edges with angle π between them by finite boundary values $\pm n$. The boundary symmetry lines corresponding to the two horizontal edges at height $\pm n$

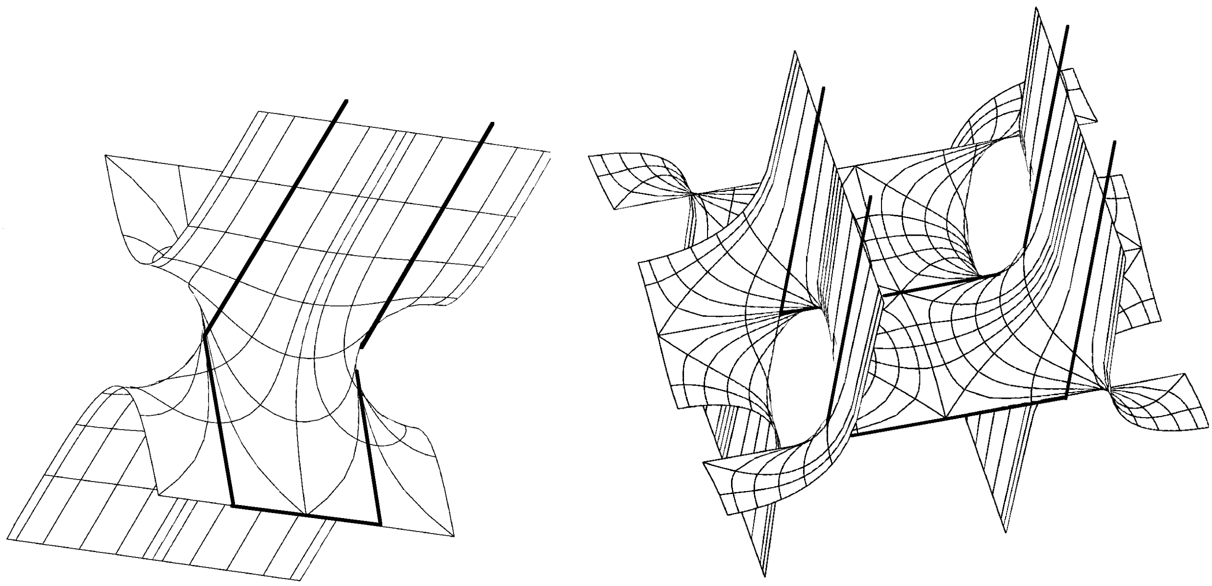
are no longer straight. Therefore they connect two horizontal symmetry lines the distance of which is *less* than the distance between all other neighboring horizontal symmetry lines (which are all equal to the edge length of the equilateral polygon). This has to be remedied by making the edges with finite Dirichlet data a bit longer. If we make the convex polygon *symmetric* (with respect to the symmetry line of the angle π) then this is a 1-parameter problem which is solved with the intermediate value argument, effortless if we permit ourselves to take n very large. The following figures are taken from [KP] p.2098, note that these surfaces are computed by Polthier as discrete minimal surfaces in the sense of Pinkall and Polthier. They suggest further modifications: instead of double connections one can take triple connections or more. These higher parameter problems have not been treated.



Examples of saddle towers with many wings.

To obtain the singly periodic examples to the left one has to imagine the polygonal contours to be infinitely high and get minimal graphs with [JS]. The conjugate of such a patch is a fundamental piece for the saddle tower to its left. – In the first case one can extend the Jenkins-Serrin piece by repeated rotation around its two horizontal edges to a graph over a square, then reduce the height of two adjacent strips from $\pm\infty$ to $\pm n$. This is still a Jenkins-Serrin contour. The conjugate of its minimal graph generates a saddle tower where one pair of wings has finite length. Two such surfaces fit together along the pair of symmetry lines between them after one adjusts one parameter, see text above.

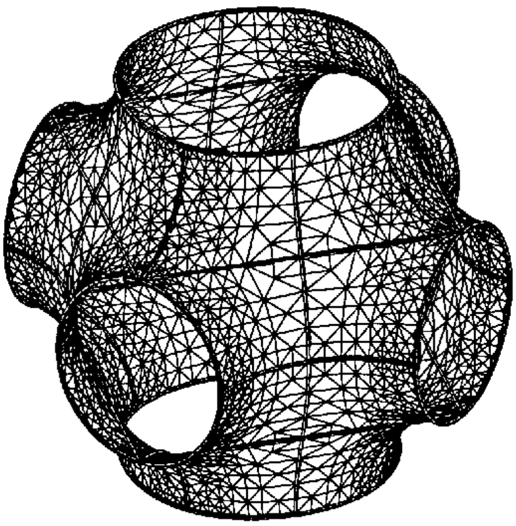
Conjugate construction of some doubly periodic examples in $\mathbb{R}^3$. The top figure above (taken from [KP] p.2098) suggests also 1-parameter problems to obtain doubly periodic surfaces. We only have to take two such pairs of edges with angle π in symmetric position, e.g. by subdividing a rectangle with edgelengths $2+\epsilon$ and k. All the neigboring infinite horizontal symmetry lines then have vertical distance 1 and the distance of the finite horizontal symmetry lines from their neighbors is controlled by ϵ. Higher parameter versions of this idea, suggested by the bottom figure, have not been treated. Another possibility is to increase e.g. the parameter a in the contours P_1, P_2 of section 1, see [Ka1] pp.102-107. Also, the idea of gluing catenoid like handles into already known doubly periodic surfaces has succeeded, for one handle in Scherk's doubly periodic surface see [HKW] pp.138-141. Many handles have been constructed by Wei, Thayer, Wohlgemuth or Weber, mostly unpublished, mostly using the Weierstraß representation.

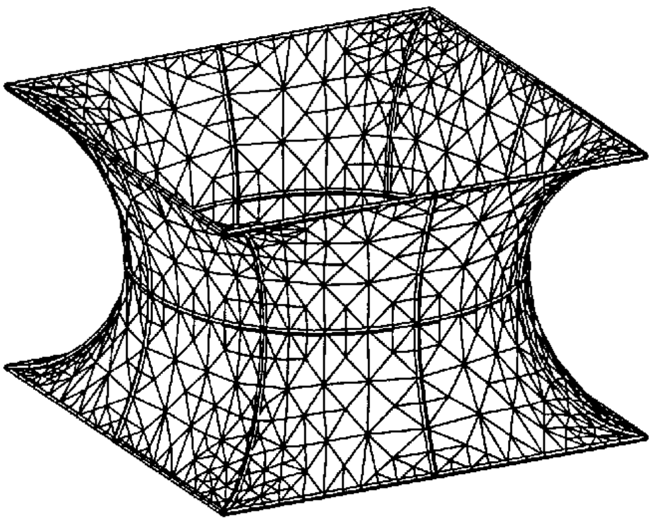


Two minimal surfaces that are analytically continued from Jenkins-Serrin graphs.
Left: The Dirichlet data on the edges of a rectangle are $0, 0, 0, \infty$.
Right: The Dirichlet data on the edges of a rectangle are $0, \infty, 0, \infty$.
The conjugate surfaces of both of these are also doubly periodic embedded minimal surfaces. If ∞ is replaced by some finite height then one obtains triply periodic surfaces.

Description of many triply periodic examples in $\mathbb{R}^3$. Here already the 1-parameter possibilities are richer than I can cover. I begin the detailed explanations with the contour P_1 of section 1 which needs no parameter adjustment. The disk type Plateau solution has a hexagonal polygon as boundary, i.e., a boundary made up of six geodesic asymptote

lines. The conjugate simply connected piece is therefore bounded by six geodesic principal curvature lines and one can analytically extend the piece by reflection in the planes of these boundary arcs. At each vertex the two boundary arcs meet with a $\pi/2$-angle; repeated reflections in the arcs which leave from one vertex therefore extend the piece to a larger one with the vertex as a smooth interior point. Now we consider the six symmetry planes of the boundary arcs and we claim that they are the boundary planes of a brick. For this it is crucial that a minimal surface and its conjugate have the same Gauß map. If we orient the contour P_1 so that the edges are parallel to the coordinate axes in $\mathbb{R}^3$ then it follows that the symmetry planes of the conjugate piece are orthogonal to the coordinate axes. Since this nonplanar conjugate piece also sits in the convex hull of its boundary we have that the symmetry planes are indeed the boundary planes of a brick which contains the six symmetry arcs on its six faces and the interior of the conjugate piece in its interior.
The next claim is that each of the six boundary arcs has a monotonely rotating normal and the total rotation is $\pi/2$, i.e., each is one quarter of a closed convex curve with two orthogonal symmetry axes. We look again at the polygonal Plateau contour. It follows from [Ni] that the interior is a graph over the interior of the convex projection of the boundary *also in the limiting case where certain edges project to a point.* This implies that the tangent planes of the Plateau piece along such a vertical edge must rotate monotonely since otherwise the interior could not be a graph. And since the Plateau piece is also in the convex hull of its boundary this monotone rotation can only be through $\pi/2$, which is the angle of the projected contour (at the vertex to which the vertical edge projects). This proves the claim since the normal rotates through the *same angle* along the boundary arc of the conjugate piece.

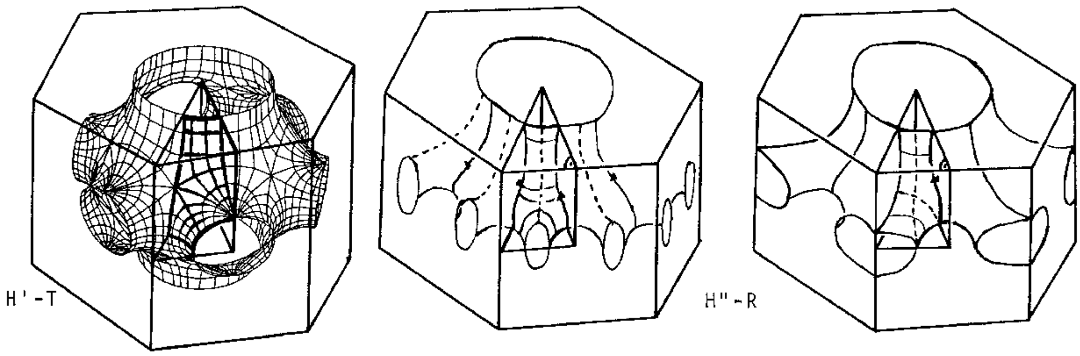




Left: translational fundamental domain of a deformation of Schwarz P-surface (the tope hole is bigger!). Right: the minimal annulus between parallel squares is part of this P-surface. Again, these are discrete minimal surfaces by Polthier.

Now we have the complete picture: eight of the conjugate pieces fit together to a translational fundamental domain of the complete surface. This larger building block sits in a brick with twice the edge lengths of the previous brick (around the conjugate piece); and each face of this brick is met by a closed convex symmetry line of the minimal surface. A. Schoen named the member of this family with cubical symmetry *Schwarz P-surface* (where the P is referring to the primitive cubical lattice). See [DHKW], fig. 22(a)-(c) and plates II(b), V(d).

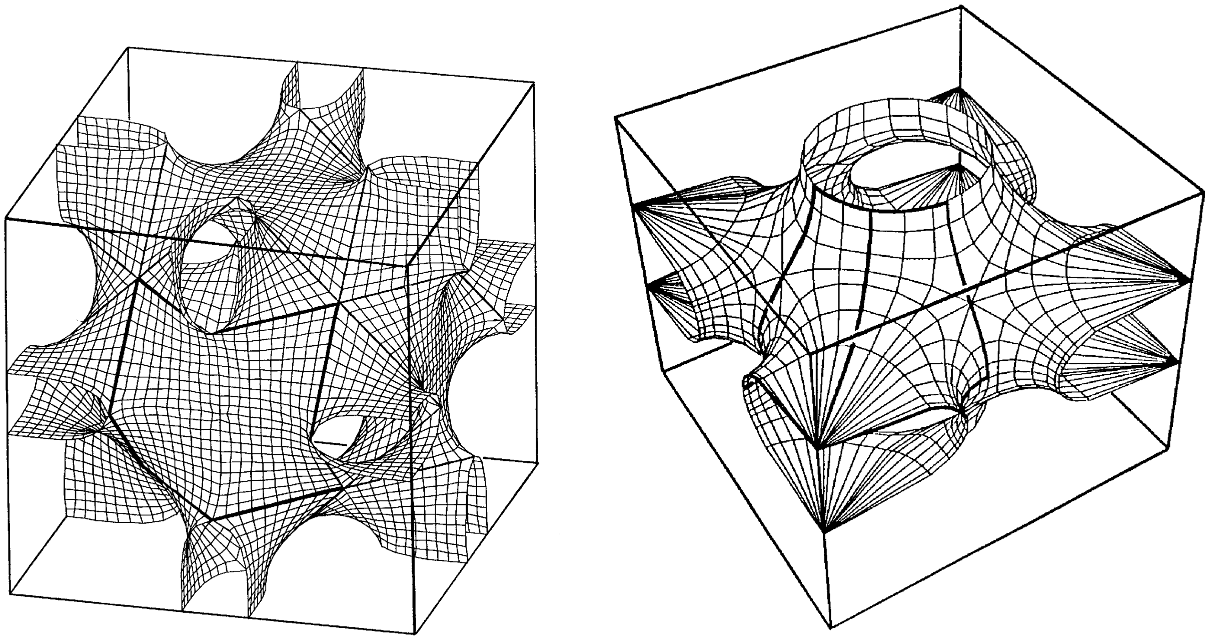
Summary of similar examples in $\mathbb{R}^3$. In the same way one can have minimal surfaces which meet all the faces of prisms over a regular hexagon resp. over a regular triangle in convex curves, [DHKW] plate III(a),(b). But these two surfaces are in fact only different views of the same surface, see the illustration [Ka2], p.298. The conjugate contour of a fundamental piece consists of five edges of the prism over a $(30, 60, 90)$-triangle, [Ka2], p.330 and below. A. Schoen named it H'-T-*surface* .



To the left and right are translational fundamental domains of two minimal surfaces of A Schoen, named H'-T- and H''-R-surface. The middle one is a 1-parameter modification. (Conjugate contours for these surfaces see below.)

The rhombic dodecahedron is another tessellating polyhedron; the conjugate construction with the other contour on [Ka2], p.330 gives a minimal surface which meets all the faces of the rhombic dodecahedron in convex curves. A. Schoen named it *F-Rd-surface.* In all these cases we call the catenoid-like connections to the neighbouring cristallographic cells *Schwarz-handles.* In what follows the emphasis is on the construction of minimal surfaces which *combine* features of better known simpler surfaces. The term Schwarz-handle is meant to direct the attention to one such feature.
E.R. Neovius, a student of H.A. Schwarz, has constructed the following minimal surface with a translational fundamental domain in a cube. It is connected with the neighbouring cristallographic cells by twelve "arms" crossing the midpoints of the edges; one can also say, a handle like a thickened cross connects the surface pieces in the four cubes around one edge. We call this connection a *Neovius-handle.* Conjugate contours are discussed in [Ka2], for more illustrations see [DHKW] plate VII(b) and [Ka2], p.300. A. Schoen's name for the *Neovius-surface* is $C(P)$-*surface.*

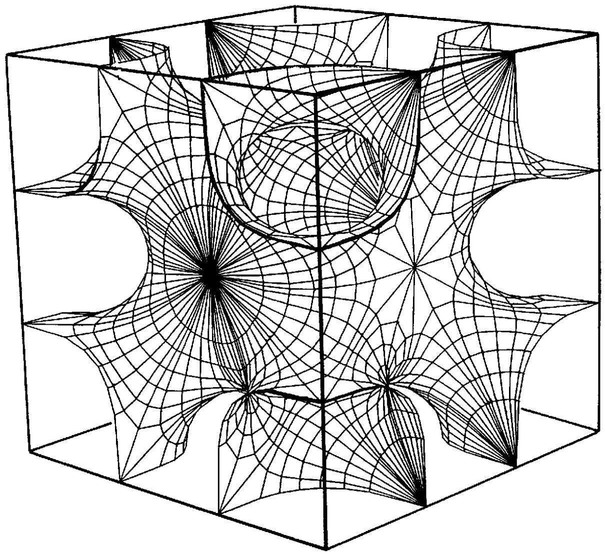
Also without parameter adjustments of the Plateau contours we can obtain minimal surfaces which continue to meet the top and bottom face of a prism over a square, over a regular hexagon or over a regular triangle in closed convex curves (Schwarz-handles), but the vertical edges are met by (4-fold, 3-fold resp. 6-fold) Neovius handles. The conjugate contours consist of five edges of the prisma over a $(\alpha, \pi/2 - \alpha, \pi/2)$-triangle with $\alpha = \pi/4, \pi/3, \pi/6$, [Ka2], p.308. For illustrations see [DHKW] plates IV(e) and [Ka2], p.299. Schoen's names are S'-S''-, H''-R-, T'-R'-*surfaces.*

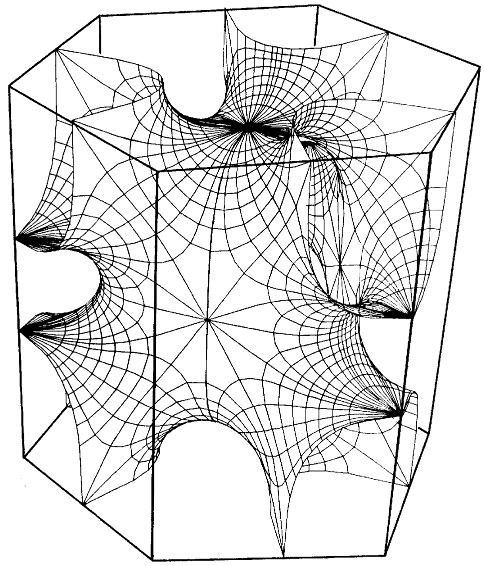


Neovius minimal surface with cubical symmetry carries the same lines as Schwarz P-surface. To the right is A. Schoen's mixture of Schwarz handles and Neovius handles, his S'-S''-surface.

We mentioned the annular minimal surface, *Schwarz' H-surface*, bounded by two parallel regular triangles. A translational fundamental domain of this surface meets the top and bottom face of a hexagonal prisma in closed convex curves and every *second* vertical edge is met by a 3-fold Neovius-handle. A contour for a conjugate construction is in [Ka2], p.328. For more illustrations see [DHKW] plate VI(a)-(d), [Ka2], p.297.
Another way to fit a minimal surface into a cristallographic cell does not occur in Schwarz' school, the first example is A. Schoen's *I-WP-surface.* The translational fundamental domain in a cube sends arms towards all the eight vertices so that each arm is cut by the cube in three convex arcs on the faces around one vertex of the cube. The conjugate contour is in [Ka2], p.331, illustrations are in [DHKW] plate VII(a), [Ka2], p.297. We call the handle *I-WP-handle.* This conjugate contour depends on an angle α, with $\alpha = \pi/4$ for Schoen's surface and with $\alpha = \pi/3$ for a surface whose translational fundamental domain sits in a hexagonal prism with Schwarz-handles to the top and bottom faces and 3-fold

Neovius handles to every *second* vertex. Illustrations: [DHKW] plate VI(e), [Ka2], p.297.





Two triply periodic minimal surfaces in cristallographic cells of $\mathbb{R}^3$.
Left: Schoen's I-Wp-surface, right: a similar hexagonal one.
The conjugate contours are quadrilateral and hexagonal polygons.

Many more examples in $\mathbb{R}^3$ via 1-parameter adjustments. From now on we consider the previous examples as construction material and we want to use them for more complicated examples, but we only want to solve 1-dimensional intermediate value problems. I cannot exhaust the possibilities, and it will also be clear that with 2-parameter problems one should expect to find a huge collection of further examples. In [Ka2], pp.350-356 I have given fifteen 1-parameter contours designed to mix the handles we have seen so far. The paper [KP] was written to explain the conjugate Plateau construction to a wider audience and with Polthier's discrete minimal surface programs we also illustrated this mixing of handles: [KP] p.2099 shows how *I-WP*-handles grow out of a Schwarz-P-surface until they are long enough to reach the faces of the cube around the P-surface; the result is a surface which combines the handles of the Schwarz-P- and the Schoen-I-WP-surfaces. On p.2100 we combine the handles of the I-WP- and the Neovius surface and on p.2101 we combine the handles of the Schwarz-P- and the Neovius surface. The contours given in [Ka2] also add to the Schwarz-handles of Schoen's F-Rd-surface either I-WP-handles to all the rhombic dodecahedron's 4-valent vertices or to all its 3-valent vertices. Other contours give (i) both types of I-WP-handles to all vertices of the rhomic dodecahedron (the Schwarz-handles are omitted since they require a further parameter) or (ii) Neovius-handles to all edges of the rhombic dodecahedron (further handles need more parameters to be adjusted.).
Another modification is illustrated in [KP], pp.2096,2097. We insert vertical catenoid-like handles into all the horizontal Schwarz-handles of the P-surface (2096) or into all the

horizontal Schwarz-handles of the H'-T-surface (see contours below). This works because the conjugate contours allow to define 1-parameter families of contours which give already known minimal surfaces (without parameter adjustments) at the endpoints of the parameter range. In the family we have in general one undesired period, a symmetry arc along which the normal rotates through 180°, but the parallel normals at the endpoints are not on the same *line*. On both parameter boundary values the normal of that arc rotates only through 90°, i.e., this symmetry arc converges (at the parameter boundary) to a *rising*, resp. *falling*, convex arc. The intermediate value theorem applies, giving one parameter value where also the symmetry line with the 180° rotating normal closes to a convex curve. – The idea of adding Schwarz-handles at suitable places can often be accomplished by adjusting just one parameter for the length of the handle.

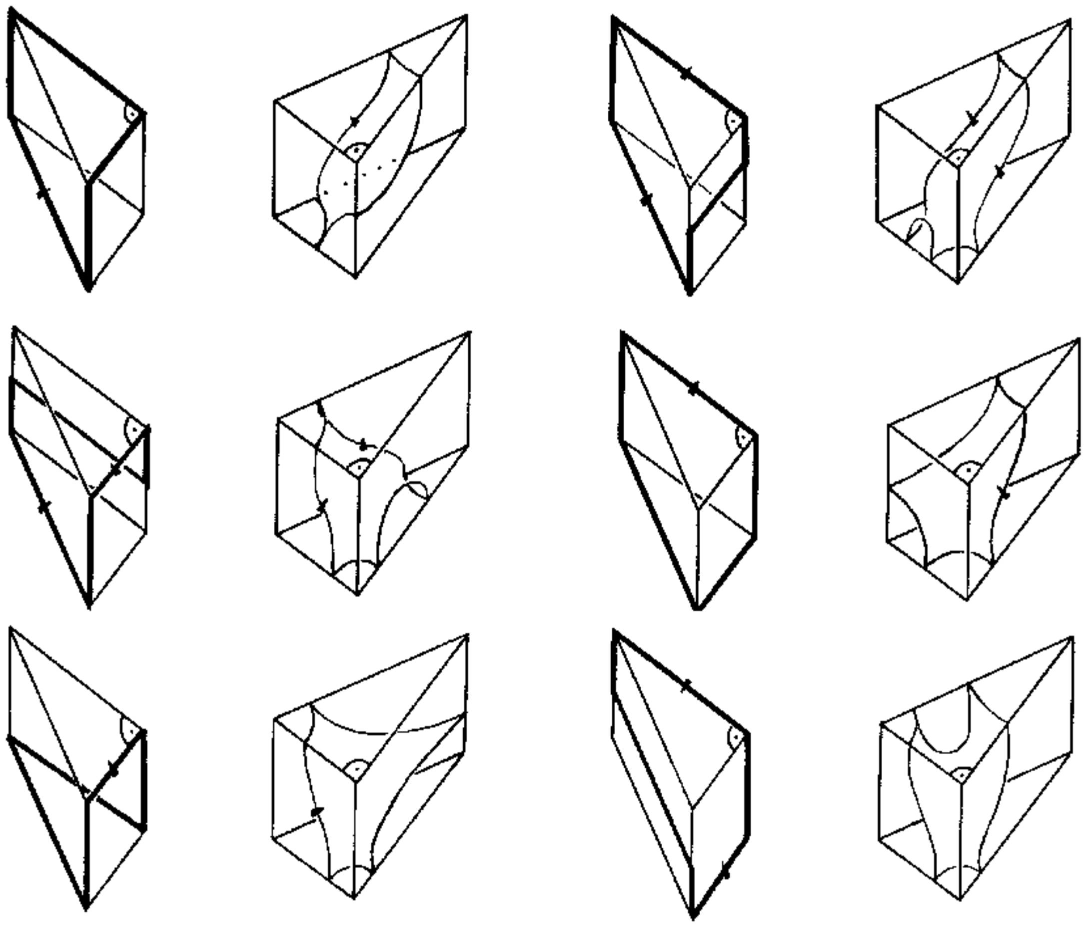
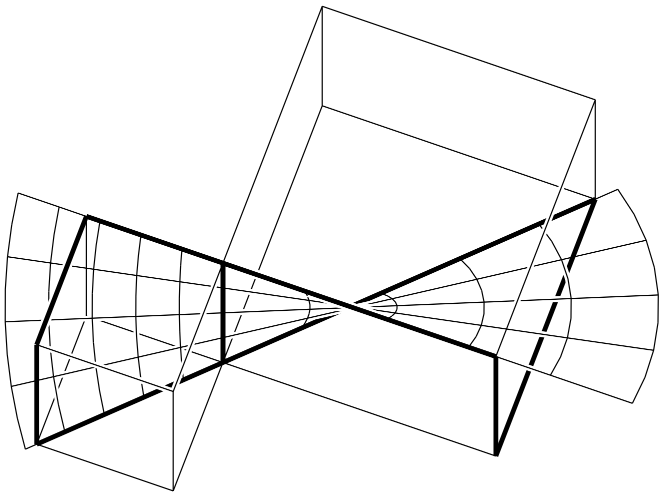


Illustration of intermediate contours

On the boundary of a 30°-60°-90°-prism we see six polygonal contours and to the right of each we see a sketch of the conjugate minimal patch in the same 30°-60°-90°-prism. These conjugate patches can be extended to triply periodic complete embedded minimal surfaces by repeated reflection in the faces of the prism. The three pentagonal contours lead to surfaces of A.Schoen without period killing. The three hexagons have one horizontal edge on a face of the prism and one has to use the intermediate value theorem to find the correct height of this edge. The contour to the right in the first row gives the surface between Schoen's H'-T- and H''-R-surfaces above. The other two intermediate contours add vertical handles towards the horizontal faces of the hexagonal prism.

One can also insert a 4-fold Neovius-handle into the vertical necks of the Schwarz-P-surface. This splits each vertical catenoid like neck into four thinner parallel ones, [KP], p.2104. Analogous modifications apply to other surfaces of the "construction material".
In all the highly symmetric cristallographic prisms we have seen surfaces with vertical Schwarz handles but with *different* types of horizontal handles (Schwarz- and Neovius-handles). In the quadratic prism we have Schwarz' P-surface and Schoen's S-S'-surface; in the hexagonal prism we have Schoen's H'-T- and H''-R- and Schwarz' H-surfaces and similar examples exist in the prisms over equilateral triangles.



Combining two contours

The Plateau solution for the pentagon in the smaller (left) quadratic prism is conjugate to a piece of Schwarz' P-surface, and the Plateau solution for the pentagon in the larger prism is conjugate to a piece of Schoen's S-S'-surface. If one removes the common edge of the two then one obtains a contour whose Plateau solution is a candidate for the surface which looks like the one obtained by stacking the two prisms (one with a P- the other with an S-S'-surface) on top of each other. Note that the new contour still has one convex projection (in the direction of one edge of the squares) so that its Plateau solution is a Nitsche graph [Ni], in particular unique and hence depending continuously on the edge lengths. – The picture also shows a helicoid (with the removed edge as axis) that is a barrier for this contour. Such barriers can be used if more precise control of the Plateau solution is needed.

One observes that the convex symmetry lines on the top and bottom faces of the prisms are almost circles, see also [KP], pp.2096,2097. This suggests to glue one of these on top of the other, minimally of course. The above figure explains why this is an easy task for the conjugate contour: just put the two contours together along the edge which corresponds to the horizontal symmetry line; then omit that edge. One problem remains to be solved by an 1-parameter adjustment: the two types of horizontal handles must have the same length, i.e., their vertical symmetry planes (which are by constrcution parallel) must coincide. If the common height of the two prisms converges to zero then the Plateau solutions converge to the union of two squares, and so do their conjugate patches. This says that those handles, which come from the part of the contour over the smaller square, are the shorter ones, at least for very small heights of the prims. Continuity and the intermediate value theorem finish the proof.
I hope this is enough to convince the reader that the conjugate Plateau method for minimal surfaces in $\mathbb{R}^3$ is so flexible that already with contours, where only one parameter has to be adjusted, we can obtain a very large collection of triply periodic minimal surfaces. In most cases embeddedness comes for free because the Plateau contour is a graph, by Krust's theorem the conjugate piece is embedded and the different congruent pieces sit in different fundamental cells for the symmetry group in question. – One would also expect that the possibilities for 2-parameter adjustments are more than one would like to describe.

3. Conjugate constant mean curvature surfaces.

Recall from section 2: if we are given minimal surface data $\{g, S\}$ (Riemannian metric g and Weingarten map S) in a space of constant curvature k, $M^3(k)$, then $\{g, S \pm c \cdot \mathrm{id}\}$ are surface data for a constant mean curvature $\pm c$ surface in $M^3(k - c^2)$. Such surface data can be integrated on simply connected domains. The constant mean curvature surface has been called the *cousin* of the minimal surface. In the examples of section 2 we have seen that the transition from a polygonally bounded Plateau piece to the conjugate surface bounded by planar lines of reflectional symmetry is the main reason for the flexibility of this method. This is true even more for the transition to constant mean curvature surfaces because only the geodesic principle curvature lines allow to use a symmetry of the space M^3, namely a reflection, to analytically extend the original piece. Then one speaks of a constant mean curvature *conjugate cousin surface* if the data $\{g, S\}$ of the minimal surface are changed to $\{g, J \cdot S \pm c \cdot \mathrm{id}\}$ (with J the complex structure, the positive 90° rotation). Quite remarkably it turns out that the derivative TF^* of the cmc1 conjugate cousin in $\mathbb{R}^3$ can be obtained explicitly from the derivative TF of the corresponding minimal surface in $\mathbb{S}^3$ and from the complex structure J; one does not have to relate these through the second order surface data. In conjugate cousin constructions one allways assumes that the minimal piece has a geodesic polygon as boundary so that the conjugate cousin is bounded by planar symmetry lines. Of course, the angles between the geodesic edges are the same as the angles between the planar symmetry arcs at corresponding vertices, and these angles have to be of the form $2\pi/k, k \in \mathbb{N}$ so that the extended surface (by repeated reflections) has the vertices as *smooth* interior points.

We first need to understand why the general case is so much more difficult than the case of conjugate minimal surfaces in $\mathbb{R}^3$ in section 2. Then we can appreciate the extra help which we get from the group structure of $\mathbb{S}^3$ when we use the conjugate cousin method to construct constant mean curvature one surfaces in $\mathbb{R}^3$. We have the simplest case possible if we start with a disk type minimal Plateau solution in a nonplanar geodesic quadrilateral with angles of the form $2\pi/k$. The conjugate minimal piece and all the conjugate cousins are then bounded by four planar symmetry arcs. Extension by reflections in the two arcs that meet at one vertex then gives a larger surface with that vertex as smooth interior point. Moreover, if the quadrilateral was chosen with some care then the four symmetry planes of the boundary arcs are *boundary planes of a simplex in M^3 that contains the conjugate cousin piece*. This is the situation in [Sm], [KPS], parts of [Po] and also in [Ka3]. For the construction of complete embedded minimal or cousin surfaces the difficulty begins now:

> We need to guarantee that the above simplex (made from the boundary symmetry planes of the cousin piece) *tessellates* $M^3(k-c^2)$.

Note that the cousin piece meets the faces of the simplex orthogonally. This means that the *dihedral angle* between any pair of symmetry planes that meet at one vertex is also the angle between the corresonding symmetry lines on the surface and therefore is one of the angles of the quadrilateral Plateau contour. In other words, four of the six dihedral angles of the simplex are known as the angles of the quadrilateral. The problem is to control the other two dihedral angles of the simplex. This is much simpler in $\mathbb{R}^3$ (where the scalar product of the normals of the planes gives the desired dihedral angle) than in a curved space. In the applications of [KPS], [Po] and [Ka3] the simplex is a fundamental domain for the symmetry group of a platonic polyhedron. This means that it has three dihedral angles equal to $\pi/2$; these and a fourth one are the angles given by the angles of the conjugate Plateau quadrilateral, as we said before. The simplification caused by the $\pi/2$-angles is that the remaining two dihedral angles of the simplex are also face angles of this simplex. But each such face angle is *the angle between the normals at the endpoints of a boundary arc* because these normals are edges of the simplex.
In $\mathbb{R}^3$ such an angle is the same as the total curvature of the symmetry arc and therefore also the same as the total rotation of the tangent plane of the Plateau piece along the edge under consideration.

> This means in particular: In $\mathbb{R}^3$ this angle is determined by the Plateau contour alone, without reference to the Plateau solution.

In spheres and hyperbolic spaces the Gauß-Bonnet theorem says that the total curvature and the desired angle between the end point normals *differ by the area of the curved triangle that is bounded by the symmetry arc and its two endpoint normals.* Therefore the remaining dihedral angles are not computable from the Plateau *contour* alone, but they can be estimated if one has good bounds for the Plateau *solution.*

A quadrilateral with prescribed angles has two free parameters in spheres, Euclidean and hyperbolic spaces and we have to choose those so that the two remaining dihedral angles of the simplex around the conjugate cousin have the correct values. One therefore has to find a closed curve in the 2-dimensional parameter space so that the image curve of the pairs of dihedral angles has winding number $\neq 0$ with respect to the correct pair of dihedral angles. This argument gets simpler for Bryant surfaces since one of the domain parameters can be specified as a function of the other such that along the corresponding curve one of the two dihedral angles is always correct while the other is too small at one end and too large at the other. The completion of these arguments requires so much work that only the mentioned simplest cases are treated in [KPS], [Po] and [Ka3]. One is far from the flexibility which we saw in the applications in section 2 to triply periodic minimal surfaces in $\mathbb{R}^3$.

Conjugate cousins in $\mathbb{R}^3$. The preceding discussion directs some extra attention to the case of constant mean curvature one surfaces in $\mathbb{R}^3$, considered as conjugate cousins of minimal surfaces in $\mathbb{S}^3$. In these cases the remaining dihedral angles are now known to be computable from the spherical Plateau *contour* without reference to the Plateau *solution*. But how explicitly can this be done? The answer is very nice. Consider $\mathbb{S}^3$ as a group, e.g. as the unit quaternions. The parallel translations of $\mathbb{R}^3$ are replaced by the *left translations* $L_q : \mathbb{S}^3 \to \mathbb{S}^3,\ L_q(p) := q \cdot p$. With these isometries we can extend every tangent vector $X \in T_{\mathrm{id}}\mathbb{S}^3$ to a left invariant vectorfield X^* by $X^*(q) := TL_q|_{\mathrm{id}}(X) = q \cdot X$. Such a left invariant vector field has constant length, the angle between two such vector fields is *constant* and the integral curves of these vector fields are great circles, e.g. if $|X| = 1$ then $t \mapsto \cos(t) \cdot q + \sin(t) \cdot X^*(q)$ is the integral curve through q. And clearly, each great circle is integral curve of exactly one unit length left invariant vector field. All this is in complete analogy to parallel vector fields on $\mathbb{R}^3$. With these notions the answer is:

> Measure the total rotation of the tangent plane of the Plateau piece along a great circle edge against left invariant vector fields then this total rotation is the same as the angle between the normals at the endpoints of the symmetry arc (which corresponds to the edge under consideration) of the conjugate cousin surface.

This says in particular that we can explicitly determine the great circle Plateau contours if we have decided which angles between symmetry planes we want to achieve. For example all the conjugate contours for triply periodic minimal surfaces in $\mathbb{R}^3$ which did not require any parameter adjustment can now be translated into great circle polygons such that the conjugate cousins of their Plateau solutions give immersed constant mean curvature one surfaces in $\mathbb{R}^3$ with the same symmetry groups as the minimal surfaces. A slightly different picture is: Scale the conjugate cousins made from very small spherical polygons up to have the same periods as the minimal surfaces, then we get *small constant mean curvature deformations* of the minimal surfaces. Such small deformations continue to be embedded. Moreover, there are deformations which are not interesting for the minimal surface: If we let the edgelength a of the contour P_1 in section 1 shrink to 0 then we get a planar contour

and a planar minimal piece. However the corresponding spherical quadrilateral does *not* lie in a great sphere, therefore we get a nontrivial Plateau piece and the corresponding doubly periodic constant mean curvature one cousin looks like one horizontal layer of the minimal surface, but with the vertical Schwarz handles between layers having shrunk to zero size.
Since we get in this way without effort many triply periodic cmc1 surfaces together with doubly periodic degenerations, it is clear that with a little effort one can get many more families of cmc1 surfaces. Among the surfaces one obtains that way are triply periodic ones which almost look like sphere packings; the large handles of the related minimal surface have shrunk to very small catenoid like connections between the spheres. In the work of N. Kapouleas portions of his surfaces are very close to very long strings of spheres. It is therefore a natural question how close to such examples one can come with the conjugate cousin method. Here a successful idea was to solve "spherical" Plateau problems not really in the sphere but for example in the universal cover of the solid Clifford torus [Gb]. This modification indeed allows to replace the catenoid like connectors between spheres by long strings of spheres with tiny necks between them.

Proof of the explicit relation between the differentials of minimal surfaces in $\mathbb{S}^3$ and the differentials of their cmc1 conjugate cousins in $\mathbb{R}^3$. We need three steps. The first treats

Cross product and complex structure. For a 2-dimensional surface M^2 with normal field N in a 3-dimensional space M^3 we have a simple relation between the complex structure J of M^2 and the normal N via the cross product in the tangent spaces of M^3. Usually the orientations are chosen so that for each tangent vector X of M^2 holds

$$X \times JX = N \qquad \text{or} \qquad JX = N \times X.$$

An immediate application is the fact that J is a covariantly parallel endomorphism field. Recall that a tangent vectorfield $t \mapsto X(t)$ to M^2 along a curve $t \mapsto c(t)$ is called *covariantly parallel along* c, iff the covariant derivative $\frac{\nabla}{dt}X(t)$ in M^3 is orthogonal to M^2. As in $\mathbb{R}^n$ extend this definition and call an *endomorphism field covariantly parallel* iff it maps parallel vector fields to parallel vector fields. The endomorphism field J has this property because of

$$\frac{\nabla}{dt}(JX) = \frac{\nabla}{dt}(N \times X) = \frac{\nabla}{dt}N \times X + N \times \frac{\nabla}{dt}X \sim N + 0 \perp M^2.$$

This parallelity is important because it says that for the covariant differentiation $\frac{D}{dt}$ of M^2 (which equals the M^2-tangential component of $\frac{\nabla}{dt}$) the composition with J behaves as multiplication of complex functions by the constant i behaves:

$$\frac{D}{dt}(JX) = J\frac{D}{dt}X, \quad \text{or} \quad \frac{D}{dt}(JSX) = J(\frac{D}{dt}S)X + JS\frac{D}{dt}X.$$

Quaternions and left translation on $\mathbb{S}^3$. We consider $\mathbb{S}^3$ as the group of unit quaternions. The tangent space at $1 \in \mathbb{S}^3$ is $T_1\mathbb{S}^3 = \mathrm{Im}\mathbb{H} \perp 1$. Left invariant vector fields are given, for each $X \in T_1\mathbb{S}^3$, by using quaternion multiplication $\bullet$ as follows:

$$X^*(q) := q \bullet X.$$

The fact that the *imaginary part* of the quaternionic product of two imaginary quaternions is their *cross product* in $\mathrm{Im}\mathbb{H}$ is easily checked on the basis $\{i, j, k\}$ with $i \bullet j = k = i \times j$, $\mathrm{Im}(i \bullet i) = 0$, etc. The covariant derivative $\frac{\nabla}{dt}$ (in $\mathbb{S}^3$) of a left invariant vectorfield along a curve $t \mapsto q(t)$, by definition the tangential part of the ordinary derivative in $\mathbb{H} = \mathbb{R}^4$, can therefore be expressed by the cross product with the tangent vector $q'(t)$ of the curve as follows:

$$\frac{\nabla}{dt} X^*(q(t)) = ((q(t) \bullet X)')^{tan} = (q'(t) \bullet X)^{tan} =$$

and because the normal part of $(q'(t) \bullet X)$ is proportional to q and $q^{-1} \bullet q \in \mathbb{R}$ we have

$$= q \bullet \mathrm{Im}(\frac{q'}{q} \bullet X) = q \bullet (\frac{q'}{q} \times X).$$

Notice that this formula says that left invariant vector fields "rotate towards the right" of covariantly constant vector fields in the following sense: If we look in the direction q' of the curve then we see the vector $X^*(q)$ and the covariant derivative of the vector field X^*, namely the vector $q \bullet (\frac{q'}{q} \times X)$, points $90°$ to the right of $X^*(q)$.

The conjugate cousin relation. Now assume that $F : M^2 \to \mathbb{S}^3$ is a minimal immersion with unit normal field $N : M^2 \to T\mathbb{S}^3$, $N(p) \perp \mathrm{image}(TF_p)$. Then $\{TF, N\}$ satisfy the (minimal) surface equations:

$$\nabla N(X) = TF(S \cdot X) \qquad \nabla^2 F(X, Y) = -g(SX, Y) \cdot N, \qquad \text{trace } S = 0.$$

We left translate the vector field N and the vector valued 1-form TF to $1 \in \mathbb{S}^3$ and, also using the complex structure J, we define:

$$n(p) := -F^{-1}(p) \bullet N(p) \qquad \omega_p(\ ) := F^{-1}(p) \bullet TF_p(J\ \ ) \qquad \text{with } F^{-1} \bullet F = 1 \in \mathbb{S}^3.$$

We claim that the $\mathbb{R}^3$-valued 1-form ω has a symmetric derivative and is therefore integrable. The integral surfaces clearly have n as normal field and their Riemannian metric $g(.,.)$ is the same as that of the minimal surface in $\mathbb{S}^3$ since left translation is an isometry. And we also claim that the shape operator of the integral surfaces is $(JS - \mathrm{id})$, i.e., they are surfaces of constant mean curvature -1. The following computations prove these claims by showing that $\{n, \omega\}$ satisfy the surface equations in $\mathbb{R}^3$ for the given Riemannian metric g and Weingarten map $JS - \mathrm{id}$. First we use the product rule to get the derivative of the quaternionic inverse F^{-1}:

$$\frac{\nabla}{dt} \left(F^{-1}(p(t)) \bullet F(p(t))\right) = 0 \ \Rightarrow\ TF_p^{-1}(p') = -F^{-1}(p) \bullet TF_p(p') \bullet F^{-1}(p).$$

Next we differentiate n using the covariant product rule. Since $n(p)$ is in the fixed tangent space $T_{\mathrm{id}}\mathbb{S}^3$ =Im$\mathbb{H}$ the ordinary derivative in that Euclidean space agrees with the covariant derivative of $\mathbb{S}^3$ (which in turn is the tangential part of the ordinary derivative in $\mathbb{H}$). Abreviate $X := p' \in T_p\mathbb{S}^3$.

$$\nabla n_p(X) = \mathrm{Im}\left(F^{-1}(p) \bullet TF_p(X) \bullet F^{-1}(p) \bullet N_p\right) \; - \; F^{-1}(p) \bullet \nabla N_p(X)$$

Use the connection with the cross product from above and the surface equation for ∇N :

$$= (F^{-1}(p) \bullet TF_p(X)) \times (F^{-1}(p) \bullet N_p) \; - \; F^{-1}(p) \bullet TF_p(SX)$$

Use $X \times N = -JX$ and $J \cdot J = -\mathrm{id}$:

$$= -F^{-1}(p) \bullet TF_p(JX) \; + \; F^{-1}(p) \bullet TF_p(J \cdot JSX)$$

Finally insert the definition of ω_p :

$$= \omega_p((JS - \mathrm{id})X).$$

Which gives the first surface equation for $\{n, \omega\}$, with shape operator $JS - \mathrm{id}$:

$$\nabla n_p = \omega_p \circ (JS - \mathrm{id}).$$

Similarly we use the covariant product rule to differentiate ω:

$$\nabla_X \omega(Y) = -\mathrm{Im}\left(F^{-1}(p) \bullet TF_p(X) \bullet F^{-1}(p) \bullet TF_p(JY)\right) + F^{-1} \bullet \nabla^2 F(X, JY)$$

Observe $X \times JY = \det(X, JY) \cdot N$ and insert the surface equation for $\nabla^2 F$:

$$= -(F^{-1}(p) \bullet (\det(X, JY) \cdot N(p)) \; - \; F^{-1}(p) \bullet (g(X, SJY) \cdot N)$$

Use $\det(X, JY) = g(X, Y)$, the symmetry of S and the skew symetry of J :

$$= -g(X, Y) \cdot ((F^{-1}(p) \bullet N(p)) \; + \; F^{-1}(p) \bullet (g(JSX, Y) \cdot N)$$

Finally insert the definition of $n(p)$:

$$= -g((JS - \mathrm{id})X, Y) \cdot n(p),$$

which is the second surface equation for $\{n, \omega\}$, with shape operator $JS - \mathrm{id}$:

$$\nabla_X \omega(Y) = -g((JS - \mathrm{id})X, Y) \cdot n.$$

Other signs come from other conventions, e.g. between J and N or between S and ∇N.

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