Ordinal Computability Theory

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A standard TURING machine

		SPACE									
		0	1	2	3	4	5	6	7	...	...
	0	1	0	0	1	1	1	0	0	0	
	1	0	0	0	1	1	1	0	0		
T	2	0	0	0	1	1	1	0	0		
I	3	0	0	1	1	1	1	0	0		
M	4	0	1	1	1	1	1	0	0		
E	⋮										
⇓	n	1	1	1	1	0	1	1	1		
	$n+1$	1	1	1	1	1	1	1	1		
	⋮										

A Hamkins-Kidder-Lewis *infinite time* Turing *machine* (ITTM)

		S P A C E						ω		
		0	1	2	3	4	5	6	7	…	…
	0	1	0	0	1	1	1	0	0	0	
	1	0	0	0	1	1	1	0	0		
T	2	0	0	0	1	1	1	0	0		
I	3	0	0	1	1	1	1	0	0		
M	4	0	1	1	1	1	1	0	0		
E	$\vdots$										
$\Downarrow$	n	0	1	1	1	0	1	1	1		
	$n+1$	0	1	1	1	1	1	1	1		
Ord	$\vdots$										
	ω	0	1	1	1	1	0	0	1		
	$\omega+1$	1	1	1	1	1	1	1	1		
	$\vdots$										
	ξ	0	1	0	1	0	1	0	1		
	$\vdots$										

An α-β-Turing machine

		S	P	A	C	E	$\Rightarrow$				β				
		0	1	2	3	4	5	6	7	…	ω	$\omega+1$	…	ζ	…
	0	**1**	0	0	1	1	1	0	0	…	0				
	1	0	**0**	0	1	1	1	0	0	…					
T	2	0	0	**0**	1	1	1	0	0						
I	3	0	**0**	1	1	1	1	0	0						
M	4	0	1	**1**	1	1	1	0	0						
E	⋮														
$\Downarrow$	n	0	1	1	1	**0**	1	1	1						
	$n+1$	0	1	1	1	1	**1**	1	1						
α	⋮														
	ω	0	1	1	1	1	0	0	1	…	**0**	1	…		
	$\omega+1$	1	1	1	1	1	1	1	1	…	1	**1**	…		
	⋮														
	ξ	0	1	0	1	0	1	0	1	…	1	1	…	**1**	
	⋮														

Limit rules

$$H(t+1) = \begin{cases} H(t)+1, \text{ if } m=1 \text{ (“go right”)}; \\ H(t)-1, \text{ if } m=0 \text{ (“go left”) and } H(t) \text{ is a successor ordinal}; \\ 0, \text{ else; (“go left”) and } H(t) \text{ is a limit ordinal.} \end{cases}$$

If $t < \alpha$ is a limit ordinal, the machine constellation at t is determined by taking inferior limits:

$$\begin{aligned} \forall \xi \in \mathrm{Ord}\; T(t)_\xi &= \liminf_{r \to t} T(r)_\xi\,; \\ S(t) &= \liminf_{r \to t} S(r); \\ H(t) &= \liminf_{s \to t, S(s)=S(t)} H(s). \end{aligned}$$

Ordinal Turing computability

Turing	space ω	space admissible α	space Ord
time ω	standard Turing machine computable $= \Delta_1^0$	-	-
time admissible α	?	α Turing machine (α-recursion theory) computable $= \mathbf{\Delta}_1(L_\alpha)$	-
time Ord	ITTM $\mathbf{\Delta}_1^1 \subsetneq$ computable $\subsetneq \mathbf{\Delta}_2^1$	?	Ordinal Turing machine computable $= L \cap \mathcal{P}(\mathrm{Ord})$

Ordinal register computability

Register machines	space ω	space admissible α	space Ord
time ω	standard register machine computable $= \Delta^0_1$	-	-
time admissible α	?	α register machine (α recursion theory) computable $= \boldsymbol{\Delta}_1(L_\alpha)$	-
time Ord	ITRM Infinite time register machine computable $= \Delta^1_1$	?	Ordinal register machine computable $= L \cap \mathcal{P}(\mathrm{Ord})$

Ordinal stack computability ...

Recursion schemes on ordinals ...

Representations of classes of sets via ordinal computability:

Infinite time register machines: $\Delta^1_1 \cap \mathcal{P}(\omega)$, hyperarithmetic reals,

α-Turing machines: $\mathbf{\Delta}^1_1(L_\alpha)$, α-recursion theory,

Ordinal Turing machines: L, the constructible sets of ordinals

Computing the constructible sets of ordinals

A recursive truth predicate:

$\alpha \in T$ iff φ_α is a bounded sentence and $(\alpha, <, G, T \cap \alpha) \vDash \varphi_\alpha$

where G: Ord $\times$ Ord $\leftrightarrow$ Ord is the GÖDEL pairing function

or

$\alpha \in T$ iff φ_α is a bounded sentence and $(\alpha, <, G, T \cap \alpha) \vDash \alpha$

A recursive definition of χ_T

$$\chi_T(\alpha) = \begin{cases} 1 \text{ iff } \exists \nu < \alpha \; H(\alpha, \nu, \chi_T(\nu)) = 1 \\ 0 \text{ else} \end{cases}$$

with

$$\begin{array}{lll} H(\alpha, \nu, \chi) = 1 & \text{iff} & \alpha \text{ is an } L_T\text{-sentence and} \\ & & \exists \xi, \zeta < \alpha \, (\alpha = c_\xi \equiv c_\zeta \wedge \xi = \zeta) \\ & \text{or} & \exists \xi, \zeta < \alpha \, (\alpha = c_\xi < c_\zeta \wedge \xi < \zeta) \\ & \text{or} & \exists \xi, \zeta, \eta < \alpha \, (\alpha = \dot{G}(c_\xi, c_\zeta, c_\eta) \wedge \eta = G(\xi, \zeta)) \\ & \text{or} & \exists \xi < \alpha \, (\alpha = \dot{R}(c_\xi) \wedge \nu = \xi \wedge \chi = 1) \\ & \text{or} & \exists \varphi < \alpha \, (\alpha = \neg \varphi \wedge \nu = \varphi \wedge \chi = 0) \\ & \text{or} & \exists \varphi, \psi < \alpha \, (\alpha = (\varphi \vee \psi) \wedge (\nu = \varphi \vee \nu = \psi) \wedge \chi = 1) \\ & \text{or} & \exists n < \omega \exists \xi < \alpha \exists \varphi < \alpha \, (\alpha = \exists v_n < c_\xi \, \varphi \wedge \exists \zeta < \xi \; \nu = \varphi \dfrac{c_\zeta}{v_n} \wedge \chi = 1). \end{array}$$

Hence χ_T is ordinal register computable.

T codes a model of the set theory SO

For ordinals μ and α define

$$T(\mu,\alpha)=\{\beta<\mu\,|\,T(G(\alpha,\beta))=1\}.$$

Set

$$\mathcal{S}=\{T(\mu,\alpha)\,|\,\mu,\alpha\in\mathrm{Ord}\}.$$

$(\mathrm{Ord},\mathcal{S},\in,<,G)$ is a model of a set theoretic axiom system SO, which describes the class of sets of ordinals in a ZFC-model.

The theory SO of sets of ordinals

- Well-ordering: $\forall\alpha,\beta,\gamma(\neg\alpha<\alpha\wedge(\alpha<\beta\wedge\beta<\gamma\rightarrow\alpha<\gamma)\wedge$ $(\alpha<\beta\vee\alpha=\beta\vee\beta<\alpha))\wedge\forall a(\exists\alpha(\alpha\in a)\rightarrow\exists\alpha(\alpha\in a\wedge\forall\beta(\beta<\alpha\rightarrow\neg\beta\in a)))$

- Infinity: $\exists\alpha(\exists\beta(\beta<\alpha)\wedge\forall\beta(\beta<\alpha\rightarrow\exists\gamma(\beta<\gamma\wedge\gamma<\alpha)))$

- Extensionality: $\forall a,b(\forall\alpha(\alpha\in a\leftrightarrow\alpha\in b)\rightarrow a=b)$

- Initial segments: $\forall\alpha\exists a\forall\beta(\beta<\alpha\leftrightarrow\beta\in a)$

- Boundedness: $\forall a\exists\alpha\forall\beta(\beta\in a\rightarrow\beta<\alpha)$

- Pairing axiom:
  $\forall\alpha,\beta,\gamma\,(g(\beta,\gamma)\leq\alpha\leftrightarrow\ \forall\delta,\epsilon((\delta,\epsilon)<^*(\beta,\gamma)\ \rightarrow g(\delta,\epsilon)<\alpha))$;
  Here $(\alpha,\beta)<^*(\gamma,\delta)$ stands for
  $\exists\eta,\theta(\eta=\max(\alpha,\beta)\wedge\theta=\max(\gamma,\delta)\wedge(\eta<\theta\vee$
  $(\eta=\theta\wedge\alpha<\gamma)\vee(\eta=\theta\wedge\alpha=\gamma\wedge\beta<\delta)))$

- G is onto: $\forall\alpha\exists\beta,\gamma\,(\alpha=g(\beta,\gamma))$

- Separation: For all L_{SO}-formulae $\phi(\alpha,P_1,\ldots,P_n)$ postulate:
$\forall P_1,\ldots,P_n\forall a\exists b\forall\alpha\,(\alpha\in b\leftrightarrow\alpha\in a\wedge\phi(\alpha,P_1,\ldots,P_n))$

- Replacement: For all L_{SO}-formulae $\phi(\alpha,\beta,P_1,\ldots,P_n)$ postulate:
$\forall P_1,\ldots,P_n\,(\forall\xi,\zeta_1,\zeta_2(\phi(\xi,\zeta_1,P_1,\ldots,P_n)\wedge\phi(\xi,\zeta_2,P_1,\ldots,P_n)\rightarrow\zeta_1=\zeta_2)\rightarrow$
$\forall a\exists b\forall\zeta\,(\zeta\in b\leftrightarrow\exists\xi\in a\,\phi(\xi,\zeta,P_1,\ldots,P_n)))$

- Powerset:
$\forall a\exists b(\forall z(\exists\alpha(\alpha\in z)\wedge\forall\alpha(\alpha\in z\rightarrow\alpha\in a)\rightarrow\exists^{=1}\xi\forall\beta(\beta\in z\leftrightarrow g(\beta,\xi)\in b))$

∞-∞-ordinal computability $=$ constructibility

A set-theoretic model $(M', \in')$ can be canonically defined inside the model $(\mathrm{Ord}, \mathcal{S}, \in, <, G) \models \mathrm{SO}$.

SO proves that $(M', \in') \models \mathrm{ZFC}$.

$(M', \in')$ can be transitivized to an $\in$-model $(M, \in) \models \mathrm{ZFC}$.

$M = L$ since M can be formed inside the minimal transitive inner model L.

$\mathcal{P}(\mathrm{Ord}) \cap L = \mathcal{S}$, i.e., $\mathcal{S}$ consists of the constructible subsets of Ord.

$x \subseteq \mathrm{Ord}$ is computable by an Ordinal TURING Machine (or by an Ordinal Register Machine) iff $x \in L$.

References:

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