Techniques for Getting Large Cardinals in Inner Models, Part 1

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Tutorial at PhDs in Logic III

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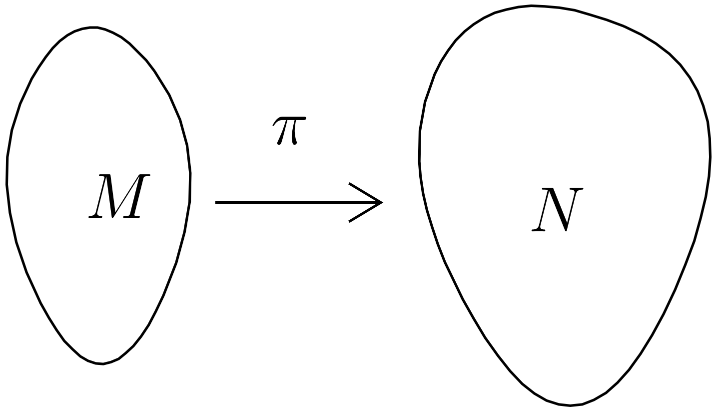
Uncountable combinatorics

Many notions in uncountable combinatorics have the form:

for all premises ...

there is an embedding $\pi: M \to N$

with properties ...

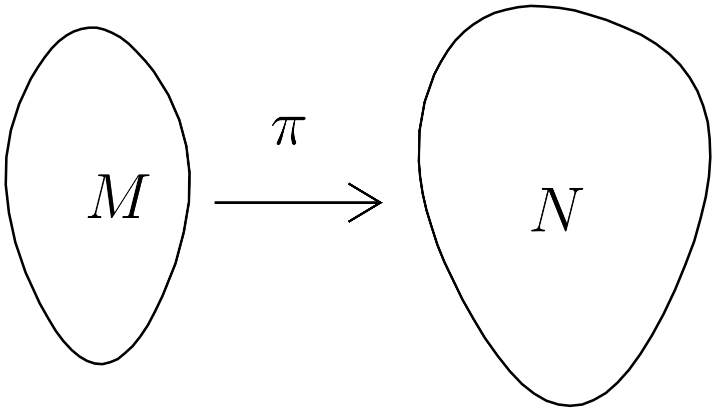


Example: downward Löwenheim-Skolem

for every infinite structure N

there is an elementary embedding $\pi : M \to N$

with M being countable

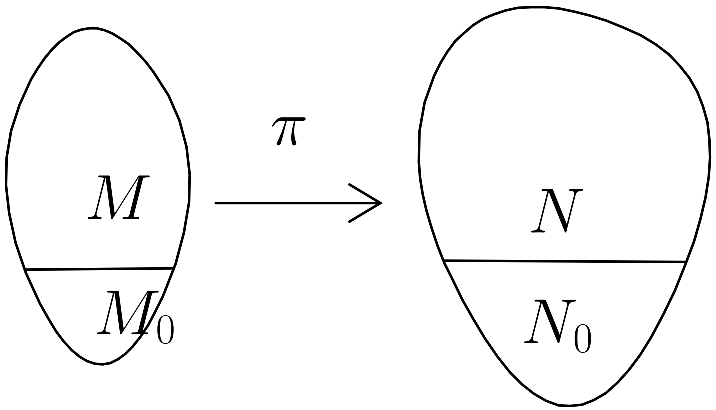


Example: Chang's conjecture (CC)

for every structure $N=(N, N_0, \ldots)$ with $\mathrm{card}(N)=\aleph_2$ and $\mathrm{card}(N_0)=\aleph_1$

there is an elementary embedding $\pi\colon M\to N$

where $M=(M, M_0, \ldots)$ and $\mathrm{card}(M)=\aleph_1$ and $\mathrm{card}(M_0)=\aleph_0$

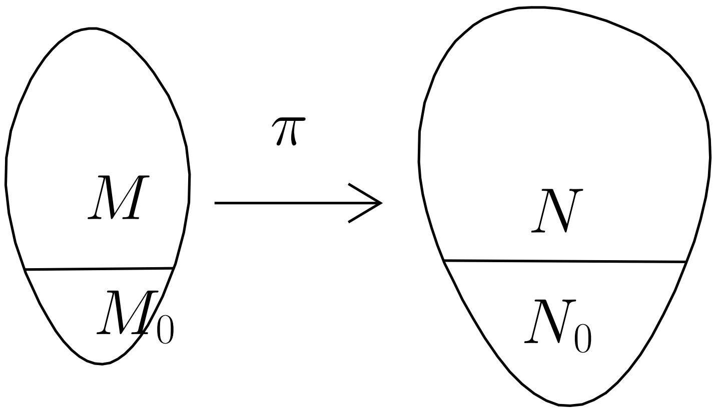


Example: Generalized Chang's conjecture ($\mathrm{CC}(\boldsymbol{\kappa},\boldsymbol{\lambda})$)

for every structure $N=(N, N_0, \ldots)$ with $\operatorname{card}(N)=\kappa^+$ and $\operatorname{card}(N_0)=\kappa$

there is an elementary embedding $\pi\colon M\to N$

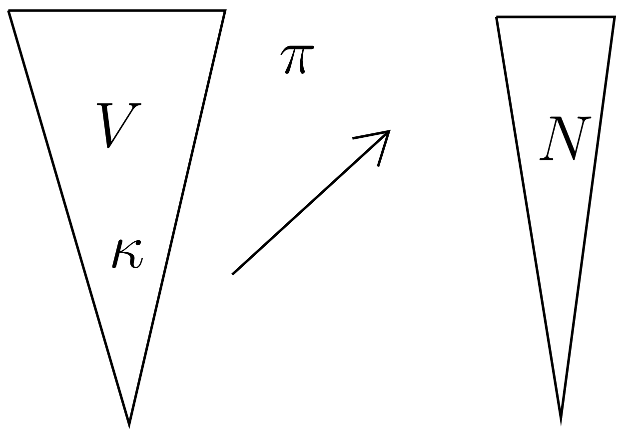
where $M=(M, M_0, \ldots)$ and $\operatorname{card}(M)=\lambda^+$ and $\operatorname{card}(M_0)=\lambda$



Example: κ is a measurable cardinal

There is an elementary embedding $\pi: (V, \in) \rightarrow (N, \in)$

with a transitive inner model N and critical point κ, i.e. $\pi \upharpoonright \kappa = \mathrm{id}$ and $\pi(\kappa) > \kappa$



Large cardinals

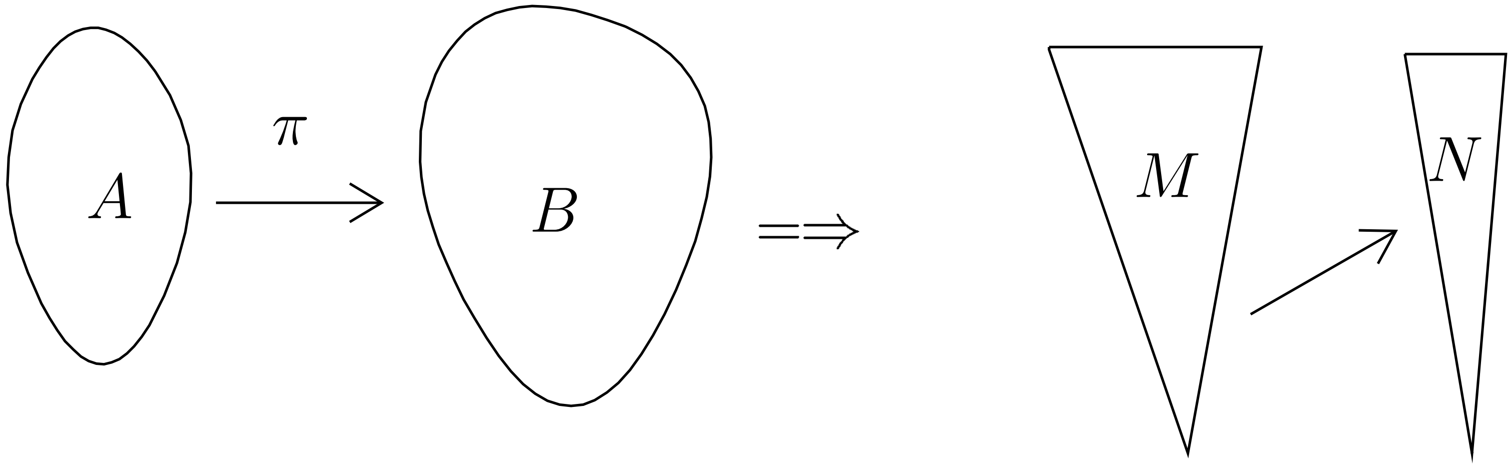
If κ is measurable then κ is (weakly) inaccessible

- κ is a regular cardinal

- κ is a limit cardinal

Getting large cardinals?

Does some combinatorial property imply that there are large cardinals in some inner model M?



(Some) large cardinals

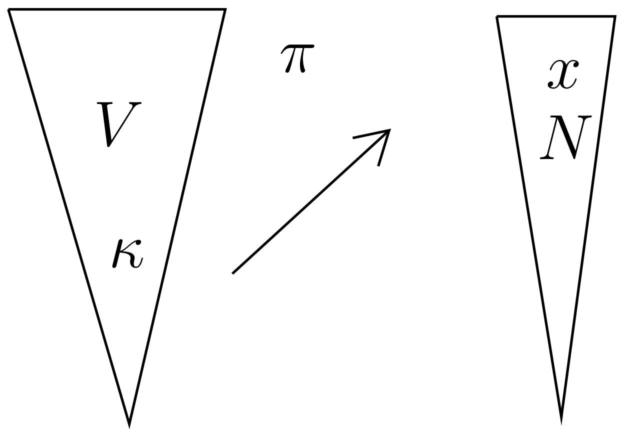
- Supercompact
- Woodin cardinals
- strong
- measurable
- Erdös cardinals
- weakly inaccessible and strongly inaccessible

Example: κ is a strong cardinal

For every set x

there is an elementary embedding $\pi: (V, \in) \to (N, \in)$

with a transitive inner model N and critical point κ such that $x \in N$

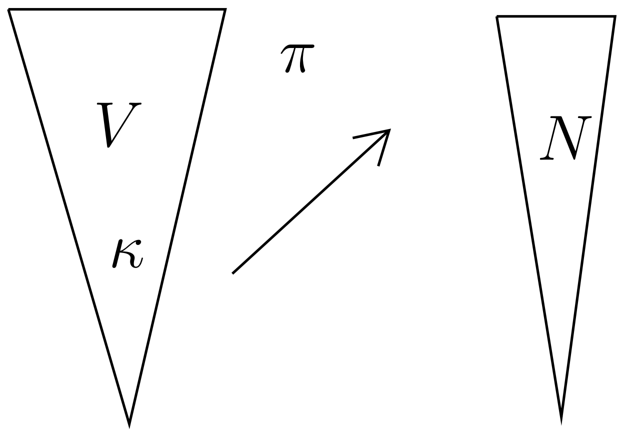


Example: κ is a supercompact cardinal

For every λ

there is an elementary embedding $\pi: (V, \in) \to (N, \in)$

with a transitive inner model N and critical point κ such that $\pi(\kappa) > \lambda$ and ${}^{\lambda}N \subseteq N$.



Formalizing large cardinal properties in ZFC

Can one replace the class quantifiers “there is a map $\pi: V \to \ldots$” by set quantifiers?

Standard method: ultrapowers modulo some ultrafilters

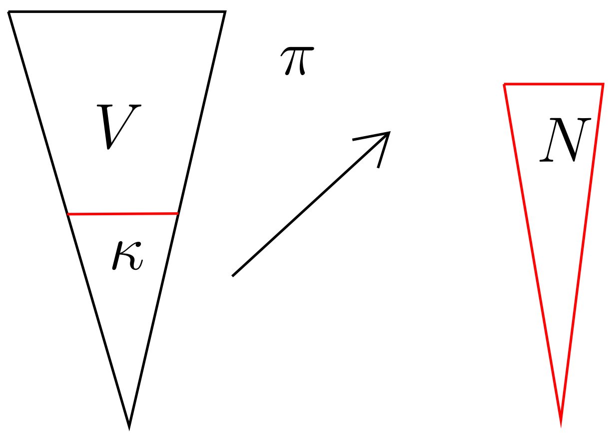
More general: extensions modulo some extender

Idea: use sufficiently long set-sized initial segments of maps instead whole maps

κ is a measurable cardinal, formalized in ZFC

There is a set-sized elementary embedding $\pi\colon (H_{\kappa^+}, \in) \to (N, \in)$

with a transitive model N and critical point κ, i.e. $\pi \upharpoonright \kappa = \mathrm{id}$ and $\pi(\kappa) > \kappa$



κ is a strong cardinal, formalized in ZFC

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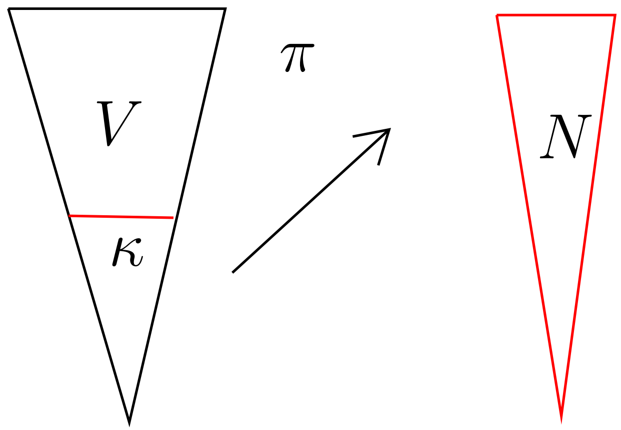
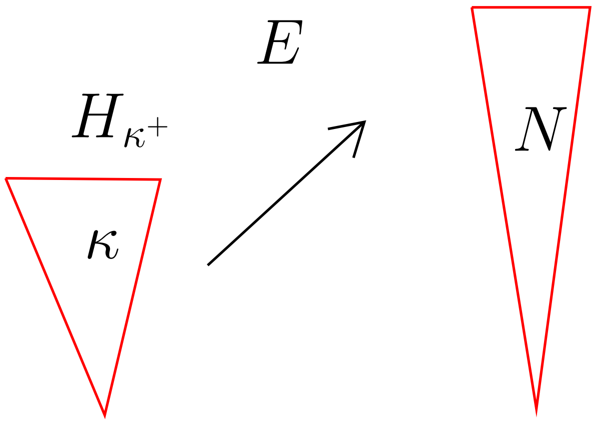


Extenders

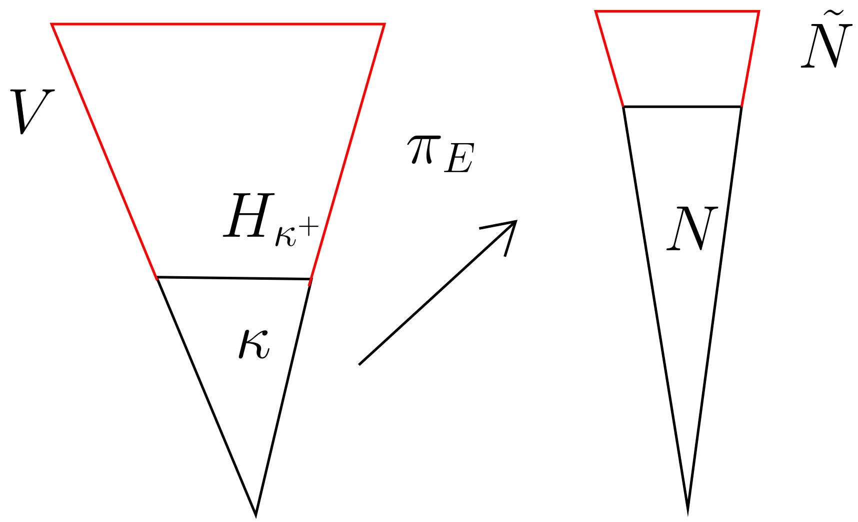
An extender at κ is a cofinal elementary map $E: (H_{\kappa^+}, \in) \to (N, \in)$ with transitive set model N and critical point κ.



Extensions determined by extenders

Let $E: (H_{\kappa^+}, \in) \to (N, \in)$ be an extender at κ.

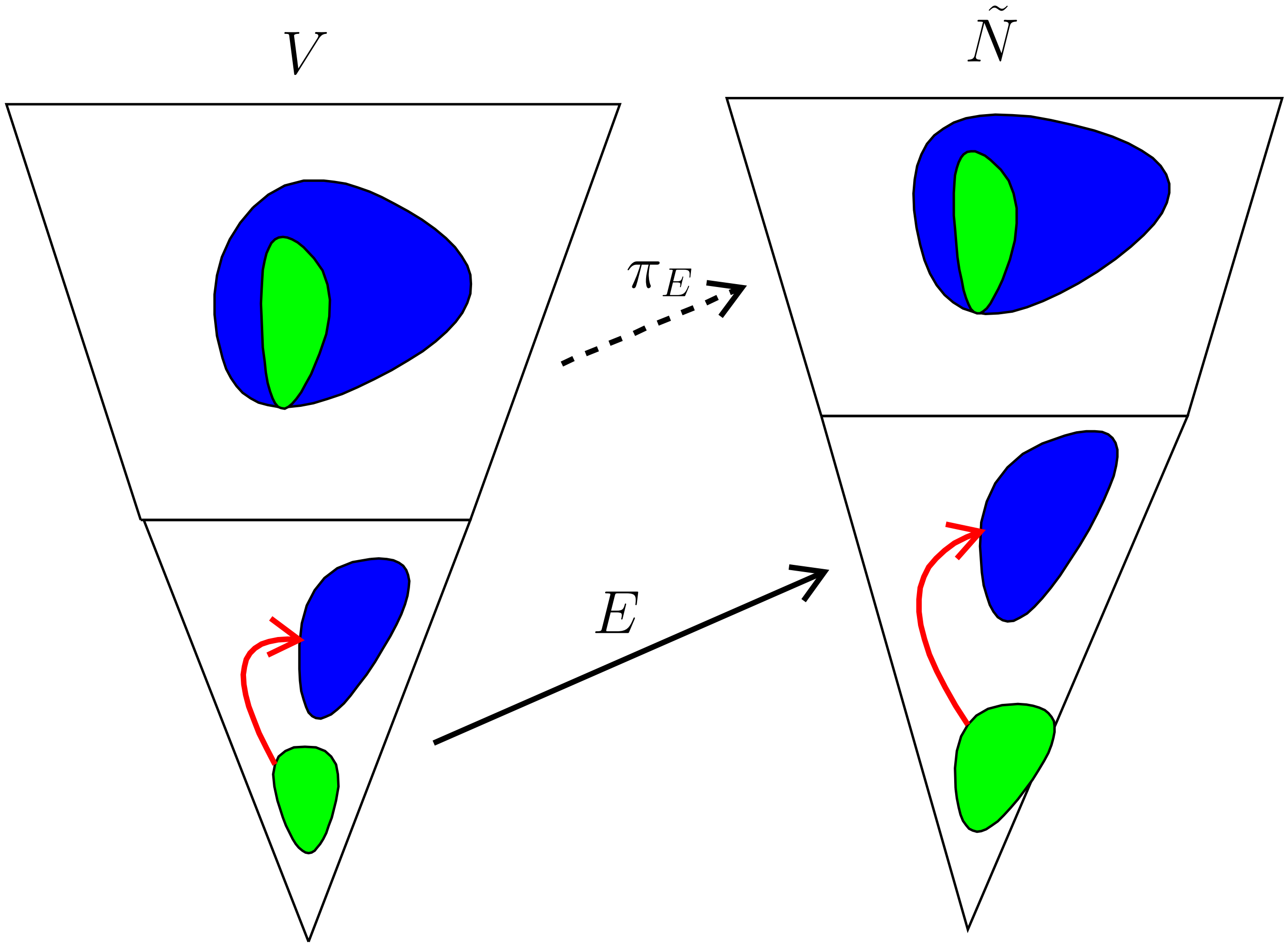
Then there is an extension $\pi_E: V \to \tilde{N}$ with transitive $\tilde{N}$ such that $\pi_E \supseteq E$.



Extensions

$$
\begin{array}{ccc}
(V, \in) & \stackrel{\pi_E}{\longrightarrow} & (\tilde{N}, \tilde{\in}) \\
\parallel & & \parallel \\
\bigcup_{A \in I} A & & \\
\parallel & & \parallel \\
\operatorname{dir} \lim_{A \leqslant B \in I} ((A, \in), \subseteq) & & \\
\parallel & & \parallel \\
\operatorname{dir} \lim_{A \leqslant B \in I} ((\bar{A}, \in), \pi_{AB}) & & \operatorname{dir} \lim_{A \leqslant B \in I} ((E(\bar{A}), \in), E(\pi_{AB})) \\
| & & | \\
\bar{A} & \stackrel{E}{\longrightarrow} & E(\bar{A})
\end{array}
$$

Extensions



Wellfoundedness

- we need $(\tilde{N}, \tilde{\in})$ to be wellfounded
- in general, for $\mathrm{dom}(E) \neq H_{\kappa^+}$, $(\tilde{N}, \tilde{\in})$ is not wellfounded
- there are criteria and techniques to ensure wellfoundedness
- in case $\mathrm{dom}(E) = H_{\kappa^+}$, $(\tilde{N}, \tilde{\in})$ is wellfounded

Comparing large cardinals

Theorem. Let κ be strong. Then κ is measurable and their are cofinally many measurable cardinals below κ.

Proof. By strongness take $E_0: H_{\kappa^+} \to N$ to be an extender with critical point κ.

By strongness take an elementary embedding $\pi: V \to M$ with critical point κ and $E_0 \in M$.

$M \vDash$ "κ is measurable", since $E_0 \in M$.

$M \vDash$ "$\exists \lambda < \pi(\kappa): \lambda$ is measurable".

$V \vDash$ "$\exists \lambda < \kappa: \lambda$ is measurable", since π is elementary. **Qed.**

The linear (?) hierarchy of large cardinals

For set theoretic properties A and B define $A \prec B$ iff

$$B \rightarrow \text{ there is a model of } A$$

(“B has greater consistency strength than A”)

Heuristically the “known” large cardinals are linearly ordered by $\prec$:

inaccessible $\prec$ Erdös $\prec$ measurable $\prec$ strong $\prec$ Woodin $\prec$ supercompact

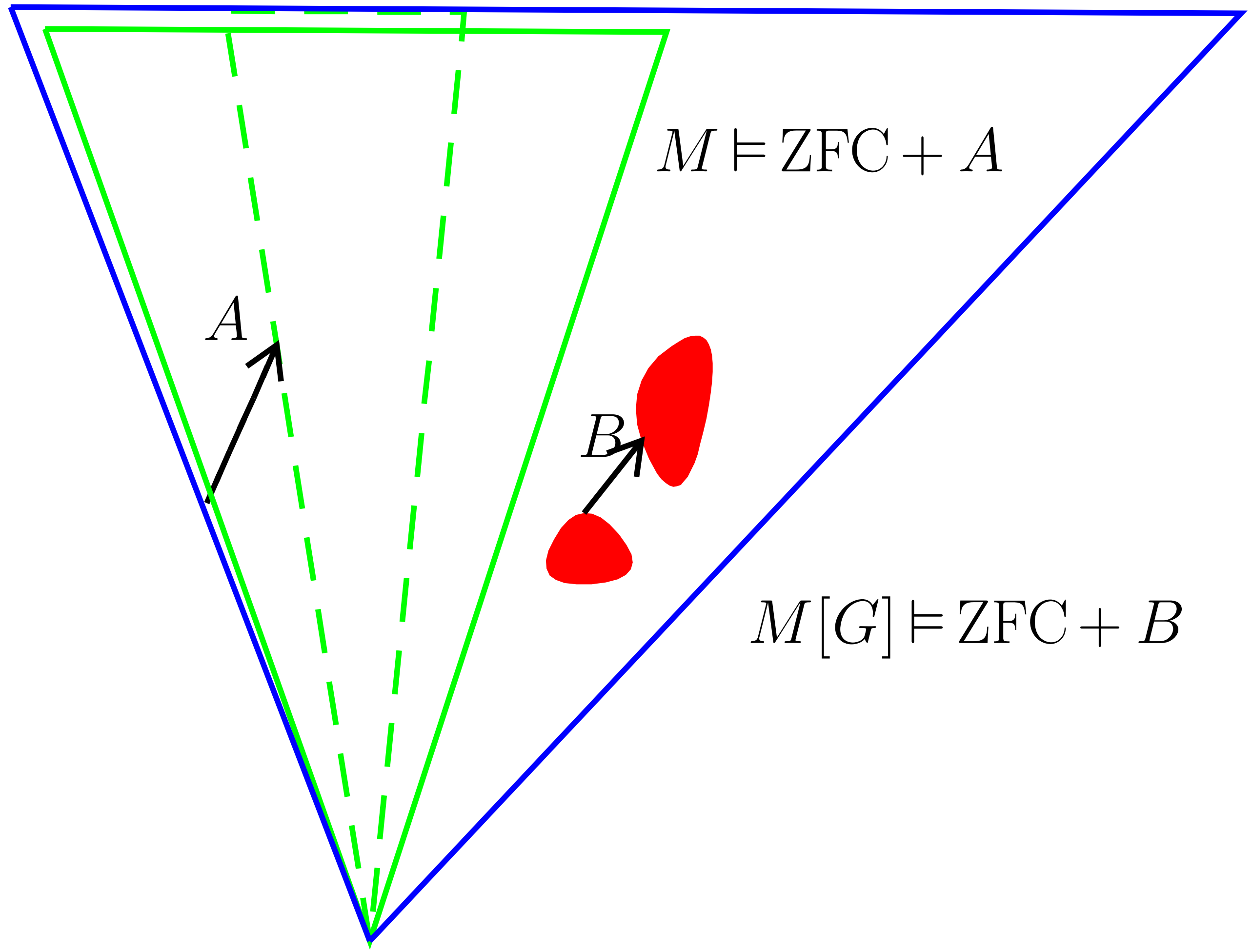
Calibrating consistency strengths by large cardinals (?)

A and B have the same consistency strength if every (countable) model of $\mathrm{ZFC}+A$ can "uniformly be transformed" into a (countable) model of $\mathrm{ZFC}+B$ and vice versa (forcing, inner models,...).

Heuristically a typical combinatorial principle has the same consistency strength as some large cardinal property.

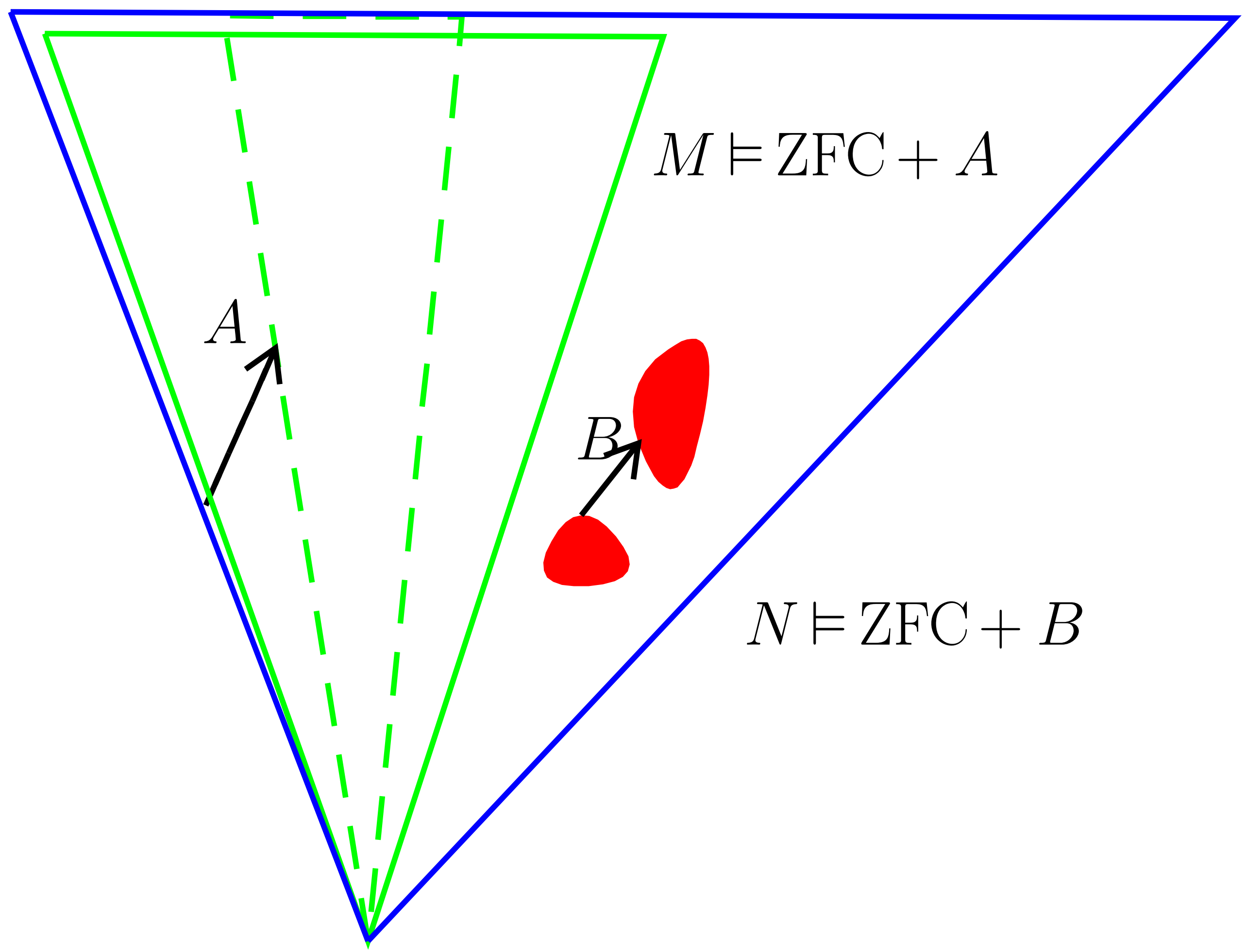
Calibrating consistency strengths by large cardinals

Forcing: ground model M $\longmapsto$ generic extension $M[G] \supseteq M$


$M \models \mathrm{ZFC} + A$
A
B
$M[G] \models \mathrm{ZFC} + B$

Calibrating consistency strengths by large cardinals

Inner model: model $N \longmapsto$ inner model $M \subseteq N$



$M \models \mathrm{ZFC} + A$
A
B
$N \models \mathrm{ZFC} + B$

The inner model L

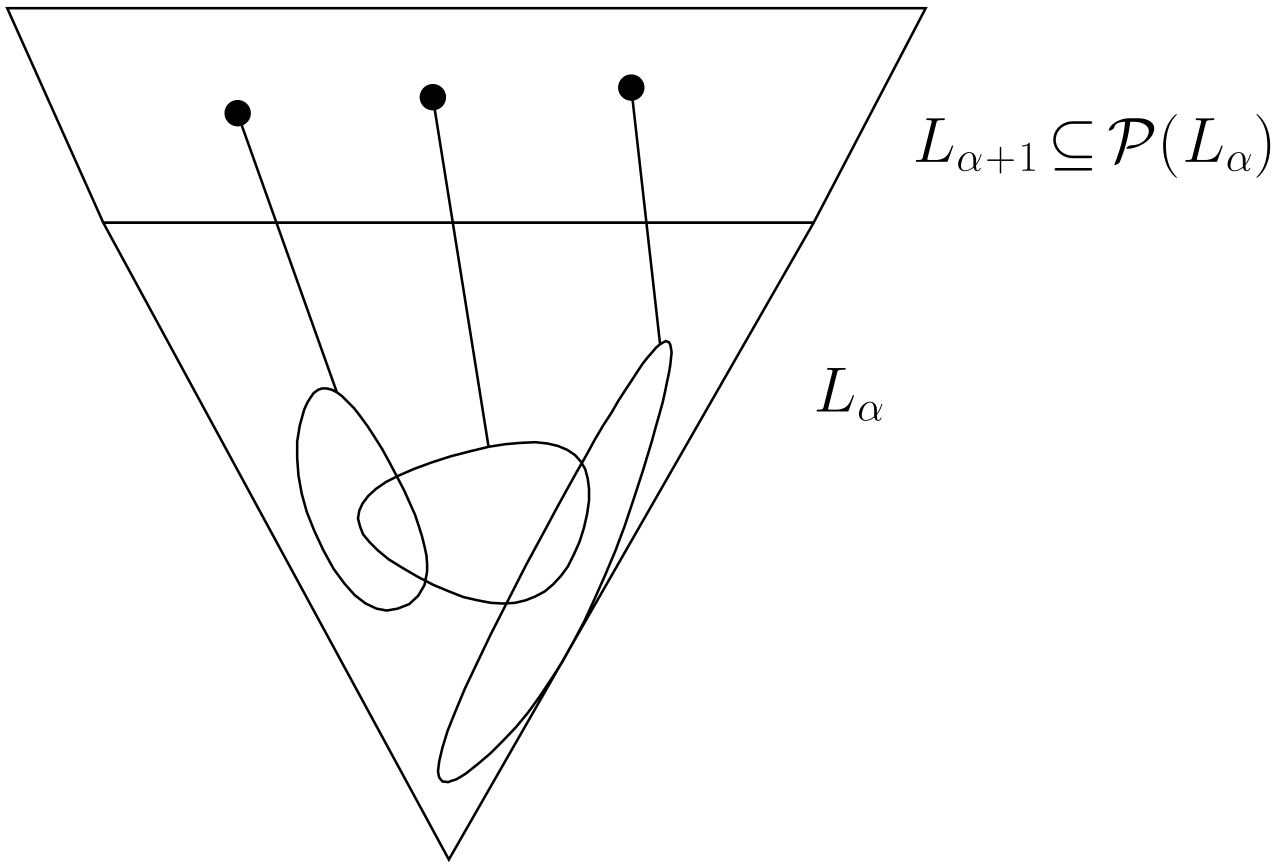
Define the constructible hierarchy

- $L_0 = \emptyset$
- $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$, i.e., the collection of all $X \subseteq L_\alpha$ which are definable over the structure $(L_\alpha, \in)$ with parameters
- $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for all limit ordinals λ

The constructible universe is the model $(L, \in)$ where

$$L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$$

The inner model L



The inner model L

- L is an inner model of set theory, i.e., L is a transitive class containing all ordinals and $L \vDash \mathrm{ZFC}$

- Condensation: If $\pi: (X, \in') \to (L_\beta, \in)$ is elementary then $(X, \in') \cong (L_\alpha, \in)$ for some α

- L is a model of combinatorial principles like the generalized continuum hypothesis (GCH), $\diamondsuit$, $\square$, ...

- L is the $\subseteq$-smallest inner model

Techniques for Getting Large Cardinals in Inner Models, Part 2

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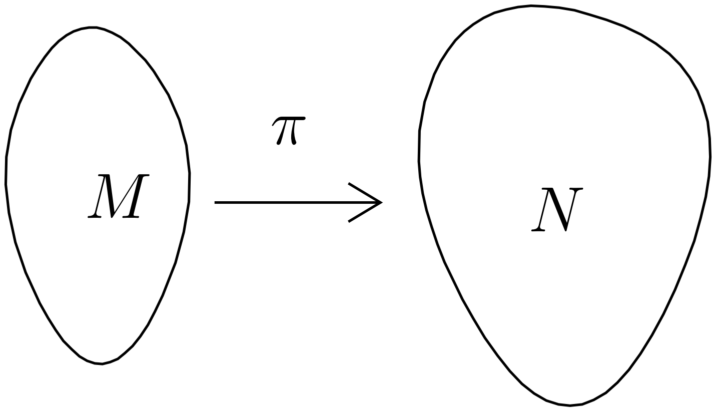
Recall: Uncountable combinatorics

Many notions in uncountable combinatorics have the form:

for all premises ...

there is an embedding $\pi: M \to N$

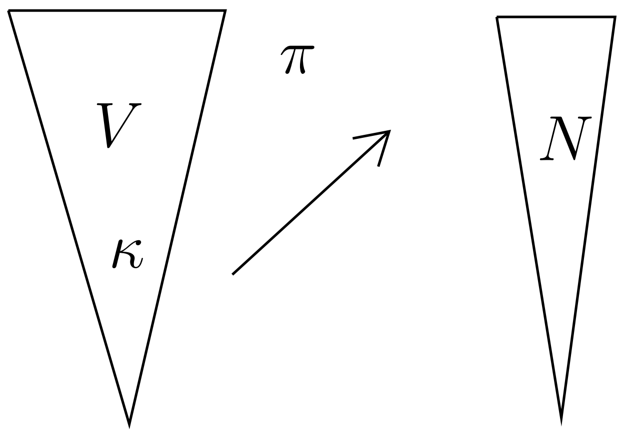
with properties ...



Recall: Large cardinals

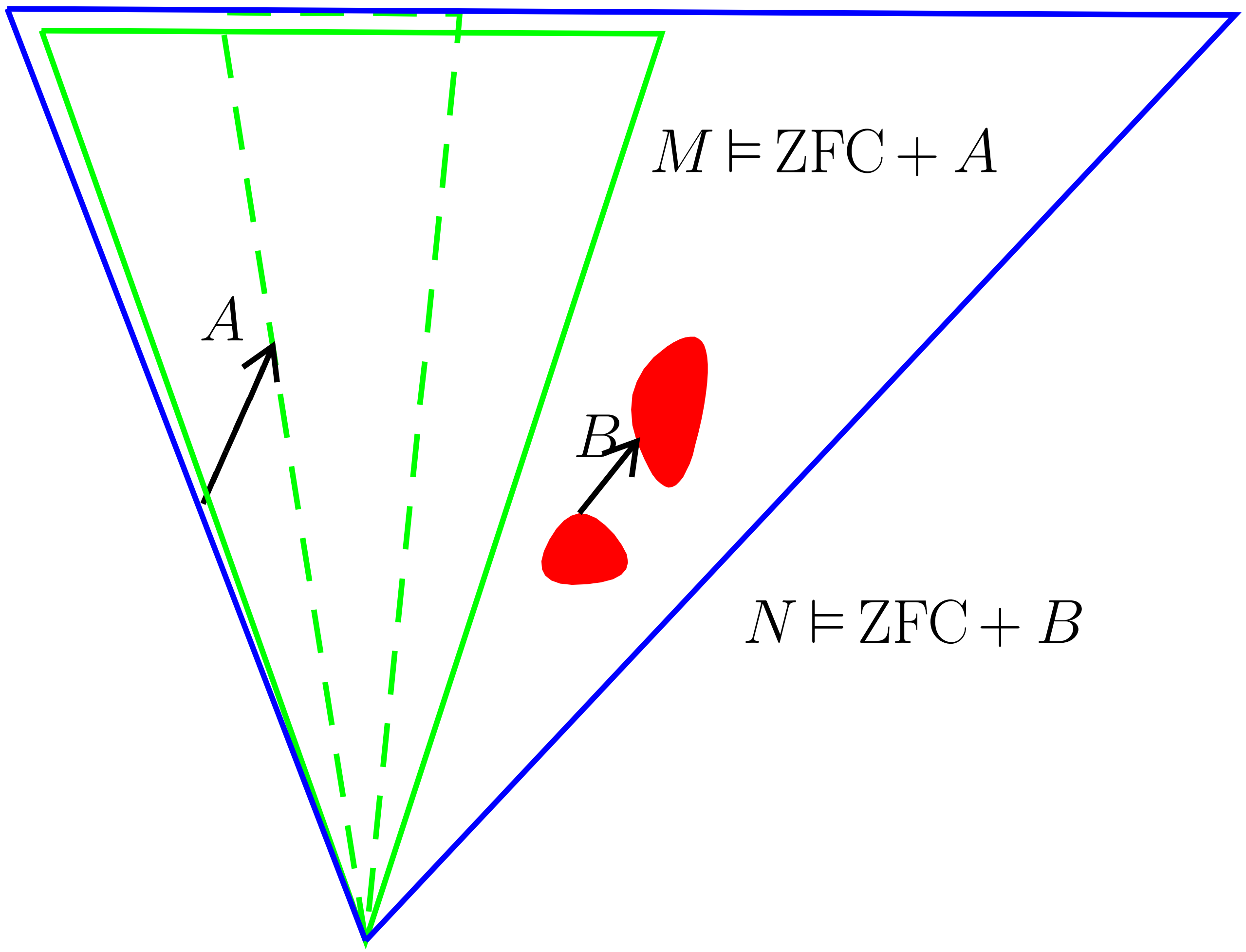
There is an elementary embedding $\pi: (V, \in) \rightarrow (N, \in)$

with a transitive inner model N and critical point κ, i.e. $\pi \upharpoonright \kappa = \mathrm{id}$ and $\pi(\kappa) > \kappa$, and ....

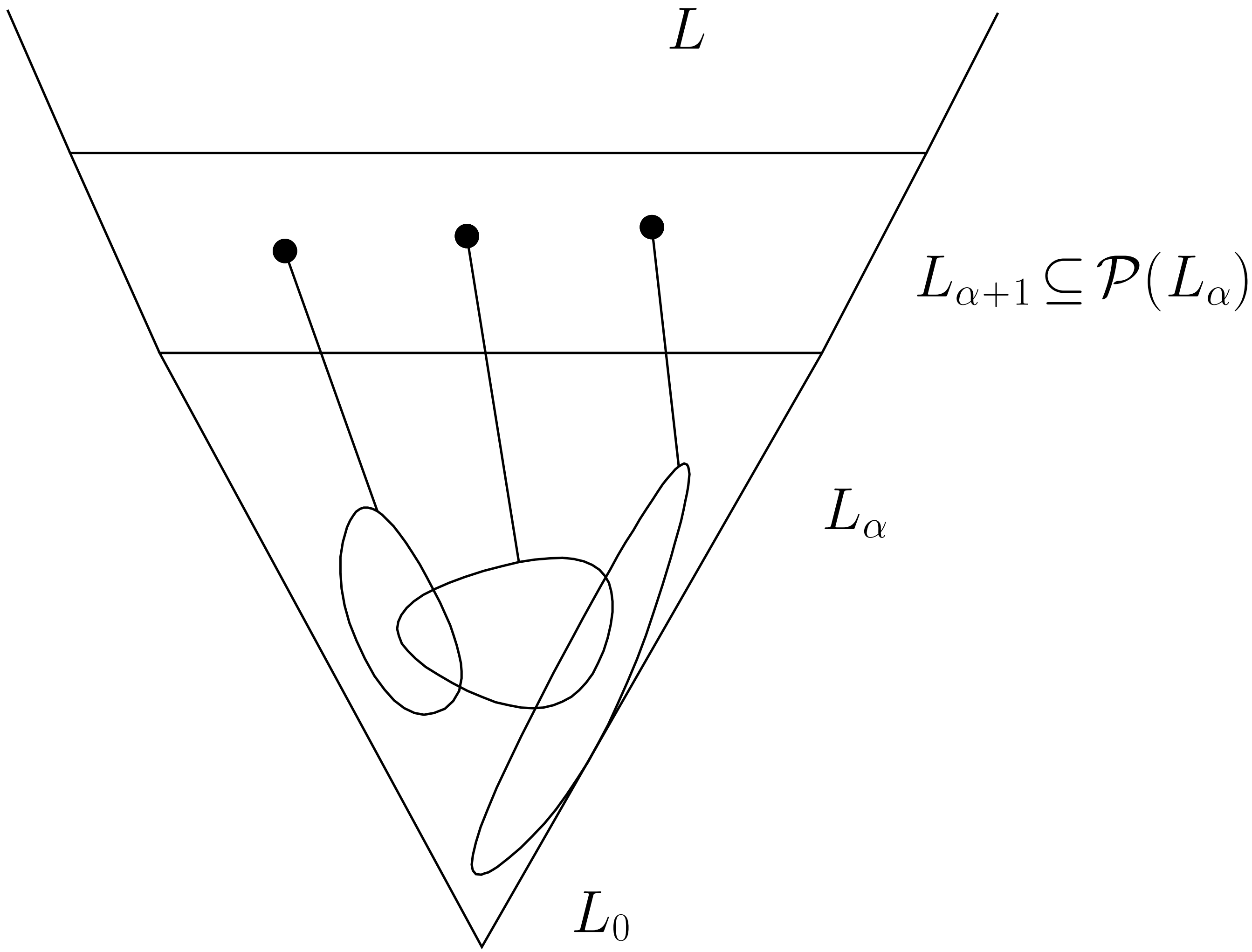


Recall: Getting large cardinals in inner models

Inner model: model $N \longmapsto$ inner model $M \subseteq N$



Recall: The inner model L

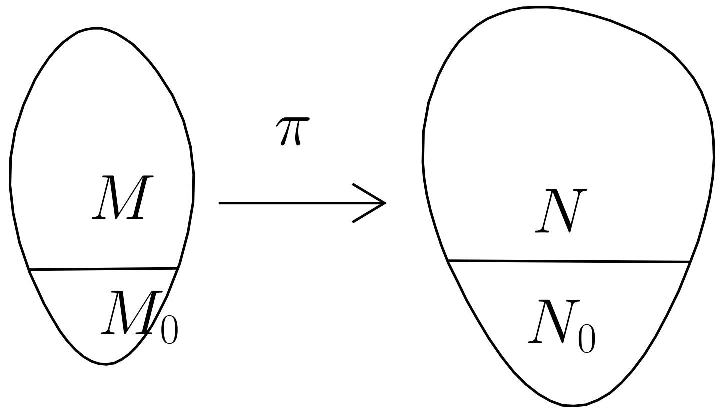


Recall: Chang's conjecture

for every structure $N = (N, N_0, \ldots)$ with $\mathrm{card}(N) = \aleph_2$ and $\mathrm{card}(N_0) = \aleph_1$

there is an elementary embedding $\pi: M \to N$

where $M = (M, M_0, \ldots)$ and $\mathrm{card}(M) = \aleph_1$ and $\mathrm{card}(M_0) = \aleph_0$



Chang's conjecture and large cardinals

For $N = (L_{\aleph_2}, \aleph_1, \in, \ldots)$ take an elementary embedding

$$E\colon (M, M_0, \in', \ldots) \to (L_{\aleph_2}, \aleph_1, \in, \ldots)$$

where $\mathrm{card}(M) = \aleph_1$ and $\mathrm{card}(M_0) = \aleph_0$

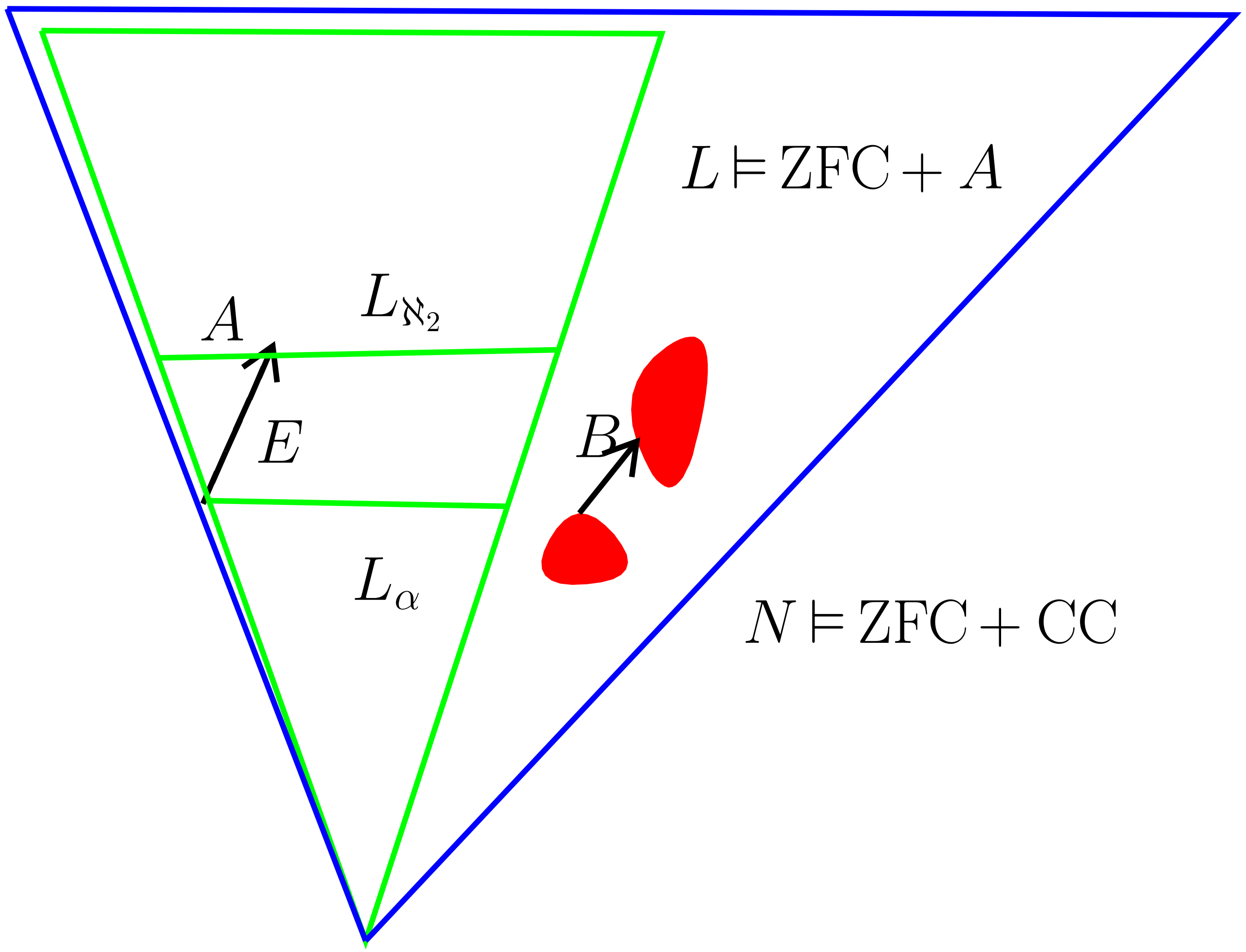
By condensation $(M, \in') \cong (L_\alpha, \in)$, and so wlog

$$E\colon (L_\alpha, M_0, \in, \ldots) \to (L_{\aleph_2}, \aleph_1, \in, \ldots)$$

where $\alpha \geqslant \aleph_1$ and E has a countable critical point κ

Chang's conjecture and large cardinals

Inner model: model $N \longmapsto$ inner model $L \subseteq N$



Chang's conjecture and large cardinals

Theorem Chang's conjecture implies that there is an inaccessible cardinal in L. Hence Chang's conjecture $\succcurlyeq$ inaccessible.

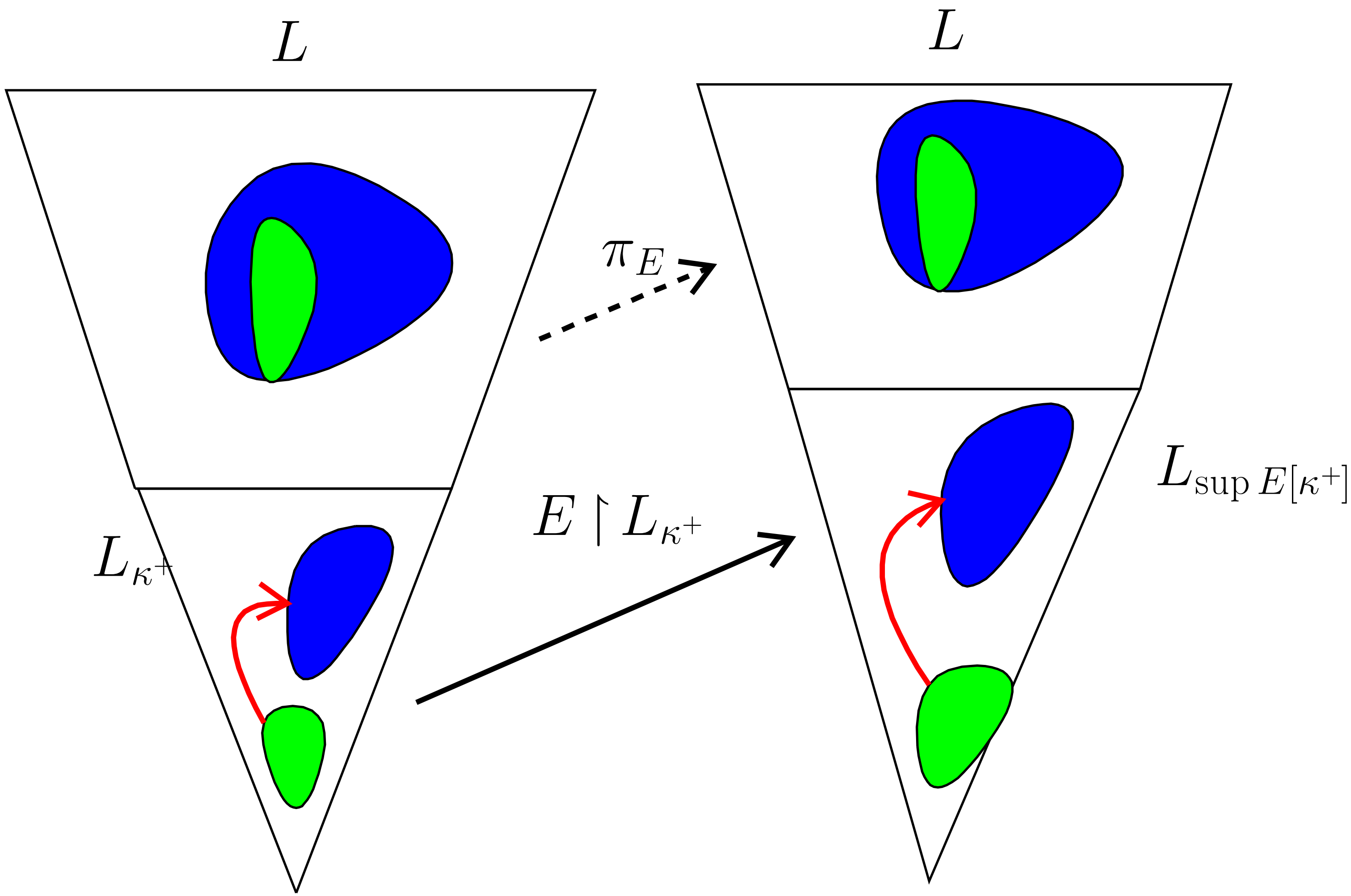
Proof Let κ be the critical point of E. κ is inaccessible in L_α.

Since $\alpha \geqslant \aleph_1$, κ is inaccessible in $L_{\aleph_1}$.

By the argument for GCH, $\mathcal{P}(\kappa) \cap L = \mathcal{P}(\kappa) \cap L_{\aleph_1}$.

Hence κ is inaccessible in L. **Qed.**

Using $E \restriction L_{\kappa^+}$ as an extender on L



$\mathbf{0}^{\#}$

Chang's conjecture implies the existence of a nontrivial elementary embedding $\pi_E: (L, \in) \to (L, \in)$.

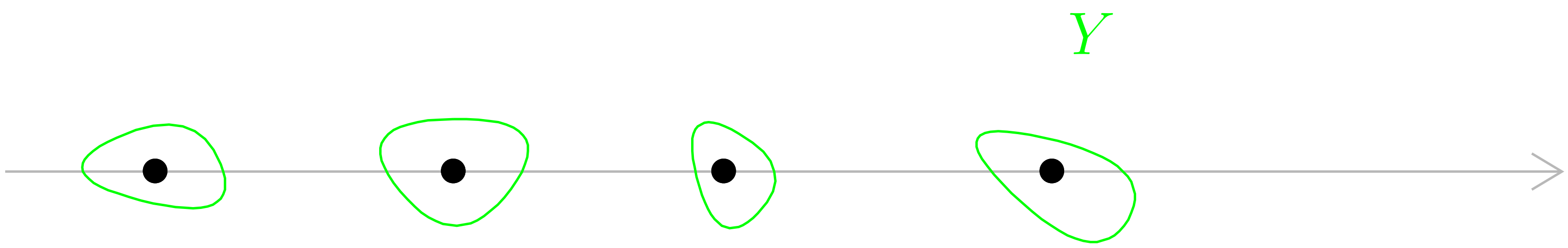
We say "$0^{\#}$ exists" for the fact that there is a nontrivial elementary embedding $\pi: (L, \in) \to (L, \in)$.

Actually, one can then define a unique set $0^{\#}$ which is the canonical, minimal extender on L which generates such an embedding.

The Jensen covering theorem

Theorem (Jensen). If $0^\#$ does not exist then L covers V, i.e., for every $X \in V$, $X \subseteq \mathrm{Ord}$ there exists $Y \in L$ such that

$$X \subseteq Y \text{ and } \mathrm{card}(Y) \leqslant \mathrm{card}(X) + \aleph_1$$



$0^{\#}$ and the singular cardinal hypothesis

Theorem (Jensen). Assume $0^{\#}$ does *not* exist and $\forall n < \omega$: $2^{\aleph_n} = \aleph_{n+1}$. Then

$$2^{\aleph_\omega} = \aleph_{\omega+1}$$

$0^{\#}$ and the singular cardinal hypothesis

Proof. Choose functions $f_n: \mathcal{P}(\aleph_n) \leftrightarrow \aleph_{n+1} \setminus \aleph_n$

For $X \in \mathcal{P}(\aleph_\omega)$ define $X' = \{f_n(X \cap \aleph_n) | n < \omega\} \in [\aleph_\omega]^\omega$

Choose $X'' \in L$ such that $X' \subseteq X'' \subseteq \aleph_\omega$ and $\mathrm{ordertype}(X'') < \aleph_2$

$$X \mapsto (X'', \{i < \aleph_2 | \text{ the } i\text{-th element of } X'' \text{ is an element of } X'\})$$

is an injection $\mathcal{P}(\aleph_\omega) \rightarrowtail \mathcal{P}^L(\aleph_\omega) \times \mathcal{P}(\aleph_2)$. Hence

$$2^{\aleph_\omega} = \mathrm{card}(\mathcal{P}(\aleph_\omega)) \leqslant \mathrm{card}(\mathcal{P}^L(\aleph_\omega)) \cdot \mathrm{card}(\mathcal{P}(\aleph_2)) \leqslant \aleph_{\omega+1} \cdot \aleph_3 = \aleph_{\omega+1}$$

Qed.

(Some) large cardinals

- Supercompact
- Woodin cardinals
- strong
- measurable
- $0^{\#}$ exists
- weakly inaccessible and strongly inaccessible

$0^{\#}$ transcends L

Theorem. $0^{\#} \notin L$.

Proof. Assume $0^{\#} \in L$.

$0^{\#}$ is an extender on L, and let $\pi: L \to L$ be the nontrivial elementary embedding induced by $0^{\#}$.

Let κ be the critical point of π.

$L \vDash$ ''there is an extender on L with critical point $< \pi(\kappa)$'' .

$L \vDash$ ''there is an extender on L with critical point $< \kappa$''

Contradiction to minimal choice of $0^{\#}$. **Qed.**

Iterating the $\#$-operation

$$L \mapsto 0^{\#} \mapsto L^{0^{\#}} \mapsto (0^{\#})^{\#} \mapsto L^{(0^{\#})^{\#}} \mapsto ((0^{\#})^{\#})^{\#} \mapsto \ldots ???$$

Constructible extender models

Define a core model with extender sequence $\mathcal{E}$

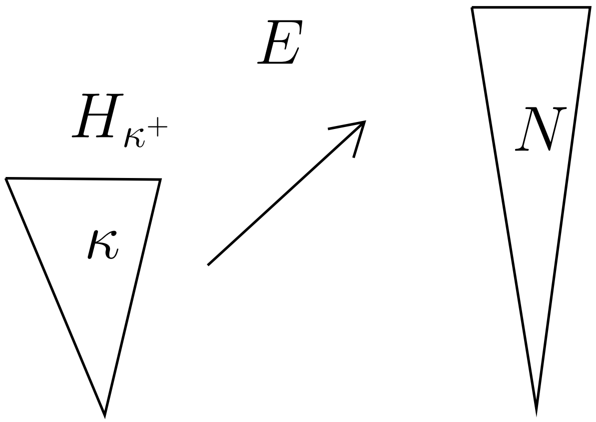
- $K_0 = \emptyset$, $\mathcal{E}(0) = \emptyset$
- $K_{\alpha+1} = \mathrm{Def}(K_\alpha, \in, \mathcal{E} \upharpoonright \alpha)$, $\mathcal{E}(\alpha+1) = \emptyset$
- for limit ordinals λ: $K_\lambda = \bigcup_{\alpha<\lambda} K_\alpha$; moreover if there is a uniquely determined "good" extender $E: (K_\gamma, \in, \mathcal{E} \upharpoonright \gamma) \to (K_\lambda, \in, \mathcal{E} \upharpoonright \lambda)$ then let $\mathcal{E}(\lambda) = E$; otherwise $\mathcal{E}(\lambda) = \emptyset$

Then the core model is the model $(K, \in)$ where

$$K = \bigcup_{\alpha \in \mathrm{Ord}} K_\alpha$$

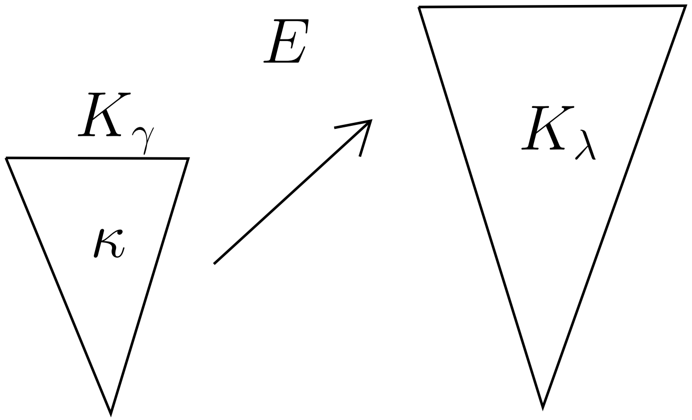
Recall: Extenders

An extender at κ is a cofinal elementary map $E: (H_{\kappa^+}, \in) \rightarrow (N, \in)$ with transitive set model N and critical point κ.



"Good" Extenders

$$E: (K_\gamma, \in, \mathcal{E} \upharpoonright \gamma) \to (K_\lambda, \in, \mathcal{E} \upharpoonright \lambda)$$



"Good" extenders

$E: (K_\gamma, \in, \mathcal{E} \upharpoonright \gamma) \to (K_\lambda, \in, \mathcal{E} \upharpoonright \lambda)$ is good if

- $(K_\lambda, \in, \mathcal{E} \upharpoonright \lambda)$ is a model of ZFC except the power set axiom
- E is an elementary map with critical point κ
- $K_\gamma = (H_{\kappa^+})^{K_\lambda}$
- certain extensions and iterated extensions formed from E are wellfounded
- moreover, these extensions and iterations have to be finestructural
- ......

The Dodd-Jensen core model

Assume that there is no inner model in which there is a measurable cardinal.

Then the model K is called the Dodd-Jensen core model, denoted by K_{DJ}

K_{DJ} is an L-like inner model of set theory

The Dodd-Jensen core model

Theorem. Assume there is no inner model with a measurable. Then

- K_{DJ} is an inner model of set theory
- (Condensation *fails* in general)
- K_{DJ} is a model of GCH, $\diamondsuit$, $\square$, ...

The Dodd-Jensen core model

Theorem (cont). Assume there is no inner model with a measurable. Then

– There is *no* nontrivial elementary embedding $\pi\colon K_{\mathrm{DJ}} \to K_{\mathrm{DJ}}$

– K_{DJ} covers V, i.e., for every $X \in V$, $X \subseteq \mathrm{Ord}$ there exists $Y \in K_{\mathrm{DJ}}$ such that

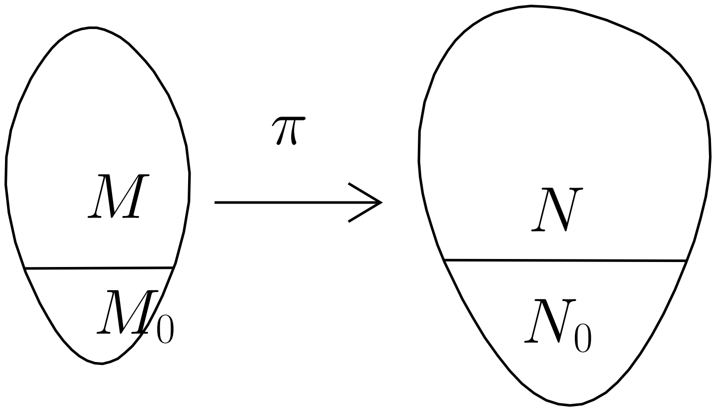
$$X \subseteq Y \text{ and } \mathrm{card}(Y) \leqslant \mathrm{card}(X) + \aleph_1$$

Recall: Chang's conjecture ($\mathrm{CC}(\kappa, \lambda)$)

for every structure $N = (N, N_0, \ldots)$ with $\mathrm{card}(N) = \kappa^+$ and $\mathrm{card}(N_0) = \kappa$

there is an elementary embedding $\pi: M \to N$

where $M = (M, M_0, \ldots)$ and $\mathrm{card}(M) = \lambda^+$ and $\mathrm{card}(M_0) = \lambda$



Chang's conjecture and Erdös cardinals

Theorem (Donder-Silver) $\mathrm{CC}(\aleph_1, \aleph_0)$ implies that, in K_{DJ}, $\aleph_2^V$ is an $\aleph_1$-Erdös cardinal. Hence $\mathrm{CC}(\aleph_1, \aleph_0)$ has the same consistency strength as an $\aleph_1$-Erdös cardinal.

Higher Chang's conjectures and measurable cardinals

Theorem (K) $\mathrm{CC}(\aleph_2, \aleph_1)$ implies that there is an inner model with a measurable cardinal. Hence $\mathrm{CC}(\aleph_2, \aleph_1) \succcurlyeq$ measurable.

The singular cardinal hypothesis and measurable cardinals

Theorem (Dodd-Jensen). Assume that there is *no* inner model with a measurable cardinal and $\forall n < \omega$: $2^{\aleph_n} = \aleph_{n+1}$. Then

$$2^{\aleph_\omega} = \aleph_{\omega+1}$$

The singular cardinal hypothesis and measurable cardinals

Theorem (Gitik-Mitchell). The consistency strength of

$$\forall n < \omega\colon 2^{\aleph_n} = \aleph_{n+1} \text{ and } 2^{\aleph_\omega} \neq \aleph_{\omega+1}$$

is equal to that of the existence of measurable cardinals of high Mitchell order.

The inner model direction of the result uses higher core models formed under the assumption that there are *no* inner models with measurable cardinals of high Mitchell order.

Resume

- combinatorial properties and large cardinals may be characterized by embedding properties
- embedding properties may be mirrored into inner models and become large cardinal properties
- this allows estimates of consistency strengths
- appropriate inner models are Gödels constructible universe and core models by Dodd-Jensen and others

Thank You!