SS 2025 Dr. Jack Davies
Dr. Tobias Lenz

Graduate Seminar on Topology (S4D2)

Lie Groups and Their Representations

Wednesdays, 2pm, SR 0.011

Preliminary meeting (joint with other topology seminars): January 28, 10am, Lipschitzsaal

Lie groups were introduced by Sophus Lie in the 1870's in order to capture symmetries of differential equations. Nowadays, they are studied as interesting objects in their own right, which involves methods from topology, geometry, analysis, algebra, and representation theory. In this seminar we want to focus on some of the topological, group theoretic, and representation theoretic aspects of Lie groups, and in particular on classification results for several classes of Lie groups and their representations. Our main reference is [Bt85], but for several advanced topics we rely on other accounts.

Note. Especially for some of the later topics, the references can contain more material than you will be able to present in your talk, and you should decide on which parts you want to focus (apart from the results explicitly mentioned in the talk description below, of course). In the case of multi-part talks it is your own responsibility to get in touch with the other speaker(s) and discuss which topics they are going to present or, conversely, which notions or results they will need for their talk.

In any case, you should meet with one of us (Talk 1–7: Jack, `davies@math.uni-bonn.de`; Talk 8–13: Tobias, `lenz@math.uni-bonn.de`) at least two weeks before your talk to go through the material you want to present and to discuss any questions you might have.

Prerequisites

You should be familiar with basic topology, in particular with the topics of the lecture "Einführung in die Geometrie und Topologie." For the later talks, knowledge of the classical representation theory of finite groups over $\mathbb{C}$ will be assumed, roughly to the amount of [Lan02, XVII.1–4 and XVIII.I–5] or [tom09, 1–2].

Talks

1. **Reminder on manifolds** .... April 9
   This introductory talk should recall some basic notions concerning manifolds. In particular, you should remind us all of the definitions of topological and smooth manifolds, smooth maps, and closed submanifolds. Characterize closed submanifolds of $\mathbb{R}^n$ as those subsets that are locally given by graphs and as those subsets that are locally zero sets of smooth functions with surjective differential. Afterwards, discuss the equivalent definitions of the tangent space in terms of germs of paths and as derivations of germs of smooth maps to $\mathbb{R}$, and construct the tangent bundle. For the latter you should also give a brief recap of (smooth) fiber bundles and vector bundles. Finally, discuss vector fields. Some of this material is spread out between [Bt85, I.1–4] (and you should follow the conventions there!), but you probably want to consider other sources as well.

2. **Lie groups and Lie algebras** .... April 16
   Define Lie groups and discuss several examples [Bt85, I.1]. Introduce abstract Lie algebras (over any field), and give three equivalent definitions of the Lie algebra $\mathfrak{g}$ associated to a Lie group G (in terms of vector fields, based on the tangent space at the unit, or as one-parameter groups). Finally, compute the associated Lie algebra for several examples of matrix groups [Bt85, I.2].

3. **Some first classification results** .... April 23
Introduce the exponential map, establish its basic properties [Bt85, I.3.1–5], and do Exercises 1 and 2 of [Bt85, I.3.13]. Afterwards, classify connected abelian and compact abelian Lie groups [Bt85, I.3.6–8]. If you have time left, 'classify' all abelian Lie groups [Bt85, I.3.13 Exercise 3].

4. **New Lie groups from old** .... April 30
Introduce Lie subgroups and prove that an abstract subgroup of a Lie group is an embedded Lie subgroup if and only if it is closed. Conclude that all continuous homomorphisms of Lie groups are smooth [Bt85, I.3.9-12]. Afterwards, discuss quotients of Lie groups [Bt85, I.4.1–12]. If time permits, introduce coverings of Lie groups [Tit83, II] and construct the Spin groups [Bt85, I.6] (either by the general theory or as a concrete construction).

5. **The Haar integral** .... May 7
The main goal of this talk is to construct a translation invariant integral on any given compact Lie group G and to establish its basic properties following [Bt85, I.5]. If time permits, mention or sketch some aspects of the theory for general locally compact topological groups, in particular existence and uniqueness; you can follow for example [vQ01, 19] or [Hal74, XI] for this part.

6. **Representation theory of compact Lie groups I: Basic results** .... May 21
Define representations of a given compact Lie group G, introduce the analogue of the group algebra, prove the existence of a G-invariant inner product, and state a version of Maschke's Theorem in this context [Bt85, II.1]. Moreover, prove Schur's Lemma and develop the theory of characters [Bt85, II.4]. Finally, give the abstract definition of representative functions and show that they are precisely the ones arising from representations [Bt85, III.1].

7. **Representation theory of compact Lie groups II: Examples** .... May 28
Classify irreducible representations of compact abelian Lie groups [Bt85, II.8] and of a non-empty subset of $\{\mathrm{SU}(2), \mathrm{SO}(3), \mathrm{U}(2), \mathrm{O}(3)\}$ [Bt85, II.5].

8. **Representation theory of Lie algebras** .... June 4
Introduce representations of Lie algebras, discuss how a representation of a Lie group G yields a representation of the associated Lie algebra $\mathfrak{g}$, and mention the relationship between the two for simply connected G (which will be proven in the next talk). Introduce solvable Lie algebras, give an example showing that not all complex representations of finite dimensional real solvable Lie algebras are completely reducible, and prove Lie's Theorem [Kna02, I.5].
Afterwards introduce semisimple Lie algebras as Lie algebras having no non-zero solvable ideal, and state Weyl's Theorem [Hum80, II.6.3] as well as the Levi decomposition [Kna02, Appendix B] (both without proof!). Finally discuss representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ [Bt85, II.9–10].

9. **The Lie Theorems I: Full faithfulness** .... June 18
The goal of this talk and the one after it is to prove that taking associated Lie algebras gives rise to an equivalence of categories between simply connected Lie groups and finite-dimensional real Lie algebras. State this result and then prove the full faithfulness part following [HN12, 3.4, 9.2, and 9.5].

10. **The Lie Theorems II: Essential surjectivity** .... June 25
The goal of this talk is to prove that every finite-dimensional real Lie algebra comes from a simply connected Lie group (also called the Cartan-Lie Theorem). This splits into three parts: the solvable case following [Tit83, IV.1] or [Kna02, I.11–12]; the semisimple case using the Integral Subgroup Theorem [HN12, Theorem 9.4.8] and that the adjoint representation of a semisimple Lie algebra is faithful; and the proof of the general statement using these two special cases together with the Levi decomposition [Kna02, Appendix B].
Give a complete account of the solvable case, and then decide on which of the remaining two steps you want to focus (however, you should of course still at least sketch the argument for the other step).

11. **Classification of semisimple Lie groups and Lie algebras I** . . . . July 2
Introduce/recall simple and semisimple Lie algebras and explain how these notions relate to each other [Kna02, I.2]; moreover, introduce semisimple Lie groups as on [Kna02, p. 61]. Prove Cartan's characterization of semisimple Lie algebras in terms of the Killing form following [Kna02, I.7] or [Hum80, II.4.3–5.1].
Afterwards, state the classification result of complex simple Lie algebras [Hum80, I.1.2, III.11.4, and V.19.2] (leaving the exceptional Lie algebras implicit), and indicate how this yields the classification of simply connected compact semisimple Lie groups [Tit83, IV.6.4]. If time permits, sketch the classification over the real numbers [Kna02, VI.8-10] or [Tit83, IV.6.1], hence of simply connected semisimple Lie groups, and very briefly indicate how this can be used to classify all connected semisimple (not necessarily simply connected) Lie groups [HN12, 9.4–5], also cf. the introduction of [Bt85, V.8].

12. **Classification of semisimple Lie groups and Lie algebras II** . . . . July 9
Introduce the abstract Jordan decomposition in a semisimple Lie algebra [Hum80, II.5.4] and compare it to the classical one for subalgebras of $\mathfrak{gl}_n(\mathbb{C})$ [Hum80, II.6.4]. Afterwards, study maximal toral subalgebras, introduce 'the' root system of a semisimple Lie algebra, and establish its basic properties following [Hum80, II.8]. Alternatively, you can follow [Kna02, II.1–4] for this talk (which uses an a priori different, but equivalent approach).

13. **Classification of semisimple Lie groups and Lie algebras III** . . . . July 16
Introduce abstract root systems as well as their Cartan matrices and Dynkyn diagrams following e.g. [Kna02, II.5] or [Hum80, III.9–11.3]. The main focus of your talk should then be a detailed account of the classification of Dynkyn diagrams as presented in [Kna02, II.7] or [Hum80, III.11.4]. Explain (without proof!) how this leads to the classification of semisimple complex Lie algebras [Hum80, V.18.4].
If you have time left, indicate why the classical Lie algebras of types A_n to D_n are indeed semisimple, and maybe explicitly calculate the root systems and Dynkyn diagrams for one of these families [Hum80, V.19.1–2].

Note. A quick tour through the main topics of the final three talks is given in [Bt85, V.5].

References

[Bt85] Theodor Bröcker and Tammo tom Dieck, *Representations of Compact Lie Groups*, Grad. Texts Math., vol. 98, Springer, New York, NY, 1985.

[Hal74] Paul R. Halmos, *Measure Theory*, 2nd ed., Grad. Texts Math., vol. 18, Springer, New York, NY, 1974.

[HN12] Joachim Hilgert and Karl-Hermann Neeb, *Structure and Geometry of Lie Groups*, Springer Monogr. Math., Berlin: Springer, 2012.

[Hum80] James E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, 3rd revised ed., Grad. Texts Math., vol. 9, Springer, New York, NY, 1980.

[Kna02] Anthony W. Knapp, *Lie Groups Beyond an Introduction*, 2nd ed., Prog. Math., vol. 140, Boston, MA: Birkhäuser, 2002.

[Lan02] Serge Lang, *Algebra*, 3rd revised ed., Grad. Texts Math., vol. 211, New York, NY: Springer, 2002.

[Tit83] Jacques Tits, *Liesche Gruppen und Algebren. Unter Mitarbeit von M. Kraemer und H. Scheerer*, Hochschultext, Berlin-Heidelberg-New York-Tokyo: Springer-Verlag, 1983 (German).

[tom09] Tammo tom Dieck, *Representation Theory*, Lecture notes, available at `https://www.uni-math.gwdg.de/tammo/rep.pdf`, 2009.

[vQ01] Boto von Querenburg, *Mengentheoretische Topologie*, 3rd revised ed., Berlin: Springer, 2001 (German).