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Summer term 2020

Exercises, Algebra I (Commutative Algebra) – Week 3

Exercise 9. (Adjunction, 3 points)
Assume $f\colon A \to B$ is a ring homomorphism. Show that for an A-module M and an B-module N there exists a natural isomorphism

$$\mathrm{Hom}_B(M \otimes_A B, N) \cong \mathrm{Hom}_A(M, {}_AN).$$

This can be viewed as an isomorphism of abelian groups or as an isomorphism of A or B-modules. What is the B-module structure on the two sides? In categorical terms, this can be expressed by saying that the functor

$$\otimes_B \colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$$

is left adjoint to the functor

$$\mathrm{Mod}(B) \to \mathrm{Mod}(A),\ N \mapsto {}_AN.$$

Exercise 10. (Deducing exactness, 2 points)
Consider a sequence of A-module homomorphisms

$$M_1 \xrightarrow{g} M_2 \xrightarrow{f} M_3 \longrightarrow 0 \tag{1}$$

and assume that for all A-modules N the induced sequence

$$0 \longrightarrow \mathrm{Hom}(M_3, N) \xrightarrow{\circ f} \mathrm{Hom}(M_2, N) \xrightarrow{\circ g} \mathrm{Hom}(M_1, N)$$

is exact. Show at then (1) is exact.

Exercise 11. (Examples of exact sequences, 4 points)
Decide which of the following sequences of A-modules are exact:

(i)
$$0 \to M_1 \cap M_2 \xrightarrow{\alpha} M_1 \oplus M_2 \xrightarrow{\beta} M_1 + M_2 \to 0$$
for two A-sub-modules $M_1, M_2 \subset M$ of an A-module M, with $\alpha : m \mapsto (m, m)$ and $\beta : (m_1, m_2) \mapsto m_1 - m_2$.

For the next two questions, let A be the polynomial ring $k[x, y, z]$, $f \in A$ and $\mathfrak{a} \subset A$ the ideal generated by $(x + z, y, f)$. Consider the following sequence of A-modules:

$$0 \to \wedge^3 A^3 \xrightarrow{\varphi_3} \wedge^2 A^3 \xrightarrow{\varphi_2} A^3 \xrightarrow{\varphi_1} \mathfrak{a} \to 0$$

where, denoting (e_1, e_2, e_3) a basis of the free A-module A^3, φ_1 defined by (extend linearly):

$$e_1 \mapsto x + z,\ e_2 \mapsto y,\ e_3 \mapsto f,$$

φ_2 by:

$$e_1 \wedge e_2 \mapsto (x + z)e_2 - ye_1,\ e_1 \wedge e_3 \mapsto (x + z)e_3 - fe_1,\ e_2 \wedge e_3 \mapsto ye_3 - fe_2$$

and φ_3 by $e_1 \wedge e_2 \wedge e_3 \mapsto (x + z)e_2 \wedge e_3 - ye_1 \wedge e_3 + fe_1 \wedge e_2$.

Solutions to be handed in before Monday April 27, 4pm.

(ii) with $f = z$.

(iii) with $f = x - y^2 + z$.

Exercise 12. (Flat, free, projective, 4 points)
Decide whether the following A-modules M are flat, free, or projective:

(i) A an integral domain, $a \in A \setminus \{0\}$ and $M = (a)$.

(ii) $A = k[x]$ and $M = k(x)$ the field of fractions of A.

(iii) k a field, $f \in k[x]$ a degree $d > 0$ polynomial, $A = k[x]/(f) \times k$ with the product ring structure and $M = 0 \times k$.

Exercise 13. (Long exact cohomology sequences, 2 points)
Consider short exact sequences

$$0 \longrightarrow M^i \xrightarrow{f_i} N^i \xrightarrow{g_i} P^i \longrightarrow 0$$

of A-modules and module homomorphisms

$$a_i\colon M^i \to M^{i+1},\ b_i\colon N^i \to N^{i+1},\ \text{and } c_i\colon P^i \to P^{i+1}$$

such that $a_{i+1} \circ a_i = b_{i+1} \circ b_i = c_{i+1} \circ c_i = 0$, $b_i \circ f_i = f_{i+1} \circ a_i$ and $c_i \circ g_i = g_{i+1} \circ b_i$. (In short: 'a short exact sequences of complexes' $0 \to M^\bullet \to N^\bullet \to P^\bullet \to 0$.)

Define $H^i(M^\bullet) := \mathrm{Ker}(a_i)/\mathrm{Im}(a_{i-1})$ (the 'cohomology of the complex $M^\bullet$') and similarly for $N^\bullet$ and $P^\bullet$. Imitate the proof of the snake lemma and prove that there exists a natural exact sequence

$$H^i(M^\bullet) \to H^i(N^\bullet) \to H^i(P^\bullet) \to H^{i+1}(M^\bullet) \to H^{i+1}(N^\bullet) \to H^{i+1}(P^\bullet).$$

Exercise 14. (Direct limit, 3 points)
Let I be a partially ordered directed set, i.e. for all $i, j \in I$ there exists $k \in I$ with $i, j \leq k$. Consider a family of A-modules M_i, $i \in I$ and homomorphisms $f_{ij}\colon M_i \to M_j$ for all $i \leq j$ such that $f_{ii} = \mathrm{id}$ and $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$. (This is called a 'directed system of A-modules'.)

Let $\varinjlim M_i$ be the quotient of $\bigoplus M_i$ by the submodule generated by all elements of the form $m_i - f_{ij}(m_i)$, where $m_i \in M_i$ and $f_{ij}\colon M_i \to M_j$. In particular, there exist natural homomorphisms $f_i\colon M_i \to \varinjlim M_i$.

(i) Show that every element of $\varinjlim M_i$ is the image of an element of the form $m_i \in M_i \subset \bigoplus M_i$.

(ii) Show that $\varinjlim M_i$ has the following universal property: For an A-module N and homomorphisms $g_i\colon M_i \to N$ there exists a unique $g\colon \varinjlim M_i \to N$ with $g \circ f_i = g_i$ for all i if and only if $g_j \circ f_{ij} = g_i$ for all $i \leq j$.

(iii) Show that $\varinjlim$ is an exact functor. More precisely, this means the following: Assume $(M_i), (N_i), (P_i)$ are three directed systems of A-modules over the same directed set I. Furthermore assume that for all i there exist short exact sequences $0 \to M_i \to N_i \to P_i \to 0$ such that the maps commute with the homomorphisms in the directed systems. Then there exists a natural short exact sequence

$$0 \to \varinjlim M_i \to \varinjlim N_i \to \varinjlim P_i \to 0.$$