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SS 2020

Retry exam: Commutative Algebra (V3A1, Algebra I)

Exercise A. (Points: 3)
Let M be an A-module and $\mathfrak{a} \subset A$ an ideal such that $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{a} \subset \mathfrak{m} \subset A$. Show that then $M = \mathfrak{a}M$.

Exercise B. (Points: 3)
Show that a finitely generated ideal $\mathfrak{a} \subset A$ is a principal ideal and generated by an idempotent element if and only if $\mathfrak{a}^2 = \mathfrak{a}$.

Exercise C. (Points: 5)
Consider the ring $A \coloneqq k[x, y, z]/(xy, z^2 - (x + y))$. Describe all irreducible components of $\mathrm{Spec}(A)$, i.e. the maximal closed irreducible subsets, and decide which of them have a non-empty intersection with $\mathrm{Spec}(A_x)$.

Exercise D. (Points: 2+2)
Describe explicitly a Noether normalization for the two k-algebras $k[x, y]/(x^2 + y^2)$ and $k[x, y, z]/(y - z^2, xz - y^2)$.

Exercise E. (Points: 2+4)
Consider the ring $A = k[x, y, z]/(xy^2 - xz^2, x^2)$ where $char(k) \neq 2$.

(i) Show that the ideals $(\bar{z} - \bar{y}) \subset A$ and $(\bar{z} + \bar{y}) \subset A$ are both primary ideals and determine their radicals.

(ii) Determine a primary decomposition of the ideal $(0) \subset A$ and decide which associated prime ideals are isolated and which are embedded.

Exercise F. (Points: 5)
Compute $\mathrm{Ass}(M)$ and $\mathrm{Ann}(M)$ of the kernel $\ker(\psi)$ of the following A-module homomorphism $\psi\colon A^{\oplus 2} \to A$, $(a, b) \mapsto a\bar{x} + b\bar{y}$, where $A \coloneqq k[x, y]/(x^2 y)$.

Exercise G. (Points: 4+4)
Consider $A = k[x, y, z]/(xyz, z^2)$ as a graded ring with $\deg(\bar{x}) = \deg(\bar{y}) = \deg(\bar{z}) = 1$.

(i) Compute the Poincaré series $P(A, t)$ and determine the dimension of A[1]

(ii) Is $A_{(x,y,z)}$ regular or Cohen–Macaulay?

All rings are commutative with a unit and $1 \neq 0$.

[1] You will have to use that there are $\binom{2+n}{2}$ monomials of degree n in three variables.