Dr. René Mboro

GRADUATE SEMINAR ON ALGEBRAIC GEOMETRY (S4A1) CLASSIFICATION OF COMPLEX ALGEBRAIC SURFACES

Winter term 2019
Thursday 2:15-3:45 pm, N 0.006 - Neubau

The seminar is intended to be an introduction to the Enriques classification of complex algebraic surfaces. Complex algebraic surfaces are (here) smooth projective varieties of dimension 2 over $\mathbb{C}$ and the classification is with respect to birational equivalence. Recall that two (irreducible) varieties X and Y are birational if there are non-empty (Zariski) open subsets $U \subset X$, $V \subset Y$ which are isomorphic.

For curves, there is a unique (up to isomorphism) smooth projective model for each birational class but for surfaces, the smooth projective model is not unique. However, there is still a way to standardize by considering relatively minimal model. For (smooth projective) curves, the main numerical invariant was the (geometric) genus; for (smooth projective) surfaces, the main birational invariant used in the classification is the Kodaira dimension, which is for a surface X a measure of the growth with r of the dimension of the space of global sections of powers $K_X^{\otimes r}$ of the canonical line bundle of X, and which is at most 2. Surfaces of Kodaira dimension < 2 can be classified more explicitly.

The seminar will allow the students to see in action various techniques of algebraic geometry and to encounter important examples of varieties, such as ruled, abelian and K3 surfaces.

Prerequisities: We assume familiarity with the basic concepts of Algebraic Geometry, roughly in the amount of chapters II and III of Hartshorne's book [4].

The main reference is [3].

Should you be interested in the seminar, please send an email to René Mboro (rmboro@uni-bonn.de) naming two or three of the talks you would be interested to give. The talks will be distributed by early March.

Every speaker is to contact René Mboro at least two weeks before the talk to discuss his topic with him. He will also be the person to contact should you encounter any problems in yours or anyone elses talk or for references. Each talk is 90 min and they should be prepared accordingly. Questions during and after the talks are welcome.

1-Divisors.

Date: 24.10.19

Recall (without proof) the relation between Weil and Cartier divisors and line bundles, see [4, II.6], especially explain when the equalities $Pic(X) = CaCl(X)$ resp. $CaCl(X) = Cl(X)$ hold and define the canonical divisor of a smooth projective variety. Explain how divisors can be pulled back or pushed forward [3, I.1]. Define the degree of a line bundle on a curve ([4, II 6.10]). Introduce linear systems and explain quickly the link to effective divisors and discuss [4, IV 1.2]. and introduce rational morphisms [3, II.4-6], see also [4, II, 7.1]. Recall the notions of ample and very ample line bundles and dis- cuss their relations [4, pp 150-156]. Prove Bertinis theorem [4, II, 8.18] and deduce that any divisor on a smooth projective variety (over an algebraically closed field) is linearly equivalent to the difference of two smooth hypersurface sections.

2-Intersection theory and the Riemann-Roch theorem on surfaces.
Date: 31.10.19
Prove Riemann-Roch theorem for curves ([4, Thm 1.3]) and compute the degree of the canonical divisor of a curve ([4, Example 1.3.3]). Prove [4, V, Theorem 1.1] (see also [3, I.1-I.7]) and the Riemann-Roch theorem for surfaces ([4, V, Theorem 1.6], see also [3, I.12]). Recall the adjunction formula and discuss [4, Proposition. 1.5].

3-Blow-ups.
Date: 7.11.19
Define the blow-up of a scheme in a closed subscheme [4, p. 162 ff.] and discuss the universal property of blow-ups [4, II Prop. 7.14]. Discuss the particular case of surfaces [3, II.1-3; II. 8]. Finally discuss how blow-ups can be used for the elimination of points of indeterminacy of a rational map [3, II.7].

4-Birational maps between surfaces.
Date: 14.11.19
Describe the structure of a birational morphism between surfaces [3, II.11,12]. Discuss the exponential sequence over the complex numbers and the Néron-Severi group [3, I.10]. Compute the Néron-Severi group of the blow-up of a surface X in terms of the Néron-Severi group of X [3, II.13]. Define the irregularity, the geometric genus and the plurigenera of a surface and prove that they are birational invariants ([3, Proposition III.20]). Introduce the Kodaira dimension (see [1, 5.4-5.8] and [3, VII]).

5-Minimal models.
Date: 21.11.19
Introduce the notion of minimal surfaces [3, II.15-16]. Prove Castelnuovo's contractibility criterion [3, II.17]. If time permits give a reformulation in the language of the minimal model program of Mori [5, Prop. 3.13, Ref.3.14].

6-Ruled surfaces.
Date: 28.11.19
Introduce Ruled surfaces and geometrically ruled surfaces [3, III.1-3]. Prove the theorem of Noether-Enriques [3, III.4] and discuss the structure of geometrically ruled surfaces [3, III.7]. Prove that ruled surfaces are the minimal models for surfaces birational to $C \times \mathbb{P}^1$, where C is a non-rational curve, see [3, III.10].

7-Castenuovo's criterion for rationality.
Date: 5.12.19
Present [3, V.1-10]. Especially show that minimal surfaces with $q = 0$ and $p_2 = 0$ are rational. Show that the minimal rational surfaces are $\mathbb{P}^2$ and $\mathbb{F}_n$, for $n \neq 1$.

8-The Albanese variety and minimal models for surfaces of Kodaira dimension ≥ 0.
Date: 12.12.19
Introduce the Picard variety [1, p.70-72] and the Albanese variety [1, p. 73, Def. 5.2-p. 75, 5.4]. Prove that there is a unique minimal model for surfaces of non-negative Kodaira-dimension [3, V.15-19]. Finally state the theorem about the characterization of ruled surfaces [1, Thm. 13.2] (see also [3, Proposition VI.2, Theorem VI.17]).

9-Surfaces of Kodaira dimension 0.
Date: 19.12.19
Present [3, VIII.1-6]. Especially define bi-elliptic surfaces.

10-K3 surfaces and Enriques surfaces.
Date: 9.01.20
Show that K3 surfaces are simply connected, [1, Thm. 10.3 (ii)] and give examples of K3 surfaces, see [3, VIII.8-11]. Study linear systems on K3 surfaces [3, VIII.13]. Show that a K3 surface is elliptic if and only if it contains a non-trivial divisor D with $D^2 = 0$, see [6, Theorem 2.2]. Show that Enriques surfaces can be written as quotients of K3 surfaces (and vice versa) [3, VIII.17] and give examples of Enriques surfaces.

11-Surfaces of Kodaira dimension ≥ 1.
Date: 16.01.20
Prove [3, Lemma IX.1, Propositions IX.2, IX.3] and give an example of an elliptic surface of Kodaira dimension 1. Discuss [1, Theorem 9.4]. Introduce Kodaira fibration and give an example of a Kodaira fibration of general type ([2, V.14]).

References

[1] L. Bădescu, *Algebraic surfaces*, Universitext, Springer Verlag, New York, 2001.

[2] W. P. Barth, K. Hulek, C. A. M. Peters, A. Van de Ven, *Compact Complex surfaces*, 2nd edition, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, A series of modern surveys in Mathematics, vol. 4, Springer-Verlag, Berlin, 2004.

[3] A. Beauville, *Complex algebraic surfaces*, 2nd editionm London Math. Soc. Students texts, vol. 34, Cambridge University Press, 1996.

[4] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Math., no. 52 (1977), Springer.

[5] C. Peters, *Complex surfaces* (long version), online lecture notes

[6] J. Sawon, Abelian fibred holomorphic symplectic manifolds, Turkish J. Math. 27 (2003), no. 1, pp. 197-230.