# On distributions with $\mathrm{GL}_2(\mathbb{R})$ dilation symmetry

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# Introduction

This thesis studies $\mathrm{GL}_n^+(\mathbb{R})$-homogeneous tempered distributions defined on the space $\mathbb{R}^{n^2}$ for $n \geq 1$. For a function $\varphi \in \mathcal{S}(\mathbb{R}^{n^2})$, a matrix $A \in \mathrm{GL}_n(\mathbb{R})$ and a parameter $\alpha \in \mathbb{C}$, we define the dilated function $D_A^\alpha \varphi \in \mathcal{S}(\mathbb{R}^{n^2})$

$$D_A^\alpha \varphi(V) \coloneqq \frac{1}{|\det A|^{\alpha+n}} \varphi(A^{-1}V),$$

where we identify $V \in \mathbb{R}^{n \times n}$ with the point $v = (v_1, \ldots, v_{n^2}) \in \mathbb{R}^{n^2}$ by $V_{i,j} = v_{i+(j-1)n}$. From now on we will use the notation $\mathbb{R}^{n \times n}$ for $\mathbb{R}^{n^2}$ with this intended identification.
Let $\mathrm{GL}_n^+(\mathbb{R})$ be the subset of $\mathrm{GL}_n(\mathbb{R})$ defined by

$$\mathrm{GL}_n^+(\mathbb{R}) \coloneqq \{A \in \mathrm{GL}_n(\mathbb{R})\colon\ \det A > 0\}.$$

**Definition 1.** *For $\alpha \in \mathbb{C}$, we say that a tempered distribution $\Lambda \in \mathcal{S}'(\mathbb{R}^{n \times n})$ is $\mathrm{GL}_n^+(\mathbb{R})$-**homogeneous of degree** $\alpha$ if*

$$\Lambda(D_A^\alpha \varphi) = \Lambda(\varphi),$$

*for every matrix $A \in \mathrm{GL}_n^+(\mathbb{R})$ and every function $\varphi \in \mathcal{S}(\mathbb{R}^{n \times n})$.*

*We denote by $\mathbf{X}_\alpha^\mathbf{n}$ the vector space of $\mathrm{GL}_n^+(\mathbb{R})$-homogeneous tempered distributions of degree $\alpha$.*

In dimension $n = 1$, our definition coincides with the one of homogeneity under scalar matrices $\{\lambda \mathbb{I}_n\colon \lambda > 0\}$, see [7], since $\mathrm{GL}_1^+(\mathbb{R}) = \{\lambda \in \mathbb{R}\colon \lambda > 0\}$.

**Definition 2.** *For $\alpha \in \mathbb{C}$, we say that a tempered distribution $\Lambda \in X_\alpha^n$ is*

- ***even**, if for some matrix $A \in \mathrm{GL}_n(\mathbb{R})$, $\det A < 0$,*

$$\Lambda(D_A^\alpha \phi) = \Lambda(\varphi);$$

- ***odd**, if for some matrix $A \in \mathrm{GL}_n(\mathbb{R})$, $\det A < 0$,*

$$\Lambda(D_A^\alpha \phi) = -\Lambda(\varphi).$$

*We denote by $\mathbf{X}_{\alpha,\mathrm{even}}^\mathbf{n}, \mathbf{X}_{\alpha,\mathrm{odd}}^\mathbf{n}$ the vector subspaces of even and odd elements of $X_\alpha^n$.*

It is easy to observe that

$$X_\alpha^n = X_{\alpha,\mathrm{even}}^n \oplus X_{\alpha,\mathrm{odd}}^n,$$

and we will elaborate on this in Lemma 20. Therefore, in order to give a complete classification for $X^n_\alpha$, it is enough to describe the elements of $X^n_{\alpha,\mathrm{even}}$ and $X^n_{\alpha,\mathrm{odd}}$.

The candidates are the tempered distributions associated to the homogeneous functions $|\det V|^\alpha$, $\operatorname{sgn}(\det V)|\det V|^\alpha$. Let $I^n_{\alpha,\mathrm{even}}, I^n_{\alpha,\mathrm{odd}}$ be defined, when it makes sense, by

$$\begin{aligned} I^n_{\alpha,\mathrm{even}}(\varphi) &:= \int_{\mathbb{R}^{n\times n}} |\det V|^\alpha \varphi(V)\,\mathrm{d}V, \\ I^n_{\alpha,\mathrm{odd}}(\varphi) &:= \int_{\mathbb{R}^{n\times n}} \operatorname{sgn}(\det V)|\det V|^\alpha \varphi(V)\,\mathrm{d}V, \end{aligned} \tag{1}$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^{n\times n})$.

In dimension $n = 1$, the spaces $X^1_{\alpha,\mathrm{even}}$ and $X^1_{\alpha,\mathrm{odd}}$ are known to have dimension 1. The result stated in Theorem 3 below can be recovered from [4] or [7]. We will focus on the case of dimension $n = 2$, for which we are able to provide a complete description of the spaces $X^2_\alpha$ for arbitrary $\alpha \in \mathbb{C}$, see Theorem 5 below. The techniques and the arguments that we develop in this context are expected to be useful also to tackle the problem in the general case of an arbitrary $n \geq 1$. However, this is beyond the scope of this work.

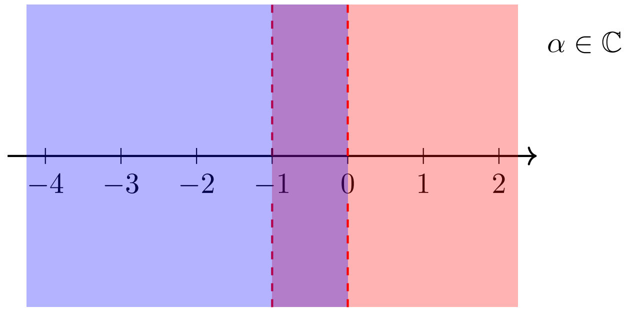
In the following statements, the Fourier transform has to be intended in the sense of tempered distributions, namely as the map

$$\widehat{\cdot} : \mathcal{S}'(\mathbb{R}^{n\times n}) \to \mathcal{S}'(\mathbb{R}^{n\times n}),\ \widehat{\Lambda}(\varphi) := \Lambda(\widehat{\varphi}).$$

**Theorem 3.** *For $\alpha \in \mathbb{C}$, both $X^1_{\alpha,\mathrm{even}}$ and $X^1_{\alpha,\mathrm{odd}}$ have dimension 1. In particular,*

- $X^1_{\alpha,\mathrm{even}}$ *is generated by*
  - *for* $\mathrm{Re}(\alpha) > -1$, $\quad I^1_{\alpha,\mathrm{even}}$;
  - *for* $\mathrm{Re}(\alpha) \leq -1$, $\quad (I^1_{-\alpha-1,\mathrm{even}})^\wedge$;
- $X^1_{\alpha,\mathrm{odd}}$ *is generated by*
  - *for* $\mathrm{Re}(\alpha) > -1$, $\quad I^1_{\alpha,\mathrm{odd}}$;
  - *for* $\mathrm{Re}(\alpha) \leq -1$, $\quad (I^1_{-\alpha-1,\mathrm{odd}})^\wedge$.

The Fourier transform defines a bijection $X^1_\alpha \to X^1_{-\alpha-1}$. The key observation is that for $\alpha \in \mathbb{C}$ we either have $\mathrm{Re}(\alpha) > -1$ or $\mathrm{Re}(-\alpha-1) > -1$, hence either $|x|^\alpha \in L^1_{\mathrm{loc}}(\mathbb{R})$ or $|x|^{-\alpha-1} \in L^1_{\mathrm{loc}}(\mathbb{R})$. It holds the stronger property that the integral of their absolute value over the ball $B_R(0)$ grows at most polynomially in $R$. Therefore at least one between $I^1_{\alpha,\mathrm{even}}$ and $I^1_{-\alpha-1,\mathrm{even}}$ is well-defined as an element of $\mathcal{S}'(\mathbb{R})$, and it has the appropriate degree of $\mathrm{GL}^+_1(\mathbb{R})$-homogeneity. Moreover, the two regions $\{\mathrm{Re}(\alpha) > -1\}$ and $\{\mathrm{Re}(\alpha) < 0\}$ overlap and cover the whole complex plane, thus we have an existence result for every $\alpha \in \mathbb{C}$. The result about having exactly dimension 1 is heavily based on the study of the tempered distributions $\Gamma \in X^1_\alpha$ supported at the origin. In particular, we will prove in Lemma 32 that $\Gamma \neq 0$ only if $\alpha \in \mathbb{Z}$, $\alpha \leq -1$. As a consequence, if $\Lambda \in X^1_{\alpha,\mathrm{even}}$, then either $\Lambda = cI^1_{\alpha,\mathrm{even}}$ or $\widehat{\Lambda} = cI^1_{-\alpha-1,\mathrm{even}}$, $c \in \mathbb{C}$, and an analogous property holds for $\Lambda \in X^1_{\alpha,\mathrm{odd}}$.




For $-1 < \mathrm{Re}(\alpha) < 0$, we can define nonzero elements of $X^1_{\alpha,\mathrm{even}}$ both via $I^1_{\alpha,\mathrm{even}}$ and, through the Fourier transform, via $I^1_{-\alpha-1,\mathrm{even}}$. Since the space has dimension 1, the two tempered distributions are linearly dependent. An analogous result holds for $X^1_{\alpha,\mathrm{odd}}$, and the constants of linear dependence are established by the following lemma.

**Lemma 4.** *For $-1 < \mathrm{Re}(\alpha) < 0$,*

$$\pi^{-\frac{\alpha+1}{2}}\ \Gamma\left(\frac{\alpha+1}{2}\right)(I^1_{-\alpha-1,\mathrm{even}})^\wedge = \pi^{\frac{\alpha}{2}}\ \Gamma\left(-\frac{\alpha}{2}\right) I^1_{\alpha,\mathrm{even}},$$
$$\pi^{-\frac{\alpha+2}{2}}\ \Gamma\left(\frac{\alpha+2}{2}\right)(I^1_{-\alpha-1,\mathrm{odd}})^\wedge = -i\pi^{\frac{\alpha-1}{2}}\ \Gamma\left(-\frac{\alpha-1}{2}\right) I^1_{\alpha,\mathrm{odd}}.$$

*Moreover, for a fixed function $\varphi \in \mathcal{S}(\mathbb{R})$, the functions in the variable $\alpha \in \mathbb{C}$ defined by*

$$\begin{cases} \pi^{-\frac{\alpha+1}{2}}\ \dfrac{1}{\Gamma\left(-\frac{\alpha}{2}\right)}\ I^1_{-\alpha-1,\mathrm{even}}(\widehat{\varphi}) & \textit{if } \mathrm{Re}(\alpha) < 0, \\ \pi^{\frac{\alpha}{2}}\ \dfrac{1}{\Gamma\left(\frac{\alpha+1}{2}\right)}\ I^1_{\alpha,\mathrm{even}}(\varphi) & \textit{if } \mathrm{Re}(\alpha) > -1, \end{cases}$$
$$\begin{cases} i\pi^{-\frac{\alpha+2}{2}}\ \dfrac{1}{\Gamma\left(-\frac{\alpha-1}{2}\right)}\ I^1_{-\alpha-1,\mathrm{odd}}(\widehat{\varphi}) & \textit{if } \mathrm{Re}(\alpha) < 0, \\ \pi^{\frac{\alpha-1}{2}}\ \dfrac{1}{\Gamma\left(\frac{\alpha+2}{2}\right)}\ I^1_{\alpha,\mathrm{odd}}(\varphi) & \textit{if } \mathrm{Re}(\alpha) > -1, \end{cases}$$

*are holomorphic on $\mathbb{C}$.*

For $n = 2$, the problem gets more complicated, ultimately because the critical variety

$$M \coloneqq \{V \in \mathbb{R}^{2\times 2} \colon\ \det V = 0\}$$

is not given only by the origin, and it has an interesting geometry. In fact, $V \in M$ may have rank 0 or 1. The only matrix with rank 0 is the zero matrix, while $M \setminus \{0\}$, the subset of matrices with rank 1, defines a smooth 3-dimensional submanifold of $\mathbb{R}^{2\times 2}$. The orbits of the action of $\mathrm{GL}_2^+(\mathbb{R})$ onto $\mathbb{R}^{2\times 2}$, given by left multiplication by the inverse matrix, fibrate $M \setminus \{0\}$ into punctured planes. We have a 2-1 map

$$\nu\colon (\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z}) \to M \setminus \{0\},\ \nu(w,\theta) = \begin{pmatrix} w\cos\theta & w\sin\theta \end{pmatrix} = \begin{pmatrix} w_1\cos\theta & w_1\sin\theta \\ w_2\cos\theta & w_2\sin\theta \end{pmatrix}.$$

To state our theorem, we have to consider the differential operator $\det(\partial_{i,j})$ of second order algebraically mirroring the determinant, namely

$$\det(\partial_{i,j}) \coloneqq \partial_{1,1}\partial_{2,2} - \partial_{1,2}\partial_{2,1},$$

which defines a map on the space of tempered distributions

$$\det(\partial_{i,j}) \colon \mathcal{S}'(\mathbb{R}^{2\times 2}) \to \mathcal{S}'(\mathbb{R}^{2\times 2}),\ \det(\partial_{i,j})\Lambda(\varphi) = \Lambda(\det(\partial_{i,j})\varphi).$$

Our main theorem is the following complete classification result.

**Theorem 5.** *For $\alpha \in \mathbb{C}$, $X^2_{\alpha,\mathrm{odd}}$ has dimension 1. For $\alpha \neq -1$, $X^2_{\alpha,\mathrm{even}}$ has dimension 1, while $X^2_{-1,\mathrm{even}}$ is infinite dimensional. In particular,*

- *$X^2_{\alpha,\mathrm{even}}$ is generated by*
  - *for* $\mathrm{Re}(\alpha) > -1$, $\quad I^2_{\alpha,\mathrm{even}}$;
  - *for* $\mathrm{Re}(\alpha) < -1$, $\quad (I^2_{-\alpha-2,\mathrm{even}})^\wedge$;
  - *for* $\mathrm{Re}(\alpha) = -1,\ \alpha \neq -1$, $\quad \det(\partial_{i,j}) I^2_{\alpha+1,\mathrm{odd}}$;
- *there is a bijection $\mu \colon \mathcal{D}'(\mathbb{R}/\pi\mathbb{Z}) \to X^2_{-1,\mathrm{even}}$ given by*

  $$(\mu(F))(\varphi) := F(\psi_\varphi),$$

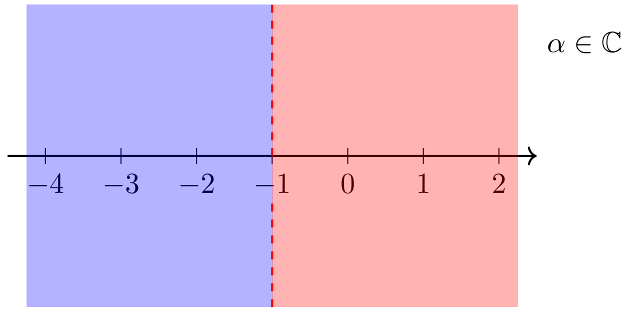
  *where, for every $\varphi \in \mathcal{S}(\mathbb{R}^{2\times 2})$,*

  $$\psi_\varphi(\theta) := \int_{\mathbb{R}^2\setminus\{0\}} \varphi(\nu(w,\theta))\,\mathrm{d}w = \int_{\mathbb{R}^2\setminus\{0\}} \varphi\left(w\cos\theta \quad w\sin\theta\right)\mathrm{d}w,$$

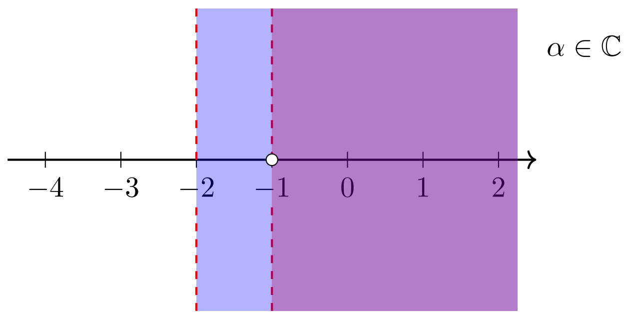
  *is a smooth $\pi$-periodic function on $\mathbb{R}$;*
- *$X^2_{\alpha,\mathrm{odd}}$ is generated by*
  - *for* $\mathrm{Re}(\alpha) > -1$, $\quad I^2_{\alpha,\mathrm{odd}}$;
  - *for* $\mathrm{Re}(\alpha) < -1$, $\quad (I^2_{-\alpha-2,\mathrm{odd}})^\wedge$;
  - *for* $\mathrm{Re}(\alpha) = -1,\ \alpha \neq -1$, $\quad \det(\partial_{i,j}) I^2_{\alpha+1,\mathrm{even}}$;
  - *for* $\alpha = -1$, $\quad \mathrm{p.\,v.}\,\frac{1}{\det V}$,
    *where*

$$\mathrm{p.\,v.}\,\frac{1}{\det V}(\varphi) := \mathrm{p.\,v.}\int_{\mathbb{R}^{2\times 2}} \frac{\varphi(V)}{\det V}\,\mathrm{d}V = \lim_{\varepsilon\to 0}\int_{|\det V|>\varepsilon} \frac{\varphi(V)}{\det V}\,\mathrm{d}V. \tag{2}$$

The Fourier transform defines a bijection $X^2_\alpha \to X^2_{-\alpha-2}$. Unlike in the case $n = 1$, the two reflected half planes defined by the conditions $\mathrm{Re}(\alpha) > -1$ and $\mathrm{Re}(-\alpha-2) > -1$, namely $\{\mathrm{Re}(\alpha) > -1\}$ and $\{\mathrm{Re}(\alpha) < -1\}$, don't cover the whole plane $\mathbb{C}$. Instead, there is a "gap" given by the line $\{\mathrm{Re}(\alpha) = -1\}$. Therefore, while it is enough to restrict to the case $\mathrm{Re}(\alpha) \geq -1$, the study of $X^2_\alpha$ for $\mathrm{Re}(\alpha) = -1$ requires a deeper analysis. Outside the critical line we either have $|\det V|^\alpha$ or $|\det V|^{-\alpha-2}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{2\times 2})$, and the integral over the ball $B_R(0)$ grows at most polynomially in $R$. Hence either $I^2_{\alpha,\mathrm{even}}$ or $I^2_{-\alpha-2,\mathrm{even}}$ is well-defined as an element of $\mathcal{S}'(\mathbb{R}^{2\times 2})$. An analogous argument holds for the elements of $X^2_{\alpha,\mathrm{odd}}$.




To define the elements of $X^2_\alpha$ for $\mathrm{Re}(\alpha) = -1$, $\alpha \neq -1$, we observe that the differential operator $\det(\partial_{i,j})$ gives a map $X^2_\alpha \to X^2_{\alpha-1}$. In particular, if $\alpha \notin \{-1, 0\}$ this map is nonzero, as we will prove in Lemma 26. Therefore we can use $\det(\partial_{i,j})$ to describe nonzero elements of $X^2_\alpha$ for $\mathrm{Re}(\alpha) = -1$, $\alpha \neq -1$.




For $-2 < \mathrm{Re}(\alpha) < -1$, we can use the differential operator $\det(\partial_{i,j})$ to define nonzero elements of the spaces $X^2_{\alpha,\mathrm{even}}$ and $X^2_{\alpha,\mathrm{odd}}$, which have dimension 1. The following lemma establishes the constants of linear dependence between these alternative generators and the ones enlisted in Theorem 5.

**Lemma 6.** *For $-2 < \mathrm{Re}(\alpha) < -1$,*

$$4\pi^{-\alpha-2}\,\Gamma\left(\frac{\alpha+4}{2}\right)\Gamma\left(\frac{\alpha+3}{2}\right)(I^2_{-\alpha-2,\mathrm{even}})\hat{} = \\ = \pi^{\alpha}\,\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(-\frac{\alpha+1}{2}\right)(\det(\partial_{i,j})I^2_{\alpha+1,\mathrm{odd}}),$$

$$4\pi^{-\alpha-2}\,\Gamma\left(\frac{\alpha+4}{2}\right)\Gamma\left(\frac{\alpha+3}{2}\right)(I^2_{-\alpha-2,\mathrm{odd}})\hat{} = \\ = \pi^{\alpha}\,\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(-\frac{\alpha+1}{2}\right)(\det(\partial_{i,j})I^2_{\alpha+1,\mathrm{even}}).$$

*Moreover, for a fixed function $\varphi \in \mathcal{S}(\mathbb{R}^{2\times 2})$, the functions of $\alpha \in \mathbb{C}$ defined by*

$$\begin{cases} 4\pi^{-\alpha-2}\dfrac{1}{\Gamma\left(-\frac{\alpha}{2}\right)}\dfrac{1}{\Gamma\left(-\frac{\alpha+1}{2}\right)}\; I^2_{-\alpha-2,\mathrm{even}}(\widehat{\varphi}) & \text{if } \mathrm{Re}(\alpha) < -1, \\ \pi^{\alpha}\dfrac{1}{\Gamma\left(\frac{\alpha+4}{2}\right)}\dfrac{1}{\Gamma\left(\frac{\alpha+3}{2}\right)}\; I^2_{\alpha+1,\mathrm{odd}}(\det(\partial_{i,j})\varphi) & \text{if } \mathrm{Re}(\alpha) > -2, \end{cases}$$

$$\begin{cases} 4\pi^{-\alpha-2}\dfrac{1}{\Gamma\left(-\frac{\alpha}{2}\right)}\dfrac{1}{\Gamma\left(-\frac{\alpha+1}{2}\right)}\; I^2_{-\alpha-2,\mathrm{odd}}(\widehat{\varphi}) & \text{if } \mathrm{Re}(\alpha) < -1, \\ \pi^{\alpha}\dfrac{1}{\Gamma\left(\frac{\alpha+4}{2}\right)}\dfrac{1}{\Gamma\left(\frac{\alpha+3}{2}\right)}\; I^2_{\alpha+1,\mathrm{even}}(\det(\partial_{i,j})\varphi) & \text{if } \mathrm{Re}(\alpha) > -2, \end{cases}$$

*are holomorphic on* $\mathbb{C}$.

To conclude the classification we observe that (2) defines an element of $X^2_{-1,\mathrm{odd}}$, while for the even ones the claim follows by the analysis of elements of $X^2_{-1}$ supported on $M$ and the fact that the function

$$\frac{1}{|\det V|}$$

cannot be extended to an element of $X^2_{-1,\mathrm{even}}$.

An important difference between the cases of dimension $n = 1$ and $n = 2$ is encoded by the following statement, which will be proven as part of Corollary 45.

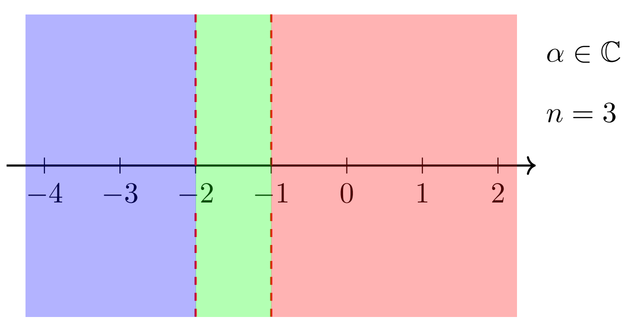
**Proposition 7.** *For* $\alpha \in \mathbb{Z}$, $\alpha \leq -2$, *let* $\Lambda \in X^2_\alpha$, *or let* $\Lambda \in X^2_{-1,\mathrm{even}}$. *Then* $\operatorname{supp}(\Lambda) \subset M$.

In fact, in dimension $n = 1$, for $\alpha \in \mathbb{Z}$, $\alpha \leq -1$, one between $\Lambda \in X_{\alpha,\mathrm{even}}$ and $\widetilde{\Lambda} \in X_{\alpha,\mathrm{odd}}$, $\Lambda, \widetilde{\Lambda} \neq 0$, is not supported at the origin.

It is also interesting to consider the effect of the operator $\det(\partial_{i,j})$ on the spaces $X^2_\alpha$ for $\alpha \in \mathbb{Z}$. In Corollary 47 we will prove it defines a bijection if and only if $\alpha \notin \{-1, 0\}$. In the diagram, the arrows stand for the maps $\det(\partial_{i,j})$, the hook denotes the injectivity, the double pointer the surjectivity.

$$\begin{tikzcd}
\mathbf{-3} & \mathbf{-2} & \mathbf{-1} & \mathbf{0} & \mathbf{1} & \\
\det(\partial_{i,j})\delta & (\operatorname{sgn}(\det V))\widehat{\ } \arrow[dl, hook, two heads] & \mathrm{p.\,v.}\,\frac{1}{\det V} \arrow[dl, hook, two heads] & \operatorname{sgn}(\det V) \arrow[dl, hook] & \det V \arrow[dl, hook, two heads] & \mathbf{X^2_{\alpha,\mathrm{odd}}} \\
(|\det V|)\widehat{\ } & \delta \arrow[ul, hook, two heads] & \{\mu(F)\} \arrow[ul] & 1 \arrow[ul] & |\det V| \arrow[ul, hook, two heads] & \mathbf{X^2_{\alpha,\mathrm{even}}}
\end{tikzcd}$$

For $n \geq 3$, the Fourier transform defines a bijection $X^n_\alpha \to X^n_{-\alpha-n}$. The "gap" between the two reflected half planes given by the conditions $\mathrm{Re}(\alpha) > -1$ and $\mathrm{Re}(-\alpha - n) > -1$ widens to a strip, where neither $|\det V|^\alpha$ nor $|\det V|^{-n-\alpha}$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^{n\times n})$, hence not even in $\mathcal{S}'(\mathbb{R}^{n\times n})$. For $\alpha \in \{-n+1, \ldots, -1\}$, we define elements of $X^n_{\alpha,\mathrm{even}}$ via integrals of the restriction of $\varphi$ over $n(n+\alpha)$-dimensional subspaces of $\mathbb{R}^{n\times n}$ contained in $M$. Moreover, through a principal value integral, we define an element of $X^n_{-1,\mathrm{odd}}$, and by means of the Fourier transform, an element of $X^n_{-n+1,\mathrm{odd}}$. However, the problem of existence is unsolved for $-n + 1 \leq \mathrm{Re}(\alpha) \leq -1$ except for the stated cases.




Our interest in $\mathrm{GL}^+_n(\mathbb{R})$-homogeneous distributions arose from the study of the multilinear singular integral forms with determinantal kernel in [1] and in [8]. These forms generalize the Hilbert transform and are connected to the restriction problem for the Fourier transform, in particular in the case of the sphere, as explained in [2] and [3]. We want to have a better understanding of their invariances in order to tackle effectively the conjecture about their boundedness.

A consequence of the classification theorem in dimension $n = 2$ is that the tempered distribution studied in [1] is identified by its invariance properties. In fact, let $\Lambda \in \mathcal{S}'(\mathbb{R}^6)$ be defined by

$$\begin{aligned}\Lambda(\varphi) &\coloneqq \mathrm{p.\,v.} \int_{\mathbb{R}^6} \frac{1}{\det \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \end{pmatrix}} \, \varphi(x,y,z) \ \delta(x+y+z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \\ &= \lim_{\varepsilon \to 0} \int_{|\det| > \varepsilon} \frac{1}{\det \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \end{pmatrix}} \, \varphi(x,y,z) \ \delta(x+y+z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,\end{aligned} \tag{3}$$

where $x, y, z \in \mathbb{R}^2$. It possesses the following invariance properties:

- (modulation invariance) for each vector $b \in \mathbb{R}^2$ define

  $$M_b \varphi(x,y,z) \coloneqq e^{2\pi i b \cdot (x+y+z)} \varphi(x,y,z).$$

  Then

  $$\Lambda(M_b \varphi) = \Lambda(\varphi),$$

  for every vector $b \in \mathbb{R}^2$ and every function $\varphi \in \mathcal{S}(\mathbb{R}^6)$;

- for each matrix $A \in \mathrm{GL}_2(\mathbb{R})$ define

  $$B_A \varphi(x,y,z) \coloneqq \frac{1}{\det A} \varphi(A^{-1}x, A^{-1}y, A^{-1}z).$$

  Then

  $$\Lambda(B_A \varphi) = \Lambda(\varphi), \tag{4}$$

  for every matrix $A \in \mathrm{GL}_2(\mathbb{R})$ and every function $\varphi \in \mathcal{S}(\mathbb{R}^6)$.

**Corollary 8.** *Up to multiplication by a constant, the tempered distribution $\Lambda$ defined in* (3) *is the unique nonzero element of $\mathcal{S}'(\mathbb{R}^6)$ satisfying the modulation invariance and the property* (4).

The thesis is organised as follows.

In the first chapter, we briefly recall some preliminaries, definitions and results from the theory of distributions. Then, we study the effect on $X^n_\alpha$ of the Fourier transform, the operator associated to the multiplication by $\det V$ and $\det(\partial_{i,j})$. After that, for arbitrary $n \geq 1$, we prove an existence result for nonzero elements of $X^n_{\alpha,\mathrm{even}}$ for $\mathrm{Re}(\alpha) \notin [-n+1, -1]$ or $\alpha \in \{-n+1, \dots, -1\}$, and of $X^n_{\alpha,\mathrm{odd}}$ for $\mathrm{Re}(\alpha) \notin [-n+1, -1]$ or $\alpha \in \{-n+1, -1\}$. Finally, we prove an uniqueness result for $n \geq 1$, namely that, away from $M$, a tempered distribution $\Lambda \in X^n_{\alpha\mathrm{even}} \cup X^n_{\alpha,\mathrm{odd}}$ coincides with the definitions in (1). The idea behind our approach to obtain this result is simple enough to warrant discussion in the Introduction. We differentiate the equalities given by the $\alpha$-homogeneity conditions for functions $\phi \in \mathcal{D}(\mathbb{R}^{n \times n} \setminus M)$ and for matrices arbitrarily "$\varepsilon$-close" to the identity $\mathbb{I}_n$, e.g.

$$A = (\mathbb{I}_n + \varepsilon E^{i,j})^{-1}, \ (E^{i,j})_{l,m} = \delta_{i,l}\delta_{j,m},$$

where $\delta_{i,l}$ is the Kronecker delta of the couple $(i, l)$. We divide the associated equality by $\varepsilon$, and we take the limit as $\varepsilon$ goes to zero, which yields

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Lambda(D_A^\alpha \phi - \phi) = 0$$

By continuity of $\Lambda$, we move the limit to the argument of the distribution. In this way we translate the information coming from homogeneity condition into properties of the derivatives of $\Lambda$.

The second chapter is devoted to the proof of the classification results stated in Theorem 3 and Theorem 5. In particular, the complete classification in the case $n = 2$ is established by investigating $\alpha$-homogeneous tempered distributions supported on $M$. In Lemma 39, we prove that if $\Gamma \in X_\alpha^2$ is supported on $M$, then $\alpha \in \mathbb{Z}$, $\alpha \leq -1$. In particular, if $\alpha = -1$, then $\Gamma$ is even and of the form appearing in Theorem 5. To conclude the classification we show that the function defined on $\mathbb{R}^{2\times 2} \setminus M$

$$\frac{1}{|\det V|}$$

cannot be extended to an element of $X^2_{-1,\text{even}}$. In the last part of the chapter we prove Lemma 6 and Corollary 8.

**Acknowledgements.** I would like to express my gratitude to my advisor Prof. Christoph Thiele for his guidance, the inspiring discussions, and for giving me a lot of motivation on the subject and plenty of helpful remarks to improve the content and the exposition of this thesis.

# Chapter 1

# Existence and uniqueness results for $n \geq 1$

## 1.1 Notation and preliminaries for tempered distributions

From now on, $\mathbb{N} = \{0, 1, 2, \dots\}$ will denote the set of nonnegative integers.
For a multi-index $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$ we define

- $|\beta| = \beta_1 + \cdots + \beta_d$, $\beta! = \beta_1! \dots \beta_d!$;
- $x^\beta = x_1^{\beta_1} \dots x_d^{\beta_d}$, $\partial^\beta = \partial_1^{\beta_1} \dots \partial_d^{\beta_d}$.

Let $\Omega$ be an open subset of $\mathbb{R}^d$. We have the following definitions.

**Definition 9.** *We denote by $\mathcal{D}(\boldsymbol{\Omega})$ the set of complex-valued smooth functions with compact support in $\Omega$. A sequence $\{\phi_k\}_k \subset \mathcal{D}(\Omega)$ is said to* **converge to** *$\phi \in \mathcal{D}(\Omega)$ if the following two conditions hold:*

- *there is a compact set $K \subset \Omega$ containing the supports of all $\phi_k$;*
- *for each multi-index $\beta$ we have $\partial^\beta \phi_k \to \partial^\beta \phi$ uniformly in $x$ as $k \to \infty$.*

*This defines a topology on $\mathcal{D}(\Omega)$.*

*A* **distribution** *is a continuous linear functional $\Lambda\colon \mathcal{D}(\Omega) \to \mathbb{C}$. We denote by $\mathcal{D}'(\boldsymbol{\Omega})$ the vector space of distributions on $\Omega$.*

For a smooth function $\varphi\colon \mathbb{R}^d \to \mathbb{C}$, $N \in \mathbb{N}$, we define the norm

$$\|\varphi\|_N \coloneqq \sup_{\substack{x \in \mathbb{R}^d \\ |\beta|,|\gamma| \leq N}} \left| x^\gamma (\partial^\beta \varphi)(x) \right|, \tag{1.1}$$

where $\beta, \gamma \in \mathbb{N}^d$ are multi-indices.

We note that, for $N \in \mathbb{N}$, there exists $C_{N,d}$ such that for every $x \in \mathbb{R}^d$ we have

$$|x|^N \leq C_{N,d} \sum_{|\gamma|=N} |x^\gamma|.$$

In fact, the function $F\colon \mathbb{S}^{d-1} \to (0,\infty)$, $F(x) = \sum_{|\gamma|=N} |x^\gamma|$ is continuous and strictly positive. Since it attains a minimum, there exists $C_{N,d}$ such that

$$\sum_{|\gamma|=N} |x^\gamma| = F(x) \geq \frac{1}{C_{N,d}} > 0.$$

We get the claim for an arbitrary $x$ by considering $F$ for $\frac{x}{|x|}$.

Therefore, for every $N \in \mathbb{N}$, there exists $C_N$ such that

$$\|\varphi\|_N \leq \sup_{\substack{x\in\mathbb{R}^d \\ |\beta|,k\leq N}} |x|^k \big|(\partial^\beta \varphi)(x)\big| \leq C_N \|\varphi\|_N. \tag{1.2}$$

**Definition 10.** *We denote by $\mathcal{S}(\mathbb{R}^d)$ the set of complex-valued smooth functions such that $\|\varphi\|_N < \infty$ for every $N$. The elements of $\mathcal{S}(\mathbb{R}^d)$ are called* **Schwartz functions***. A sequence $\{\varphi_k\}_k \subset \mathcal{S}(\mathbb{R}^d)$ is said to* **converge to** *$\varphi \in \mathcal{S}(\mathbb{R}^d)$ if $\|\varphi - \varphi_k\|_N \to 0$, as $k \to \infty$, for every $N \in \mathbb{N}$. This defines a topology on $\mathcal{S}(\mathbb{R}^d)$.*

*A* **tempered distribution** *is a continuous linear functional $\Lambda\colon \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$. We denote by $\mathcal{S}'(\mathbb{R}^{\mathbf{d}})$ the vector space of tempered distributions on $\mathbb{R}^d$.*

For every open subset $\Omega \subset \mathbb{R}^d$, a tempered distribution $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$ defines a distribution $\widetilde{\Lambda} \in \mathcal{D}'(\Omega)$ by

$$\widetilde{\Lambda}(\phi) \coloneqq \Lambda(\phi),$$

for every $\phi \in \mathcal{D}(\Omega)$, where we identify $\phi \in \mathcal{D}(\Omega)$ with the function $\phi \in \mathcal{S}(\mathbb{R}^d)$ obtained by letting $\phi(x) = 0$ for $x \in \mathbb{R}^d \setminus \Omega$. A priori $\widetilde{\Lambda}$ is only a linear functional, the continuity is implied by the following statement.

**Proposition 11.** *Let $\{\phi_k\}_k \subset \mathcal{D}(\Omega)$ be a sequence converging to $\phi \in \mathcal{D}(\Omega)$. Then $\{\phi_k\}_k$ converges to $\phi$ also in $\mathcal{S}(\mathbb{R}^d)$, under the identification of the functions $\phi_k, \phi$ with elements of $\mathcal{S}(\mathbb{R}^d)$ described above.*

*Proof.* By the definition of convergence in $\mathcal{D}(\Omega)$, there exists a compact set $K \subset \Omega$ such that $\operatorname{supp}(\phi_k), \operatorname{supp}(\phi) \subset K$. Moreover, for each multi-index $\beta$, we have

$$\sup_{x\in K} \Big|(\partial^\beta(\phi_k - \phi))(x)\Big| \to 0, \qquad \text{as } k \to \infty.$$

Now let $N \in \mathbb{N}$. Since $K \subset B_R(0)$ for some $R \geq 1$, then, for every multi-index $\gamma$ such that $|\gamma| \leq N$,

$$\sup_{x\in K} |x^\gamma| \leq (\sup_{x\in K} |x|)^{|\gamma|} \leq R^N.$$

This implies that

$$\|\phi_k - \phi\|_N \leq R^N \max_{|\beta|\leq N} \sup_{x\in K} \Big|(\partial^\beta(\phi_k - \phi))(x)\Big| \to 0, \qquad \text{as } k \to \infty,$$

proving the claim. $\square$

A well known characterization of the tempered distributions in terms of the norms (1.1) is given by the following result.

**Proposition 12.** *Let $\Lambda$ be a complex-valued linear functional on $\mathcal{S}(\mathbb{R}^d)$. Then it is a tempered distribution if and only if there exist $N \in \mathbb{N}$ and a constant $c > 0$ such that*

$$|\Lambda(\varphi)| \leq c\|\varphi\|_N,$$

*for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$.*

*Proof.* The sufficient condition trivially follows from the definition of $\mathcal{S}'(\mathbb{R}^d)$.

The necessary condition is proven by contradiction. Assume otherwise. Then the conclusion fails and, for each $k$, there is $\psi_k \in \mathcal{S}(\mathbb{R}^d)$ with $\|\psi_k\|_k = 1$, while $|\Lambda(\psi_k)| \geq k$. Take $\varphi_k = \frac{\psi_k}{\sqrt{k}}$. Then $\|\varphi_k\|_N \leq \|\varphi_k\|_k$ as soon as $k \geq N$, and thus $\|\varphi_k\|_N \leq \frac{1}{\sqrt{k}} \to 0$ as $k \to \infty$. Instead, $|\Lambda(\varphi_k)| \geq \sqrt{k} \to \infty$, contradicting the continuity of $\Lambda$. □

In an analogous way we can prove the following characterization of the distributions.

**Proposition 13.** *Let $\Lambda$ be a complex-valued linear functional on $\mathcal{D}(\Omega)$. Then it is a distribution if and only if for every compact set $K \subset \Omega$, there exist $N_K \in \mathbb{N}$ and a constant $c_K > 0$ such that*

$$|\Lambda(\phi)| \leq c_K \sup_{|\gamma| \leq N_K} \sup_{x \in K} \left|\partial^\gamma \phi(x)\right|,$$

*for every $\phi \in \mathcal{D}(\Omega)$, $\operatorname{supp}(\phi) \subset K$.*

**Definition 14.** *For a distribution $\Lambda \in \mathcal{D}'(\Omega)$, we say that $\Lambda$* **vanishes** *in an open subset of $\Omega$ if $\Lambda(\phi) = 0$ for all test functions $\phi \in \mathcal{D}(\Omega)$ which have their supports in that open set. We define the* **support of a distribution** *$\Lambda \in \mathcal{D}'(\Omega)$ as the complement of the largest open set on which $\Lambda$ vanishes.*

*We define the* **support of a tempered distribution** *$\Lambda \in \mathcal{S}'(\mathbb{R}^d)$ as the support of the associated distribution $\widetilde{\Lambda} \in \mathcal{D}'(\mathbb{R}^d)$.*

A rather classical result, which will be useful in our analysis, is the classification of tempered distributions supported at the origin. In proving it we will follow [7].

**Proposition 15.** *Let $\Gamma \in \mathcal{S}'(\mathbb{R}^d)$, $\operatorname{supp}(\Gamma) = \{0\}$. Then there exist $N \in \mathbb{N}$, $c_\beta \in \mathbb{C}$, such that*

$$\Gamma = \sum_{|\beta| \leq N} c_\beta \partial^\beta \delta,$$

*where $\delta$ is the Dirac delta function.*

*Proof.* The argument is based on the following result.

**Lemma 16.** *Let $\Gamma_1 \in \mathcal{S}'(\mathbb{R}^d)$, $\operatorname{supp}(\Gamma_1) = \{0\}$, satisfy, for some $N \in \mathbb{N}$, the following conditions:*

- *$|\Gamma_1(\varphi)| \leq c\|\varphi\|_N$, for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$;*
- *for a function $\eta \in \mathcal{D}(\mathbb{R}^d)$ with $\eta(x) = 0$ on $(B_2(0))^c$, and $\eta(x) = 1$ on $B_1(0)$,*

$$\Gamma_1(x^\gamma \eta) = 0,$$

*for every multi-index $\gamma$ with $|\gamma| \leq N$.*

*Then* $\Gamma_1 = 0$.

*Proof of Lemma.* In fact, let $\eta \in \mathcal{D}(\mathbb{R}^d)$ as in the statement and write $\eta_{\varepsilon^{-1}}(x) = \eta(\frac{x}{\varepsilon})$. Then, since $\Gamma_1$ is supported at the origin, $\Gamma_1(\eta_{\varepsilon^{-1}}\varphi) = \Gamma_1(\varphi)$. Moreover, by the same token $\Gamma_1(\eta_{\varepsilon^{-1}}x^\beta) = \Gamma_1(\eta x^\beta) = 0$ for every $|\beta| \leq N$, and hence

$$\Gamma_1(\varphi) = \Gamma\left(\eta_{\varepsilon^{-1}}\left(\varphi(x) - \sum_{|\beta|\leq N} \frac{\partial^\beta\varphi(0)}{\beta!}x^\beta\right)\right).$$

If $R(x) = \varphi(x) - \sum_{|\beta|\leq N} \frac{\partial^\beta\varphi(0)}{\beta!}x^\beta$ is the remainder, then $|R(x)| \leq c|x|^{N+1}$ and $|\partial^\gamma R(x)| \leq c_\gamma |x|^{N+1-|\gamma|}$, when $|\gamma| \leq N$. However $|\partial^\gamma \eta_{\varepsilon^{-1}}(x)| \leq c_\gamma \varepsilon^{-|\gamma|}$ and $\partial^\gamma\eta_{\varepsilon^{-1}} = 0$ if $|x| \geq 2\varepsilon$. Thus by Leibnitz's rule, $\|\eta_{\varepsilon^{-1}}R\|_N \leq c\varepsilon$, and our first assumption gives $|\Gamma_1(\varphi)| \leq c'\varepsilon$, which yields the desired conclusion upon letting $\varepsilon \to 0$. □

Proceeding with the proof of the proposition, we now apply the above lemma to

$$\Gamma_1 = \Gamma - \sum_{|\beta|\leq N} c_\beta \partial^\beta \delta,$$

where $N$ is the index that guarantees the conclusion of Proposition 12, while the $c_\beta$ are chosen so that

$$c_\beta = \frac{(-1)^\beta}{\beta!}\Gamma(x^\beta\eta).$$

Then since $(\partial^\beta\delta)(x^\gamma\psi) = (-1)^{|\beta|}\beta!$ if $\beta = \gamma$, and zero otherwise, we see that $\Gamma_1 = 0$, which proves the proposition. □

The following statement establishes a sufficient condition for locally integrable functions to define tempered distributions.

**Proposition 17.** *Let* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ *such that for some* $N \in \mathbb{N}$,

$$\int_{|x|\leq R} |f(x)|\,\mathrm{d}x = O(R^N), \qquad \text{as } R \to \infty.$$

*Then* $\Lambda_f \in \mathcal{S}'(\mathbb{R}^d)$, *where, for every* $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\Lambda_f(\varphi) = \int_{\mathbb{R}^d} f(x)\varphi(x)\,\mathrm{d}x.$$

*Proof.* By hypothesis, there exist $\widetilde{R} > 0$, $C > 0$ such that, for $R \geq \widetilde{R}$,

$$\int_{|x|\leq R} |f(x)|\,\mathrm{d}x \leq CR^N.$$

Then, for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$
\begin{aligned}
|\Lambda_f(\varphi)| = \left| \int_{\mathbb{R}^d} f(x)\varphi(x)\,\mathrm{d}x \right| &\leq \int_{\mathbb{R}^d} |f(x)\varphi(x)|\,\mathrm{d}x \leq \\
&\leq \int_{|x|\leq \widetilde{R}} |f(x)|\|\varphi\|_\infty + \sum_{i=1}^{\infty} \int_{2^{i-1}\widetilde{R}\leq |x|\leq 2^i\widetilde{R}} |f(x)\varphi(x)|\,\mathrm{d}x \leq \\
&\leq K\|\varphi\|_0 + \sum_{i=1}^{\infty} \int_{2^{i-1}\widetilde{R}\leq |x|\leq 2^i\widetilde{R}} |f(x)| \frac{1}{|x|^{N+1}} |x|^{N+1} |\varphi(x)|\,\mathrm{d}x \leq \\
&\leq K\|\varphi\|_0 + K\sum_{i=1}^{\infty} \frac{1}{(2^{i-1}\widetilde{R})^{N+1}} \|\varphi\|_{N+1} \int_{|x|\leq 2^i\widetilde{R}} |f(x)|\,\mathrm{d}x \leq \\
&\leq K\|\varphi\|_0 + K\sum_{i=1}^{\infty} \frac{1}{(2^{i-1}\widetilde{R})^{N+1}} \|\varphi\|_{N+1} C(2^i\widetilde{R})^N \leq C\|\varphi\|_{N+1} \sum_{i=1}^{\infty} \frac{1}{2^i} \leq \\
&\leq C\|\varphi\|_{N+1}.
\end{aligned}
$$

By Proposition 12, this condition implies the claim. □

One tool we need in our analysis is a rather classical result, which can be found in [4].

**Lemma 18.** *Let $u \in \mathcal{D}'(Y \times I)$ where $Y$ is an open set in $\mathbb{R}^{d-1}$ and $I$ an open interval on $\mathbb{R}$. If $\partial_d u = 0$ then*

$$
u(\phi) = \int u_0(\phi(\cdot, x_d))\,\mathrm{d}x_d = u_0\left( \int \phi(\cdot, x_d)\,\mathrm{d}x_d \right), \qquad \phi \in \mathcal{D}(Y \times I),
$$

*where $u_0 \in \mathcal{D}'(Y)$. Thus $u$ is a distribution $u_0$ in $x' = (x_1, \dots, x_{d-1})$ independent of $x_d$.*

*Proof.* Choose $\psi_0 \in \mathcal{D}(I)$ with $\int_I \psi_0(t)\,\mathrm{d}t = 1$ and define

$$
u_0(\chi) = u(\chi_0),
$$

where, for $\chi \in \mathcal{D}(Y)$,

$$
\chi_0(x) = \chi(x')\psi_0(x_d).
$$

It is obvious that $u_0 \in \mathcal{D}'(Y)$. For $\phi \in \mathcal{D}(Y \times I)$, we define

$$
(I\phi)(x') = \int_I \phi(x', x_d)\,\mathrm{d}x_d.
$$

Thus we have

$$
\phi(x) - (I\phi)(x')\psi_0(x_d) = \partial_d \Phi,
$$

where $\Phi \in \mathcal{C}^\infty(Y \times I)$ is defined by

$$
\Phi(x', x_d) = \int_{-\infty}^{x_d} (\phi(x', s) - (I\phi)(x')\psi_0(s))\,\mathrm{d}s.
$$

Since

$$
\int_{-\infty}^{\infty} (\phi(x', s) - (I\phi)(x')\psi_0(s))\,\mathrm{d}s = 0,
$$

we have $\Phi \in \mathcal{D}(Y \times I)$. Hence

$$u(\phi) = u(\partial_d \Phi + (I\phi)_0) = u((I\phi)_0) = u_0(I\phi) = \int_I u_0(\phi(\cdot, x_d))\, \mathrm{d}x_d,$$

since $u(\partial_d \Phi) = -\partial_d u(\Phi) = 0$. $\square$

In particular, we need the following related result.

**Lemma 19.** *Let $F \in \mathcal{D}'(\Omega)$, where $\Omega$ is an open connected set in $\mathbb{R}^d$. If $\partial_i F = 0$ for every $i \in \{1, \dots, d\}$, then there exists $c \in \mathbb{C}$ such that, for every function $\phi \in \mathcal{D}(\Omega)$,*

$$F(\phi) = c \int_\Omega \phi(x)\, \mathrm{d}x.$$

*Proof.* The proof is a standard application of Lemma 18. Let $Q = I_1 \times \cdots \times I_d \subset \Omega$, where $I_i$ is an open interval in $\mathbb{R}$. Then, by repeatedly applying Lemma 18 for every interval, we obtain that there exists $c_Q \in \mathbb{C}$ such that, for every function $\phi \in \mathcal{D}(Q)$,

$$F(\phi) = c_Q \int_Q \phi(x)\, \mathrm{d}x.$$

Moreover, let $Q' = I'_1 \times \cdots \times I'_d \subset \Omega$, $I'_i$ as above, such that $Q \cap Q' \neq \varnothing$. Then the constant $c_{Q'}$ associated to $Q'$ is equal to $c_Q$. In fact, for a function $\psi \in \mathcal{D}(Q \cap Q')$ such that $\int \psi \neq 0$, we have

$$c_Q \int_Q \psi(x)\, \mathrm{d}x = F(\psi) = c_{Q'} \int_{Q'} \psi(x)\, \mathrm{d}x.$$

Now for a function $\varphi \in \mathcal{D}(\Omega)$ such that $\operatorname{supp}(\varphi) \subset \Omega$, let $\{Q_k\}_k$ be a finite collection of cubes of the form described above such that $\operatorname{supp}(\varphi) \subset \cup_k Q_k$ and for every $i$ there exists $j$ such that $Q_i \cap Q_j \neq \varnothing$. In particular, there exists $c \in \mathbb{C}$ such that $c_{Q_k} = c$ for every $k$. There exists a partition of the unity $\{\eta_k\}_k$ such that $\eta_k \in \mathcal{D}(Q_k)$, $\sum_k \eta_k(x) = 1$ for every $x \in \operatorname{supp}(\varphi)$ (see [5]). We have

$$\begin{aligned} F(\varphi) =& F\left(\sum_k \eta_k \varphi\right) = \sum_k c_k \int_{Q_k} \eta_k(x)\varphi(x)\, \mathrm{d}x = \\ =& c \int_{\cup_k Q_k} \sum_k \eta_k(x)\varphi(x)\, \mathrm{d}x = c \int_\Omega \varphi(x)\, \mathrm{d}x. \end{aligned}$$

$\square$

## 1.2 Properties of $X^n_\alpha$ for arbitrary $n \geq 1$, $\alpha \in \mathbb{C}$

Let $\mathbb{R}^{n\times n}$ be the set of square matrices with $n$ rows and columns and coefficients in $\mathbb{R}$. We consider the bijection

$$\mathbb{R}^{n\times n} \to \mathbb{R}^{n^2},\ V \mapsto v = (v_1, \dots, v_{n^2}),\ v_{i+(j-1)n} = V_{i,j}. \tag{1.3}$$

From now on, we will use the notation $\mathbb{R}^{n\times n}$ for $\mathbb{R}^{n^2}$ with this intended identification. Thus, we let $\mathrm{d}V = \mathrm{d}V_{1,1}\, \mathrm{d}V_{2,1} \dots \mathrm{d}V_{n,n}$, and for $f\colon \mathbb{R}^{n^2} \to \mathbb{R}$ we let $\partial_{i,j} f = \partial_{i+(j-1)n} f$.

Moreover, let $\langle A, B\rangle_{\mathrm{HS}}$ denote the *Hilbert-Schmidt inner product* of the matrices $A, B \in \mathbb{R}^{n\times n}$, namely

$$\langle A, B\rangle_{\mathrm{HS}} := \mathrm{Tr}(A^T B) = \sum_{i=1}^{n}\sum_{j=1}^{n} A^T_{i,j} B_{j,i} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{j,i} B_{j,i}.$$

We note that it is equal to the inner product of the two vectors in $\mathbb{R}^{n^2}$ associated to the matrices under the identification (1.3).

**Lemma 20.** *For $n \geq 1$, $\alpha \in \mathbb{C}$,*

$$X^n_\alpha = X^n_{\alpha,\mathrm{even}} \oplus X^n_{\alpha,\mathrm{odd}}.$$

*Proof.* For $\Lambda \in \mathcal{S}'(\mathbb{R}^{n\times n})$, let $\Lambda^{\mathrm{ref}} \in \mathcal{S}'(\mathbb{R}^{n\times n})$ be defined by

$$\Lambda^{\mathrm{ref}}(\varphi) := \Lambda(\varphi^{\mathrm{ref}}),\ \varphi^{\mathrm{ref}}(V) := \varphi(B^{-1}V),\ B = B^{-1} = \begin{pmatrix} \mathbb{I}_{n-1} & 0 \\ 0 & -1 \end{pmatrix},$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^{n\times n})$. In particular, $\varphi^{\mathrm{ref}} = D^\alpha_B \varphi$ for every $\alpha \in \mathbb{C}$. Then, for $\Lambda \in X^n_\alpha$, $\Lambda \neq 0$, we have

$$\frac{\Lambda + \Lambda^{\mathrm{ref}}}{2} \in X^n_{\alpha,\mathrm{even}},\ \frac{\Lambda - \Lambda^{\mathrm{ref}}}{2} \in X^n_{\alpha,\mathrm{odd}},$$

and at least one of the two is different from 0. For example,

$$\frac{\Lambda + \Lambda^{\mathrm{ref}}}{2}(D^\alpha_B\varphi) = \frac{\Lambda(D^\alpha_B\varphi) + \Lambda^{\mathrm{ref}}(D^\alpha_B\varphi)}{2} = \frac{\Lambda(\varphi^{\mathrm{ref}}) + \Lambda^{\mathrm{ref}}(\varphi^{\mathrm{ref}})}{2} = \frac{\Lambda^{\mathrm{ref}} + \Lambda}{2}(\varphi).$$

Moreover, suppose $\Lambda \in X^n_{\alpha,\mathrm{even}} \cap X^n_{\alpha,\mathrm{odd}}$. Then

$$\Lambda = \Lambda^{\mathrm{ref}} = -\Lambda,$$

where we used first the property of being even and then the one of being odd. This implies $\Lambda = 0$. □

For a fixed $n \geq 1$, we define the differential operator $\det(\partial_{i,j})$ of $n$-th order,

$$\det(\partial_{i,j}) := \det \begin{pmatrix} \partial_{1,1} & \partial_{2,1} & \dots & \partial_{n,1} \\ \partial_{1,2} & \partial_{2,2} & \dots & \partial_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{1,n} & \partial_{2,n} & \dots & \partial_{n,n} \end{pmatrix}. \tag{1.4}$$

We also define the critical variety $M$

$$M := \{V \in \mathbb{R}^{n\times n}\colon\ \det V = 0\} = \bigcup_{k=0}^{n-1} M_k,\ M_k := \{V \in \mathbb{R}^{n\times n}\colon\ \mathrm{rk}(V) = k\},$$

and we divide $\mathbb{R}^{n\times n} \setminus M$ into

$$M_+ := \{V \in \mathbb{R}^{n\times n}\colon\ \det V > 0\},\ M_- := \{V \in \mathbb{R}^{n\times n}\colon\ \det V < 0\}.$$

We observe that $M_+$ and $M_-$ are two orbits of the action of $\mathrm{GL}_n^+(\mathbb{R})$ on $\mathbb{R}^{n\times n}$ given, for $A \in \mathrm{GL}_n^+(\mathbb{R})$, by

$$\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n},\ V \mapsto A^{-1}V. \tag{1.5}$$

Moreover, the action preserves the rank of the matrices in $M$, restricting to

$$M_k \to M_k,\ V \mapsto A^{-1}V.$$

Every $M_k$ is fibrated by the orbits of the action into Grassmanians $\mathrm{Gr}(n-k,n)$. In fact, an element $V \in M_k$ is characterized by $n-k$ and only $n-k$ independent linear conditions that are satisfied by its columns. These conditions are respected by the action, and every matrix satisfying them can be sent to every other matrix satisfying them through some invertible matrix. Therefore, $M_k$ is fibrated by the orbits into

$$M_0 = \{0\},\ M_k = \bigcup_{\substack{n-k \text{ dimensional} \\ \text{subspaces of } \mathbb{R}^n}} (\mathbb{R}^{nk} \setminus \mathbb{R}^{n(k-1)}) = \mathrm{Gr}(n-k,n) \times (\mathbb{R}^{nk} \setminus \mathbb{R}^{n(k-1)}).$$

## 1.3 The effect of the Fourier transform, $\det V$ and $\det(\partial_{i,j})$

We consider the following maps defined on $\mathcal{S}'(\mathbb{R}^{n\times n})$:

- the Fourier Transform in the sense of tempered distributions, namely

$$\widehat{\cdot} : \mathcal{S}'(\mathbb{R}^{n\times n}) \to \mathcal{S}'(\mathbb{R}^{n\times n}),\ \widehat{\Lambda}(\varphi) = \Lambda(\widehat{\varphi});$$

- the multiplication by the polynomial $\det V$,

$$\det V : \mathcal{S}'(\mathbb{R}^{n\times n}) \to \mathcal{S}'(\mathbb{R}^{n\times n}),\ (\det V\Lambda)(\varphi) = \Lambda(\overline{\varphi}),$$

  where, for a function $\varphi \in \mathcal{S}(\mathbb{R}^{n\times n})$,

$$\overline{\varphi}(V) \coloneqq \det V \varphi(V).$$

- the differential operator $\det(\partial_{i,j})$ of $n$-th order defined in (1.4),

$$\det(\partial_{i,j}) : \mathcal{S}'(\mathbb{R}^{n\times n}) \to \mathcal{S}'(\mathbb{R}^{n\times n}),\ (\det(\partial_{i,j})\Lambda)(\varphi) = (-1)^n\Lambda(\det(\partial_{i,j})\varphi).$$

The second and the third operators are parallel in the following sense. For a function $\varphi \in \mathcal{S}(\mathbb{R}^{n\times n})$, under the identification of $\Upsilon \in \mathbb{R}^{n\times n}$ with the point $\upsilon = (\upsilon_1, \dots, \upsilon_{n^2}) \in \mathbb{R}^{n^2}$ in the space of frequencies,

$$(\det(\partial_{i,j})\varphi)\widehat{\ }(\Upsilon) = (2\pi i)^n \det \Upsilon \widehat{\varphi}(\Upsilon),$$

hence

$$\det(\partial_{i,j})\widehat{\Lambda} = (-2\pi i)^n (\det \Upsilon\Lambda)\widehat{\ }. \tag{1.6}$$

The maps defined above are well behaved when restricted to the vector spaces $X^n_{\alpha,\mathrm{even}}$ and $X^n_{\alpha,\mathrm{odd}}$, as explained by the following results.

**Lemma 21.** *For $\alpha \in \mathbb{C}$, the Fourier transform restricts to a bijection $X^n_\alpha \to X^n_{-\alpha-n}$. In particular, we have the bijections*

$$\widehat{\cdot} : X^n_{\alpha,\text{even}} \to X^n_{-\alpha-n,\text{even}},$$
$$\widehat{\cdot} : X^n_{\alpha,\text{odd}} \to X^n_{-\alpha-n,\text{odd}}.$$

*Proof.* For a matrix $A \in \mathrm{GL}_n(\mathbb{R})$ and a function $\varphi \in \mathcal{S}(\mathbb{R}^{n\times n})$,

$$\begin{aligned}(D_A^{-\alpha-n}\varphi)^\frown(\Upsilon) &= \int_{\mathbb{R}^{n\times n}} \frac{1}{|\det A|^{-\alpha}}\varphi(A^{-1}V)e^{-2\pi i\langle V,\Upsilon\rangle_{\mathrm{HS}}}\,\mathrm{d}V = \\ &= \int_{\mathbb{R}^{n\times n}} \frac{1}{|\det A|^{-\alpha}}\varphi(U)e^{-2\pi i\langle AU,\Upsilon\rangle_{\mathrm{HS}}}|\det A|^n\,\mathrm{d}U = \\ &= \frac{1}{|\det A^{-1}|^{\alpha+n}}\int_{\mathbb{R}^{n\times n}} \varphi(U)e^{-2\pi i\left\langle U,A^T\Upsilon\right\rangle_{\mathrm{HS}}}\,\mathrm{d}U = \\ &= \frac{1}{|\det A^{-T}|^{\alpha+n}}\widehat{\varphi}(A^T\Upsilon) = \\ &= \frac{1}{|\det B|^{\alpha+n}}\widehat{\varphi}(B^{-1}\Upsilon) = (D_B^\alpha\widehat{\varphi})(\Upsilon),\end{aligned}$$

where $B = A^{-T}$. Then, for every matrix $A \in \mathrm{GL}_n(\mathbb{R})$ and every function $\varphi \in \mathcal{S}(\mathbb{R}^{n\times n})$,

$$\widehat{\Lambda}(D_A^{-\alpha-n}\varphi) = \Lambda((D_A^{-\alpha-n}\varphi)^\frown) = \Lambda(D_B^\alpha\widehat{\varphi}).$$

Moreover, we observe that the map from $\mathrm{GL}_n(\mathbb{R})$ to itself given by $A \mapsto A^{-T}$ is a bijection, and it respects the connected components, sending $\mathrm{GL}_n^+(\mathbb{R})$ to itself. The very definition of the Fourier transform for tempered distribution and the fact that it is a bijection complete the proof of the equivalence of the conditions $\Lambda \in X^n_{\alpha,\text{even}}$ and $\widehat{\Lambda} \in X^n_{-\alpha-n,\text{even}}$ (resp. the conditions $\Lambda \in X^n_{\alpha,\text{odd}}$ and $\widehat{\Lambda} \in X^n_{-\alpha-n,\text{odd}}$). In fact, for a tempered distribution $\Lambda \in X^n_{\alpha,\text{even}} \cup X^n_{\alpha,\text{odd}}$, let $\sigma_\Lambda \colon \mathrm{GL}_n(\mathbb{R}) \mapsto \{-1,1\}$ be defined by

$$\sigma_\Lambda(A) = (\mathrm{sgn}(\det A))^{\mathrm{par}(\Lambda)},$$

where $\mathrm{par}(\Lambda)$ is the parity of $\Lambda$, namely if $\Lambda$ is even then $\mathrm{par}(\Lambda) = 0$, if $\Lambda$ is odd then $\mathrm{par}(\Lambda) = 1$. It is easy to observe that $\sigma_\Lambda(A) = \sigma_\Lambda(A^{-T})$.

Then, for every matrix $A \in \mathrm{GL}_n(\mathbb{R})$ and every function $\varphi \in \mathcal{S}(\mathbb{R}^{n\times n})$,

$$\begin{aligned}\Lambda(D^\alpha_{A^{-T}}\widehat{\varphi}) = \sigma_\Lambda(A^{-T})\Lambda(\widehat{\varphi}) \quad &\Longrightarrow \quad \widehat{\Lambda}(D_A^{-\alpha-n}\varphi) = \sigma_\Lambda(A)\widehat{\Lambda}(\varphi),\\ \Lambda(D^\alpha_{A^{-T}}\widehat{\varphi}) = \sigma_{\widehat{\Lambda}}(A^{-T})\Lambda(\widehat{\varphi}) \quad &\Longleftarrow \quad \widehat{\Lambda}(D_A^{-\alpha-n}\varphi) = \sigma_{\widehat{\Lambda}}(A)\widehat{\Lambda}(\varphi).\end{aligned}$$

Since the Fourier transform defines a bijection on the space of tempered distributions, we obtain the claim. □

**Lemma 22.** *For $\alpha \in \mathbb{C}$, the multiplication by* $\det V$ *restricts to a map $X^n_\alpha \to X^n_{\alpha+1}$. In particular,*

$$\det V \colon X^n_{\alpha,\text{even}} \to X^n_{\alpha+1,\text{odd}},$$
$$\det V \colon X^n_{\alpha,\text{odd}} \to X^n_{\alpha+1,\text{even}}.$$

**Lemma 23.** *For $\alpha \in \mathbb{C}$, the differential operator $\det(\partial_{i,j})$ restricts to a map $X^n_\alpha \to X^n_{\alpha-1}$. In particular,*

$$\det(\partial_{i,j})\colon X^n_{\alpha,\text{even}} \to X^n_{\alpha-1,\text{odd}},$$
$$\det(\partial_{i,j})\colon X^n_{\alpha,\text{odd}} \to X^n_{\alpha-1,\text{even}}.$$

*Proof.* The proofs of these two lemmata are parallel, as explained in (1.6), therefore, by Lemma 21, we can restrict to the case of the operator $\det V$.

For a matrix $A \in \mathrm{GL}_n(\mathbb{R})$ and a function $\varphi \in \mathcal{S}(\mathbb{R}^{n\times n})$, we denote by $\varphi_{A^{-1}}$ the function

$$\varphi_{A^{-1}}(V) := \varphi(A^{-1}V).$$

Thus we have

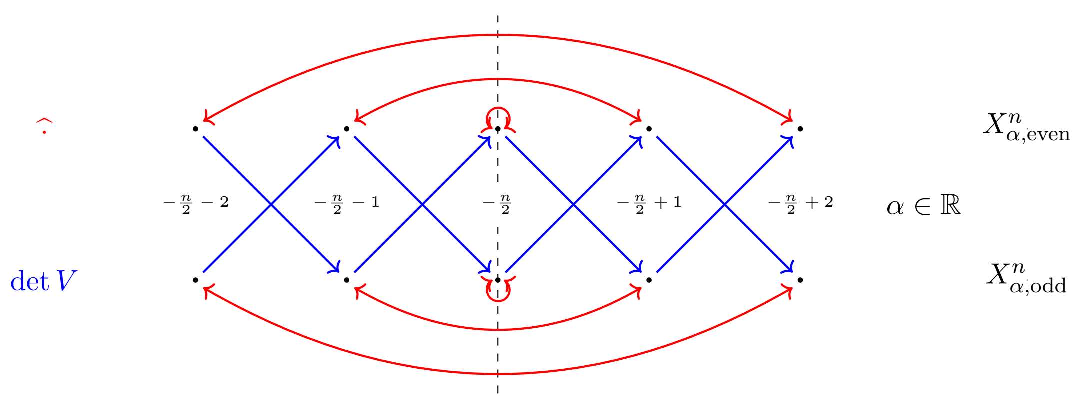
$$\overline{D^{\alpha+1}_A\varphi}(V) = \frac{1}{|\det A|^{\alpha+1+n}}\overline{\varphi_{A^{-1}}}(V) = \frac{1}{|\det A|^{\alpha+1+n}}\det V\varphi(A^{-1}V) =$$
$$= \frac{1}{|\det A|^{\alpha+n}}\operatorname{sgn}(\det A)\overline{\varphi}(A^{-1}V) = \operatorname{sgn}(\det A)D^\alpha_A\overline{\varphi}(V).$$

Then the desired implication follows from the definition of $\det V$ as a map in $\mathcal{S}'(\mathbb{R}^{n\times n})$.

Let $\sigma_\Lambda$ be defined as before. Then, for every matrix $A \in \mathrm{GL}_n(\mathbb{R})$ and every function $\varphi \in \mathcal{S}(\mathbb{R}^{n\times n})$,

$$\Lambda(D^\alpha_A\overline{\varphi}) = \sigma_\Lambda(A)\Lambda(\overline{\varphi}) \qquad \Longrightarrow \qquad (\det V\Lambda)(D^{\alpha+1}_A\varphi) = \sigma_\Lambda(A)\operatorname{sgn}(\det A)(\det V\Lambda)(\varphi).$$

$\square$




In the case of the Fourier transform we have bijections because the map

$$\hat{\cdot} : \mathcal{S}(\mathbb{R}^{n\times n}) \to \mathcal{S}(\mathbb{R}^{n\times n})$$

defines a bijection on the set of Schwartz functions.

In the other two cases we have to be more careful. Due to the parallelism between $\det V$ and $\det(\partial_{i,j})$ showed by (1.6), we can restrict to comment on $\det V$. A direct consequence of the proof is that for every matrix $A \in \mathrm{GL}_n(\mathbb{R})$ and every function $\varphi \in \mathcal{S}(\mathbb{R}^{n\times n})$,

$$\Lambda(D^\alpha_A\overline{\varphi}) = \sigma_{\det V\Lambda}(A)\Lambda(\overline{\varphi}) \;\Leftarrow\; (\det V\Lambda)(D^{\alpha+1}_A\varphi) = \sigma_{\det V\Lambda}(A)\operatorname{sgn}(\det A)(\det V\Lambda)(\varphi).$$

This implies that for $\Lambda \in \mathcal{S}'(\mathbb{R}^{n\times n})$, if $\det V\Lambda \in X^n_{\alpha+1,\text{even}}$ (resp. $\det V\Lambda \in X^n_{\alpha+1,\text{odd}}$), then $\Lambda$ is homogeneous of degree $\alpha$ and even (resp. odd) when applied to the functions of the form $\overline{\varphi}$, for $\varphi \in \mathcal{S}(\mathbb{R}^{n\times n})$. However, the map

$$\det V\colon \mathcal{S}(\mathbb{R}^{n\times n}) \to \mathcal{S}(\mathbb{R}^{n\times n}),\ \varphi \to \overline{\varphi},$$

doesn't define a bijection, in particular it is not surjective. In fact, for every $\varphi \in \mathcal{S}(\mathbb{R}^{n\times n})$, the function $\overline{\varphi}$ is zero on $M$. This may prevent $\det V$ to define a bijection. For example, the Dirac delta function $\delta$ is an element of $X^n_{-n}$, but $\det V\delta = 0$.

We conclude observing that the maps

$$\det V\colon X^n_\alpha \to X^n_{\alpha+1},\ \det(\partial_{i,j})\colon X^n_{\alpha+1} \to X^n_\alpha,$$

are trying to be one the inverse of the other.

## 1.4 Existence results

### 1.4.1 Outside the critical variety $M$

The candidates for elements of $X^n_\alpha$ are $I^n_{\alpha,\text{even}}$, $I^n_{\alpha,\text{odd}}$ defined in (1) in the Introduction.

The following lemma describes the sufficient and necessary condition on $\alpha \in \mathbb{C}$ to have local integrability of the homogeneous functions defining them, namely

$$|\det V|^\alpha,\ \operatorname{sgn}(\det V)|\det V|^\alpha.$$

**Lemma 24.** *$|\det V|^\beta \in L^1_{\text{loc}}(\mathbb{R}^{n\times n})$ if and only if $\operatorname{Re}(\beta) > -1$.*

*Proof.* Let $K \subset \mathbb{R}^{n\times n}$ be a compact set. We want to study the finiteness of

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} |\det(V_{i,1}\ \ldots\ V_{i,n})|^\beta \chi_K(V)\, \mathrm{d}V_{i,1} \ldots \mathrm{d}V_{i,n},$$

where we can assume that $\beta \in \mathbb{R}$. Without loss of generality, let $K = B_{R_1} \times \cdots \times B_{R_n}$ for some $R_i > 0$, where $B_{R_i}$ is the closed ball in $\mathbb{R}^n$ centred in the origin with radius $R_i$.

The subset $M$ has measure zero, so we can restrict to $\mathbb{R}^{n\times n}\backslash M$ to compute the integral, and therefore we can assume

$$\mathbb{R}^n = \operatorname{span}\{V_{i,1}, \ldots, V_{i,n}\}.$$

We change the variables to spherical coordinates for $V_{i,1} \in \mathbb{R}^n$, obtaining

$$\begin{aligned}
&\int_{B_{R_1}\times\cdots\times B_{R_n}} |\det(V_{i,1}\ \ldots\ V_{i,n})|^\beta\, \mathrm{d}V_{i,1} \ldots \mathrm{d}V_{i,n} = \\
&\quad = \int_{B_{R_2}\times\cdots\times B_{R_n}} \int_{B_{R_1}} |\det(V_{i,1}\ \ldots\ V_{i,n})|^\beta\, \mathrm{d}V_{i,1} \ldots \mathrm{d}V_{i,n} \leq \\
&\quad \leq \int_{B_{R_2}\times\cdots\times B_{R_n}} \int_{\mathbb{S}^{n-1}} \int_0^{R_1} |\det(e_{\theta_1}, V_{i,2}, \ldots, V_{i,n})|^\beta r_1^{(n-1)+\beta}\, \mathrm{d}r_1\, \mathrm{d}\sigma(\theta_1)\, \mathrm{d}V_{i,2} \ldots \mathrm{d}V_{i,n} \leq \\
&\quad \leq R_1^{n+\beta} \int_{B_{R_2}\times\cdots\times B_{R_n}} \int_{\mathbb{S}^{n-1}} |\det(e_{\theta_1}, V_{i,2}, \ldots, V_{i,n})|^\beta\, \mathrm{d}\sigma(\theta_1)\, \mathrm{d}V_{i,2} \ldots \mathrm{d}V_{i,n},
\end{aligned}$$

where in the last inequality we used the fact that the function $r_1^{(n-1)+\beta}$ is locally integrable if and only if $\beta > -n$. Moreover, under this condition, we observe that the integral grows polynomially in $R_1$. From now on we suppose to be in the case $\beta > -n$.

We change the variables to spherical coordinates for $V_{i,j+1} \in \mathbb{R}^n$, taking the first $j$ axes in $\operatorname{span}\{e_{\theta_1}, \dots, e_{\theta_j}\}$. An analogous argument provides

$$\int_{B_{R_2}\times\cdots\times B_{R_n}} \int_{\mathbb{S}^{n-1}} |\det(e_{\theta_1}, V_{i,2}, \dots, V_{i,n})|^\beta \, \mathrm{d}\sigma(\theta_1)\, \mathrm{d}V_{i,2} \dots \mathrm{d}V_{i,n} \leq$$
$$\leq \prod_{j=2}^n R_j^{n+\beta} \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} |\det(e_{\theta_1}, \dots, e_{\theta_n})|^\beta \, \mathrm{d}\sigma(\theta_1) \dots \mathrm{d}\sigma(\theta_n).$$

We change the angle variables in this way: for $1 < j \leq n$, let $\theta_j = \widetilde{\theta}_j + \tau_j$, where $\tau_j \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the angle between $e_{\theta_j}$ and its projection onto the $j-1$ dimensional subspace of $\mathbb{R}^n$ spanned by $e_{\theta_1}, \dots, e_{\theta_{j-1}}$. This angle is the only important part in defining the value of the determinant. In fact, the question of local integrability boils down to the study of the finiteness of

$$\int_0^{\frac{\pi}{2}} \cdots \int_0^{\frac{\pi}{2}} \prod_{j=2}^n |\sin \tau_j|^{\beta+n-j} \, \mathrm{d}\tau_2 \dots \mathrm{d}\tau_n.$$

To conclude, we observe that near 0 the integrability of $|\sin \tau_n|^\beta$ is the same of $|\tau_n|^\beta$, since

$$\lim_{\tau\to 0} \frac{\sin \tau}{\tau} = 1.$$

In particular we have local integrability if $\beta > -1$.

To prove the other implication, let $\beta \leq -1$. Let $K$ be the closure of $M_+ \cap (B_2(0) \setminus B_1(0))$. It is compact and the integral

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} |\det(V_{i,1} \ \dots \ V_{i,n})|^\beta \chi_K(V) \, \mathrm{d}V_{i,1} \dots \mathrm{d}V_{i,n},$$

is bounded from below by

$$c \int_0^{\frac{\pi}{2}} \cdots \int_0^{\frac{\pi}{2}} \prod_{j=2}^n (\sin \tau_j)^{\beta+n-j} \, \mathrm{d}\tau_2 \dots \mathrm{d}\tau_n,$$

where $c > 0$. To end the proof, we observe that $(\sin \tau_n)^\beta$ is not integrable near 0. $\square$

We recover the existence of an odd tempered distribution in $X^n_{-1}$ by means of a principal value integral, exploiting the cancellation property.

**Lemma 25.** *The definition*

$$\Lambda(\varphi) \coloneqq \mathrm{p.\,v.} \int_{\mathbb{R}^{n\times n}} \frac{\varphi(V)}{\det V} \, \mathrm{d}V = \lim_{\varepsilon\to 0} \int_{|\det V| > \varepsilon} \frac{\varphi(V)}{\det V} \, \mathrm{d}V,$$

*gives a nonzero element of $X^n_{-1,\mathrm{odd}}$.*

*Proof.* In the proof we follow the idea of [8]. For $V \in \mathbb{R}^{n\times n} \setminus M$, let $\widetilde{V} \in \mathbb{R}^{n\times n}$ be defined by

$$\begin{aligned} \widetilde{V}_{i,j} &= V_{i,j}, && \text{for } j \neq n, \\ \widetilde{V}_{i,n} &= 2\sum_{j=1}^{n-1} \frac{V_{i,j}\cdot V_{i,n}}{|V_{i,j}|^2} V_{i,j} - V_{i,n}. \end{aligned}$$

In particular, we note that

$$V_{i,n} - \sum_{j=1}^{n-1} \frac{V_{i,j}\cdot V_{i,n}}{|V_{i,j}|^2} V_{i,j}$$

is the component of $V_{i,n}$ perpendicular to the hyperplane spanned by $\{V_{i,1}, \dots, V_{i,n-1}\}$. Then

$$\det \widetilde{V} = -\det V,\ |V_{i,n}| = |\widetilde{V}_{i,n}|,\ \left(\left\langle V - \widetilde{V}, V - \widetilde{V}\right\rangle_{\mathrm{HS}}\right)^{\frac{1}{2}} = 2\frac{|\det V|}{|V_{i,1}\wedge\cdots\wedge V_{i,n-1}|}.$$

In the case $n = 1$, for $V = x$, we define $\widetilde{V} = -x$ and $|V_{i,1}\wedge\cdots\wedge V_{i,n-1}| = 1$. We observe that

$$\begin{aligned} |\Lambda(\varphi)| &= \left|\lim_{\varepsilon\to 0}\int_{|\det V|>\varepsilon} \frac{1}{\det V}\varphi(V)\,\mathrm{d}V\right| \leq \lim_{\varepsilon\to 0}\int_{\det V>\varepsilon}\left|\frac{\varphi(V)-\varphi(\widetilde{V})}{\det V}\right|\mathrm{d}V = \\ &= \lim_{\varepsilon\to 0}\int_{\substack{\det V>\varepsilon\\ |V_{i_n}|\leq 1}}\left|\frac{\varphi(V)-\varphi(\widetilde{V})}{\det V}\right|\mathrm{d}V + \lim_{\varepsilon\to 0}\int_{\substack{\det V>\varepsilon\\ |V_{i,n}|>1}}\left|\frac{|V_{i,n}|^{n+1}\varphi(V) - |\widetilde{V}_{i,n}|^{n+1}\varphi(\widetilde{V})}{|V_{i,n}|^{n+1}\det V}\right|\mathrm{d}V \leq \\ &\leq 2\lim_{\varepsilon\to 0}\int_{\substack{\det V>\varepsilon\\ |V_{i,n}|\leq 1}} \frac{|\nabla\varphi(V_{i,1}\dots V_{i,n-1}W_{i,n})|}{|V_{i,1}\wedge\cdots\wedge V_{i,n-1}|}\,\mathrm{d}V + \\ &\qquad + 2\lim_{\varepsilon\to 0}\int_{\substack{\det V>\varepsilon\\ |V_{i,n}|>1}} \frac{|\nabla(|V_{i,n}|^{n+1}\varphi)(V_{i,1}\dots V_{i,n-1}W_{i,n})|}{|V_{i,n}|^{n+1}|V_{i,1}\wedge\cdots\wedge V_{i,n-1}|}\,\mathrm{d}V = \\ &= 2\int_{\substack{\det V>0\\ |V_{i,n}|\leq 1}} \frac{|\nabla\varphi(V_{i,1}\dots V_{i,n-1}W_{i,n})|}{|V_{i,1}\wedge\cdots\wedge V_{i,n-1}|}\,\mathrm{d}V + \\ &\qquad + 2\int_{\substack{\det V>0\\ |V_{i,n}|>1}} \frac{|\nabla(|V_{i,n}|^{n+1}\varphi)(V_{i,1}\dots V_{i,n-1}W_{i,n})|}{|V_{i,n}|^{n+1}|V_{i,1}\wedge\cdots\wedge V_{i,n-1}|}\,\mathrm{d}V. \end{aligned}$$

To bound these integrals we observe that

- $|V_{i,n}|^{-n-1}$ is integrable over $(B_1(0))^c$;
- $|V_{i,1}\wedge\cdots\wedge V_{i,n-1}|^{-1}$ is integrable over
$$\{(V_{i,1}\dots V_{i,n-1}) \in \mathbb{R}^{n(n-1)} \colon |V_{i,1}\wedge\cdots\wedge V_{i,n-1}| \leq 1\}.$$
In the case $n = 2$ then $|V_{i,1}\wedge\cdots\wedge V_{i,n-1}|^{-1} = |x|^{-1}$, $x \in \mathbb{R}^2$, which is integrable over $\{x \in \mathbb{R}^2 \colon |x| \leq 1\}$. For $n \geq 3$ the same change of variables we used in Lemma 24 leads to consider the finiteness of
$$\int_0^{\frac{\pi}{2}} \dots \int_0^{\frac{\pi}{2}} \prod_{j=2}^{n-1} |\sin\tau_j|^{-1+n-j}\,\mathrm{d}\tau_2\dots\mathrm{d}\tau_{n-1},$$
which is integrable, since $j$ is at most $n-1$.

We conclude by (1.2), the well known bounds for the integrals of Schwartz functions in terms of the norms in (1.1), and Proposition 12.

Once proven that the limit is well-defined, the claim about being in $X^n_{-1,\mathrm{odd}}$ is trivial.

$$\begin{aligned}\Lambda(D_A^{-1}\varphi) &= \lim_{\varepsilon\to 0}\int_{|\det V|>\varepsilon}\frac{1}{|\det A|^{n-1}}\frac{1}{\det V}\varphi(A^{-1}V)\,\mathrm{d}V = \\ &= \lim_{\varepsilon\to 0}\int_{|\det U|>\frac{\varepsilon}{|\det A|}}\frac{\operatorname{sgn}(\det A)}{\det U}\varphi(U)\,\mathrm{d}U = \lim_{\delta\to 0}\int_{|\det U|>\delta}\frac{\operatorname{sgn}(\det A)}{\det U}\varphi(U)\,\mathrm{d}U = \\ &= \operatorname{sgn}(\det A)\Lambda(\varphi).\end{aligned}$$

□

**Lemma 26.** *For* $\mathrm{Re}(\alpha) = -1$, $\alpha \neq -1$, $X^2_{\alpha,\mathrm{even}}$ *and* $X^2_{\alpha,\mathrm{odd}}$ *have at least dimension 1.*

*Proof.* We have $I^2_{\alpha+1,\mathrm{even}} \in X^2_{\alpha+1,\mathrm{even}}$, then $\det(\partial_{i,j})I^2_{\alpha+1,\mathrm{even}} \in X^2_{\alpha,\mathrm{odd}}$. To conclude we have to show this distribution is different from zero. Let $\eta \in \mathcal{D}(\mathbb{R}^{2\times 2}\setminus M)$ such that $\eta \geq 0$, $\eta \neq 0$, and define

$$\widetilde{\eta}(V) = |\det V|^{-\alpha}\eta(V).$$

Then, for $\alpha \neq -1$,

$$\begin{aligned}(\det(\partial_{i,j})I^2_{\alpha+1,\mathrm{even}})(\widetilde{\eta}) &= \int_{\mathbb{R}^{2\times 2}\setminus M}|\det V|^{\alpha+1}(\det(\partial_{i,j})\widetilde{\eta})(V)\,\mathrm{d}V = \\ &= \int_{\mathbb{R}^{2\times 2}\setminus M}(\det(\partial_{i,j})|\det V|^{\alpha+1})\widetilde{\eta}(V)\,\mathrm{d}V = \\ &= \int_{\mathbb{R}^{2\times 2}\setminus M}(\alpha+1)(\alpha+2)|\det V|^{\alpha}|\det V|^{-\alpha}\eta(V)\,\mathrm{d}V = \\ &= (\alpha+1)(\alpha+2)\int_{\mathbb{R}^{2\times 2}\setminus M}\eta(V)\,\mathrm{d}V \neq 0,\end{aligned}$$

since

$$\begin{aligned}\det(\partial_{i,j})((\det V)^{\beta}) &= (\partial_{1,1}\partial_{2,2} - \partial_{1,2}\partial_{2,1})((V_{1,1}V_{2,2} - V_{1,2}V_{2,1})^{\beta}) = \\ &= \partial_{1,1}(\beta((V_{1,1}V_{2,2} - V_{1,2}V_{2,1})^{\beta-1}V_{2,2})) + \partial_{1,2}(\beta((V_{1,1}V_{2,2} - V_{1,2}V_{2,1})^{\beta-1}V_{1,2})) = \\ &= 2\beta(\det V)^{\beta-1} + \beta(\beta-1)(\det V)^{\beta-1} = \beta(\beta+1)(\det V)^{\beta-1}.\end{aligned}$$

An analogous argument holds for $X^2_{\alpha,\mathrm{even}}$. □

**Remark 27.** *We conjecture a more general result.*

**Conjecture 28.** *Let* $n \geq 2$. *For* $-n+1 \leq \mathrm{Re}(\alpha) \leq -1$, $\alpha \notin \mathbb{Z}$, $X^2_{\alpha,\mathrm{even}}$ *and* $X^2_{\alpha,\mathrm{odd}}$ *have at least dimension 1.*

*Upon proving the following identity on* $\mathbb{R}^{n\times n}\setminus M$

$$\det(\partial_{i,j})(\det V)^{\alpha} = \prod_{j=0}^{n-1}(\alpha+j)(\det V)^{\alpha-1},$$

*an analogous argument yields the conjectured claim.*

### 1.4.2 Inside the critical variety $M$

We recover the existence of even tempered distributions in $X_\alpha^n$ for $\alpha \in \mathbb{Z}$, $-n+1 \leq \alpha \leq -1$ by means of integrals of the restriction of $\varphi$ on $M$ over $nk$-dimensional subspaces of $R^{n\times n}$ contained in $M_k$.

**Lemma 29.** *For $n \geq 1$, for $0 \leq k \leq n-1$, $R \in \mathrm{GL}_n(\mathbb{R})$, the definition*

$$\Gamma_{R,k}(\varphi) \coloneqq \int_{\mathbb{R}^{n\times k}} \varphi((V_{i,1} \ \dots \ V_{i,k} \ 0)R)\, \mathrm{d}V_{i,1} \dots \mathrm{d}V_{i,k},$$

*gives a nonzero element of $X^n_{-n+k,\mathrm{even}}$.*

*Proof.* Without loss of generality we can assume $R = \mathbb{I}_n$. Then, for every $\mathbb{R}^n$, we split it into $B_1(0) \subset \mathbb{R}^n$ and $(B_1(0))^c \subset \mathbb{R}^n$. For the integral over the subset

$$K = B_1^1(0) \times \dots \times B_1^j(0) \times (B_1^{j+1}(0))^c \times \dots (B_1^k(0))^c$$

we have the bound

$$\begin{aligned}
&\int_K |\varphi(V_{i,1} \dots V_{i,k}\ 0)|\, \mathrm{d}V_{i,1} \dots \mathrm{d}V_{i,k} = \\
&\quad = \int_K |\varphi(V_{i,1} \dots V_{i,k}\ 0)| |V_{i,j+1}|^{n+1} \dots |V_{i,k}|^{n+1} \frac{1}{|V_{i,j+1}|^{n+1} \dots |V_{i,k}|^{n+1}}\, \mathrm{d}V_{i,1} \dots \mathrm{d}V_{i,k} \leq \\
&\quad \leq C\|\varphi\|_N \left( \int_{B_1^1(0)} \mathrm{d}V_{i,1} \right)^j \left( \int_{(B_1^{j+1}(0))^c} |V_{i,j+1}|^{-n-1}\, \mathrm{d}V_{i,j+1} \right)^{k-j},
\end{aligned}$$

for some $N \in \mathbb{N}$ big enough by (1.2). Then $|\Gamma_{R,k}(\varphi)|$ is bounded by $\|\varphi\|_N$ and is well-defined as an element of $\mathcal{S}'(\mathbb{R}^{n\times n})$ by Proposition 12.

The claim about being in $X^n_{-n+k,\mathrm{even}}$ is trivial. In fact,

$$\begin{aligned}
\Gamma_{R,k}(D_A^\alpha \varphi) &= \int_{\mathbb{R}^{n\times k}} \frac{1}{|\det A|^k} \varphi(A^{-1}(V_{i,1} \dots V_{i,k}\ 0)R)\, \mathrm{d}V_{i,1} \dots \mathrm{d}V_{i,k} = \\
&= \int_{\mathbb{R}^{n\times k}} \varphi((U_{i,1} \dots U_{i,k}\ 0)R)\, \mathrm{d}U_{i,1} \dots \mathrm{d}U_{i,k} = \Gamma_{R,k}(\varphi),
\end{aligned}$$

where, for $j \in \{1, \dots, k\}$, we changed the variables according to $U_{i,j} = A^{-1}V_{i,j}$. $\square$

For $n = 1$, the corresponding tempered distribution in the case $k = 0$ is $\delta$, the Dirac delta function, which gives rise to a one dimensional space. On the other hand, for $n \geq 2$, $1 \leq k \leq n-1$, the space $X^n_{\alpha,\mathrm{even}}$ is infinite dimensional, due to the fact that $M_k$ contains infinite distinct copies of $\mathbb{R}^{nk} \setminus \mathbb{R}^{n(k-1)}$.

We are ready to prove the existence result.

**Theorem 30.** *For $\mathrm{Re}(\alpha) \notin [-n+1, -1]$, $X^n_{\alpha,\mathrm{even}}$ and $X^n_{\alpha,\mathrm{odd}}$ have at least dimension 1. For $\alpha \in \{-n+1, \dots, -1\}$, $X^n_{\alpha,\mathrm{even}}$ has at least dimension 1. For $\alpha \in \{-n+1, -1\}$, $X^n_{\alpha,\mathrm{odd}}$ has at least dimension 1. Moreover, for $n = 2$, for $\mathrm{Re}(\alpha) = -1$, $\alpha \neq -1$, $X^2_{\alpha,\mathrm{even}}$ and $X^2_{\alpha,\mathrm{odd}}$ have at least dimension 1.*

*Proof.* By Lemma 24, the functions $|\det V|^{\alpha}$, $\operatorname{sgn}(\det V)|\det V|^{\alpha}$, are locally integrable and satisfy the property $\int_{B_R(0)}|f| = O(R^N)$ for $\mathrm{Re}(\alpha) > -1$. Therefore, by Proposition 17, they define nonzero elements of $X^n_{\alpha,\mathrm{even}}$, $X^n_{\alpha,\mathrm{odd}}$ for $\mathrm{Re}(\alpha) > -1$. By Lemma 21, we recover the same result for $\mathrm{Re}(\alpha) < -n+1$.
By Lemma 25, we define a nonzero element of $X^n_{-1,\mathrm{odd}}$, and by Lemma 21 we recover the same result for $X^n_{-n+1,\mathrm{odd}}$.
By Lemma 29, we define nonzero elements of $X^n_{\alpha,\mathrm{even}}$ for $\alpha \in \mathbb{Z}$, $-n+1 \leq \alpha \leq -1$.
Moreover, for $n = 2$, by Lemma 28, we define nonzero elements of $X^2_{\alpha,\mathrm{even}}$, $X^2_{\alpha,\mathrm{odd}}$ for $\mathrm{Re}(\alpha) = -1$, $\alpha \neq -1$. □

## 1.5 Uniqueness result

We say that, for a function $f \in \mathcal{C}^\infty(\mathbb{R}^{n\times n} \setminus M)$, the tempered distribution $\Lambda \in \mathcal{S}'(\mathbb{R}^{n\times n})$ agrees with $f$ away from $M$ if for every function $\phi \in \mathcal{S}(\mathbb{R}^{n\times n})$, $\mathrm{supp}(\phi) \cap M = \varnothing$, we have

$$\Lambda(\phi) = \int_{\mathbb{R}^{n\times n}} f(V)\phi(V)\,\mathrm{d}V.$$

The behaviour of $\Lambda \in X^n_\alpha$ on $M_+$ and $M_-$ is established by the following result.

**Proposition 31.** *Let $\alpha \in \mathbb{C}$, $\Lambda \in X^n_\alpha$. Then, away from the critical variety $M$, $\Lambda$ agrees with a function*

$$c_+ \det V^\alpha \qquad \text{on } M_+, \qquad c_-(-\det V)^\alpha \qquad \text{on } M_-, \tag{1.7}$$

*where $c_+, c_- \in \mathbb{C}$. If $\Lambda$ is even (resp. odd), then $c_+ = c_-$ (resp. $c_+ = -c_-$)*

*Proof.* We note that $\Lambda$ defines a distribution $\widetilde{\Lambda} \in \mathcal{D}'(\mathbb{R}^{n\times n} \setminus M)$ as shown in Proposition 11. Moreover, the function $|\det V|^{-\alpha}$ is well-defined and smooth on $\mathbb{R}^{n\times n} \setminus M$. Thus

$$\widetilde{\Gamma} := |\det V|^{-\alpha}\widetilde{\Lambda} \in \mathcal{D}'(\mathbb{R}^{n\times n} \setminus M).$$

The $\alpha$-homogeneity of $\Lambda$ implies a 0-homogeneity for $\widetilde{\Gamma}$. In fact, for every matrix $A \in \mathrm{GL}^+_n(\mathbb{R})$, for every function $\phi \in \mathcal{D}(\mathbb{R}^{n\times n} \setminus M)$,

$$\begin{aligned}
\widetilde{\Gamma}(D^0_A\phi) &= \left(|\det V|^{-\alpha}\widetilde{\Lambda}\right)(D^0_A\phi) = \frac{1}{|\det A|^n}\widetilde{\Lambda}\left(|\det V|^{-\alpha}\phi_{A^{-1}}\right) = \\
&= \Lambda\left(\frac{1}{(\det A)^{\alpha+n}}|\det A^{-1}V|^{-\alpha}\phi_{A^{-1}}\right) = \Lambda\left(D^\alpha_A(|\det V|^{-\alpha}\phi)\right) = \\
&= \Lambda(|\det V|^{-\alpha}\phi) = \widetilde{\Lambda}(|\det V|^{-\alpha}\phi) = (|\det V|^{-\alpha}\widetilde{\Lambda})(\phi) = \widetilde{\Gamma}(\phi).
\end{aligned}$$

Therefore for every matrix $A \in \mathrm{GL}^+_n(\mathbb{R})$ and every function $\phi \in \mathcal{D}(\mathbb{R}^{n\times n} \setminus M)$,

$$\widetilde{\Gamma}(D^0_A\phi) = \widetilde{\Gamma}(\phi).$$

We consider the homogeneity conditions for the matrices $A$ such that $A^{-1} = \mathbb{I}_n + \varepsilon E^{i,j}$, where, for $i, j \in \{1, \ldots, n\}$, $E^{i,j}$ is the matrix defined by

$$(E^{i,j})_{l,m} = \delta_{i,l}\delta_{j,m},$$

where $\delta_{i,l}$ is the Kronecker delta of the couple $(i,l)$. We have two cases for $A^{-1} = \mathbb{I}_n + \varepsilon E^{i,j}$:

- $i = j$, then $A^{-1} = \mathbb{I}_n + \varepsilon E^{i,i}$. The homogeneity condition reads

$$\widetilde{\Gamma}\left((1+\varepsilon)^n \phi_{A^{-1}} - \phi\right) = 0,$$

for every function $\phi \in \mathcal{D}(\mathbb{R}^{2\times 2} \setminus M)$. We divide by $\varepsilon$ and we take the limit as $\varepsilon$ goes to 0, obtaining

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \widetilde{\Gamma}\left((1+\varepsilon)^n \phi_{A^{-1}} - \phi\right) = 0.$$

In $\mathcal{D}(\mathbb{R}^{n\times n} \setminus M)$, an easy computation shows that

$$\lim_{\varepsilon \to 0} \frac{(1+\varepsilon)^n \phi_{A^{-1}} - \phi}{\varepsilon} \to n\phi + \sum_{k=1}^{n} V_{i,k} \partial_{i,k} \phi.$$

Since $\widetilde{\Gamma}$ is continuous in $\mathcal{D}(\mathbb{R}^{n\times n} \setminus M)$, then, for every $\phi \in \mathcal{D}(\mathbb{R}^{n\times n} \setminus M)$,

$$\widetilde{\Gamma}\left(n\phi + \sum_{k=1}^{n} V_{i,k} \partial_{i,k} \phi\right) = 0.$$

This yields the following equation in the sense of distributions on $\mathbb{R}^{n\times n} \setminus M$,

$$\sum_{k=1}^{n} V_{i,k} \partial_{i,k} \widetilde{\Gamma} = 0;$$

- $i \neq j$, then $A^{-1} = \mathbb{I}_n + \varepsilon E^{i,j}$. A similar argument yields the following equation in the sense of distributions on $\mathbb{R}^{n\times n} \setminus M$,

$$\sum_{k=1}^{n} V_{j,k} \partial_{i,k} \widetilde{\Gamma} = 0.$$

Therefore, we obtain the system of equations in the sense of distributions on $\mathbb{R}^{n\times n} \setminus M$,

$$\sum_{k=1}^{n} V_{j,k} \partial_{i,k} \widetilde{\Gamma} = 0, \qquad \text{for every } (i,j) \in \{1, \ldots, n\}^2.$$

Suppose to order both the variables $\partial_{l,m}\widetilde{\Gamma}$ and the equations associated to $(i,j)$ in the following way:

$$(1,1),\ (1,2),\ (1,3),\ \ldots,\ (1,n),\ (2,1),\ \ldots,\ (n,n).$$

Then the matrix of the coefficients is given by

$$\begin{pmatrix} V & 0 & \ldots & 0 \\ 0 & V & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & V \end{pmatrix}, \qquad \text{on } \mathbb{R}^{n\times n} \setminus M.$$

Now let co(V) denote the matrix of cofactors of $V$. Its entries are smooth functions on $\mathbb{R}^{n\times n}$. Therefore, the matrix multiplication

$$\begin{pmatrix} \mathrm{co(V)} & 0 & \dots & 0 \\ 0 & \mathrm{co(V)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathrm{co(V)} \end{pmatrix} \begin{pmatrix} V & 0 & \dots & 0 \\ 0 & V & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & V \end{pmatrix} = (\det V)\mathbb{I}_{n^2},$$

stands for algebraic operations between the equations of the system. We finally observe that the function $\det V$ is always different from 0 on $\mathbb{R}^{n\times n}\setminus M$, so that we obtain the system

$$\partial_{i,j}\widetilde{\Gamma} = 0, \qquad \text{for every } (i,j)\in\{1,\dots,n\}^2.$$

In the two connected open sets $M_+$, $M_-$ we can apply Lemma 19 to obtain that

$$\widetilde{\Gamma} = c_+ \qquad \text{on } M_+, \qquad \widetilde{\Gamma} = c_- \qquad \text{on } M_-,$$

where $c_+, c_- \in \mathbb{C}$. Therefore, away from the critical variety $M$, the tempered distribution $\Lambda$ agrees with a function

$$c_+(\det(x\ y))^\alpha \qquad \text{on } M_+, \qquad c_-(-\det(x\ y))^\alpha \qquad \text{on } M_-.$$

The even/odd condition for a matrix with determinant $-1$, e.g.

$$A = A^{-1} = \begin{pmatrix} \mathbb{I}_{n-1} & 0 \\ 0 & -1 \end{pmatrix},$$

provides us the relation between the two constants. Assume that $\phi \in \mathcal{S}(\mathbb{R}^{n\times n})$ such that $\phi \geq 0, \phi \neq 0$, $\mathrm{supp}(\phi) \subset M_+$. Let $\sigma = 1$ if $\Lambda$ is even and $\sigma = -1$ if it is odd. Then,

$$\int_{M_+} c_+(\det V)^\alpha \phi(V)\,\mathrm{d}V = \Lambda(\phi) = \sigma\Lambda(D_A^\alpha\phi) = \sigma\Lambda(\phi_{A^{-1}}) =$$
$$= \sigma\int_{M_-} c_-(-\det V)^\alpha\phi(A^{-1}V)\,\mathrm{d}V = \sigma\int_{M_+} c_-(\det U)^\alpha\phi(U)\,\mathrm{d}U,$$

implying $c_+ = \sigma c_-$. □

# Chapter 2

# Classification theorems for $n = 1, 2$

For $n = 1, 2$, Theorem 30 gives an existence result for elements of $X^n_{\alpha,\text{even}}$, $X^n_{\alpha,\text{odd}}$ for every $\alpha \in \mathbb{C}$. The problem of classification is strictly related to the study of tempered distributions $\Gamma \in X^n_\alpha$ supported on the critical variety $M$.

For example, for $\text{Re}(\alpha) > -1$, let $\Lambda \in X^n_{\alpha,\text{even}}$: by Proposition 31, there exists $c \in \mathbb{C}$ such that the tempered distribution

$$\Gamma \coloneqq \Lambda - cI^n_{\alpha,\text{even}} \in X^n_{\alpha,\text{even}}$$

is supported on the critical variety $M$. In order to prove that $X^n_{\alpha,\text{even}}$ has dimension 1, it is enough to show that $\Gamma = 0$.

An analogous argument holds for $X^2_{\alpha,\text{odd}}$.

## 2.1 The case $n = 1$

To prepare the proof of the classification result we study the case of $\Gamma \in X^1_\alpha$ supported at the origin. In particular, we have the following result.

**Lemma 32.** *Let $\alpha \in \mathbb{C}$, $\Gamma \in X^1_\alpha$ such that* $\text{supp}(\Gamma) = \{0\}$. *Then $\alpha = -1 - k$, $k \in \mathbb{N}$ and $\Gamma = c\partial^k\delta$, where $c \in \mathbb{C}$.*

*Proof.* By Proposition 15, there exist $N \in \mathbb{N}$, $\{c_i\}_{0 \le i \le N} \in \mathbb{C}$, such that

$$\Gamma = \sum_{i=0}^{n} c_i \partial^i \delta.$$

Therefore, the Fourier transform $\widehat{\Gamma}$, which by Lemma 21 belongs to $X^1_{-\alpha-1}$, is given by a polynomial function with natural exponents. By Proposition 31 there exists $k \in \mathbb{N}$ such that $-\alpha - 1 = k$, and $\widehat{\Gamma} = c'x^k$, where $c' \in \mathbb{C}$. Hence $\Gamma = c\partial^k\delta$. □

We are ready to prove the classification theorem for $n = 1$.

*Proof of Thm. 3.* By Lemma 21, without loss of generality we can assume $\text{Re}(\alpha) \ge -\frac{1}{2}$.

By Theorem 30, $X^1_{\alpha,\text{even}}$ and $X^1_{\alpha,\text{odd}}$ have at least dimension 1, in particular $I^1_{\alpha,\text{even}} \in X^1_{\alpha,\text{even}}$, $I^1_{\alpha,\text{odd}} \in X^1_{\alpha,\text{odd}}$.

Now suppose to have $\Lambda \in X^1_{\alpha,\text{even}}$. By Proposition 31, there exists $c \in \mathbb{C}$ such that

$$\Gamma = \Lambda - cI^1_{\alpha,\text{even}} \in X^1_{\alpha,\text{even}},\ \operatorname{supp}(\Gamma) \subset \{0\}.$$

By Lemma 32, we conclude $\Gamma = 0$. Therefore $X^1_{\alpha,\text{even}}$ has dimension 1.

An analogous argument prove the claim for $X^1_{\alpha,\text{odd}}$. □

*Proof of Lemma 4.* To compute the constant in the first case, we choose the function

$$\varphi(x) = e^{-\pi x^2},$$

for which

$$\widehat{\varphi}(\xi) = e^{-\pi \xi^2}.$$

To compute the constant in the second case, we choose the function

$$\phi(x) = xe^{-\pi x^2},$$

for which

$$\widehat{\phi}(\xi) = -i\xi e^{-\pi \xi^2}.$$

We observe that for $\beta \in \mathbb{C}$, $\operatorname{Re}(\beta) > -1$, under the change of variable $x \geq 0$ into $\sqrt{\frac{t}{\pi}}$,

$$\int_{\mathbb{R}} |x|^\beta e^{-\pi x^2}\,\mathrm{d}x = 2\int_0^\infty x^\beta e^{-\pi x^2}\,\mathrm{d}x = \pi^{-\frac{\beta+1}{2}} \int_0^\infty t^{\frac{\beta-1}{2}} e^{-t}\,\mathrm{d}x = \pi^{-\frac{\beta+1}{2}} \Gamma\left(\frac{\beta+1}{2}\right).$$

Then, for $-1 < \operatorname{Re}(\alpha) < 0$, in the first case we have

$$\int_{\mathbb{R}} |\xi|^{-\alpha-1} e^{-\pi\xi^2}\,\mathrm{d}\xi = \pi^{\frac{\alpha}{2}}\ \Gamma\left(-\frac{\alpha}{2}\right),$$
$$\int_{\mathbb{R}} |x|^{\alpha} e^{-\pi x^2}\,\mathrm{d}x = \pi^{-\frac{\alpha+1}{2}}\ \Gamma\left(\frac{\alpha+1}{2}\right).$$

In the second case we have

$$\int_{\mathbb{R}} \operatorname{sgn}(\xi)|\xi|^{-\alpha-1}(-i)\xi e^{-\pi\xi^2}\,\mathrm{d}\xi = -i\pi^{\frac{\alpha-1}{2}}\ \Gamma\left(-\frac{\alpha-1}{2}\right),$$
$$\int_{\mathbb{R}} \operatorname{sgn}(x)|x|^{\alpha} x e^{-\pi x^2}\,\mathrm{d}x = \pi^{-\frac{\alpha+2}{2}}\ \Gamma\left(\frac{\alpha+2}{2}\right).$$

The functions defined on the two open half planes are holomorphic because:

- the exponential function is holomorphic on $\mathbb{C}$;
- the function $\Gamma$ is holomorphic and nonzero on $\{\beta \in \mathbb{C}\colon \operatorname{Re}(\beta) > 0\}$;
- for a fixed function $\varphi \in \mathcal{S}(\mathbb{R})$, the function

  $$f\colon \{\operatorname{Re}(\alpha) < 0\} \to \mathbb{C},\ f(\alpha) := \int_{\mathbb{R}} |\xi|^{-\alpha-1}\widehat{\varphi}(\xi)\,\mathrm{d}\xi,$$

  is continuous and complex-differentiable. Let

  $$g\colon \{\operatorname{Re}(\alpha) < 0\} \to \mathbb{C},\ g(\alpha) := \int_{\mathbb{R}} |\xi|^{-\alpha-1} \lg(|\xi|^{-1})\widehat{\varphi}(\xi)\,\mathrm{d}\xi.$$

For every $\alpha$ in the domain, $g(\alpha)$ is finite. In fact, for $\alpha \in \{\mathrm{Re}(\alpha) < 0\}$ there exists $\varepsilon > 0$ such that $\alpha+\varepsilon \in \{\mathrm{Re}(\alpha) < 0\}$. Then, $|\xi|^{\varepsilon} \lg(|\xi|^{-1})$ is bounded when $|\det \xi| \leq 1$ and $|\xi|^{\varepsilon} \lg(|\xi|) \leq C|\xi|^{2\varepsilon}$ when $|\xi| \geq 1$. To conclude we observe that

$$f(\alpha) - f(\alpha_0) = g(\alpha_0)(\alpha - \alpha_0) + o(|\alpha - \alpha_0|),$$

by Lebesgue Dominated Convergence Theorem.
The same argument can be used to prove holomorphicity of the function

$$h\colon \{\mathrm{Re}(\alpha) > -2\} \to \mathbb{C},\ h(\alpha) := \int_{\mathbb{R}} \mathrm{sgn}(x)|x|^{\alpha}\varphi(x)\,\mathrm{d}x.$$

The two functions coincide on the intersection, therefore they define a holomorphic function on $\mathbb{C}$. An analogous argument can be used for the other claim about holomorphicity. $\square$

## 2.2 The case $n = 2$

Our strategy is the following:

- in Lemma 33 we characterize the elements of $X^2_\alpha$ supported at the origin;
- in Proposition 39 we prove that if $\Gamma \in X^2_\alpha$, $\mathrm{supp}(\Gamma) \subset M$, then $\alpha \in \mathbb{Z}$, $\alpha < 0$;
- in particular, for $\alpha \in \{-2, -1\}$, we produce elements of $X^2_\alpha$ supported on $M$ in Lemma 40 and 42, which we will need in proving Theorem 5 and Corollary 45;
- these results allow us to conclude that the spaces $X^2_{\alpha,\mathrm{even}}$,$X^2_{\alpha,\mathrm{odd}}$ for $\alpha \in \mathbb{C}$, $\alpha \neq -1$, and $X^2_{-1,\mathrm{odd}}$ have dimension 1, as stated in Theorem 43;
- finally, we show that there exists no $\Lambda \in X^2_{-1,\mathrm{even}}$ extending the function $\frac{1}{|\det V|}$.

**Lemma 33.** *Let $\alpha \in \mathbb{C}$, $\Gamma \in X^2_\alpha$ such that* $\mathrm{supp}(\Gamma) = \{0\}$. *Then $\alpha = -2 - k$, $k \in \mathbb{N}$ and $\Gamma = c \det(\partial_{i,j})^k \delta$. Moreover, if $k$ is even (resp. odd) then $\Gamma$ is even (resp. odd).*

*Proof.* By Proposition 15, there exist $N \in \mathbb{N}$, $\{c_\beta\}_{0 \leq |\beta| \leq N} \in \mathbb{C}$, such that

$$\Gamma = \sum_{|\beta| \leq N} c_\beta \partial^\beta \delta.$$

Therefore, the Fourier transform $\widehat{\Gamma}$, which by Lemma 21 belongs to $X^2_{-\alpha-2}$, is given by a polynomial function with natural exponents. By Proposition 31 there exists $k \in \mathbb{N}$ such that $-\alpha - 2 = k$, and $\widehat{\Gamma} = c'(\det V)^k$, where $c' \in \mathbb{C}$, which has the same parity of $k$. Hence $\Gamma = c \det(\partial_{i,j})^k \delta$, and it has the same parity of $k$ too. $\square$

### 2.2.1 The structure of $M$

**Lemma 34.** *The map*

$$\nu\colon \big(\mathbb{R}^2 \setminus \{0\}\big) \times (\mathbb{R}/2\pi\mathbb{Z}) \to M \setminus \{0\},\ \nu(w,\theta) = (w\cos\theta\ \ w\sin\theta), \tag{2.1}$$

*is 2-1. Moreover, for every fixed $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, the image of the punctured plane $\big(\mathbb{R}^2 \setminus \{0\}\big) \times \{\theta\}$ is an orbit of the action (1.5) of $\mathrm{GL}_2^+$ on $\mathbb{R}^{2\times 2}$.*

*Proof.* Let $V \in M$, $V \neq 0$. Therefore the two columns $V_{i,1}, V_{i,2} \in \mathbb{R}^2$ are linearly dependent and not both 0. In fact, there exists one and only one $\theta \in [0,\pi)$ such that

$$\sin\theta V_{i,1} = \cos\theta V_{i,2}. \tag{2.2}$$

At least one between $\cos\theta$ and $\sin\theta$ is nonzero. Without loss of generality we can assume $\cos\theta \neq 0$, and the system

$$w = \frac{1}{\cos\theta} V_{i,1} \tag{2.3}$$

has one and only one nonzero solution in $\mathbb{R}^2 \setminus \{0\}$.

Therefore the map (2.1) is surjective. Moreover, if we let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, the linear relation in (2.2) has two solutions, $\theta$ and $\theta + \pi$. In the second case, the unique solution to (2.3) is $-w$.

The action defined in (1.5) preserves the linear dependence between the columns of $V \in M \setminus \{0\}$. In fact, for $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ such that (2.2) holds,

$$\sin\theta A^{-1} V_{i,1} = \cos\theta A^{-1} V_{i,2}$$

Now let $v, w \in \mathbb{R}^2 \setminus \{0\}$. Then there exists $A \in \mathrm{GL}_2^+(\mathbb{R})$ such that $A^{-1}w = v$. Therefore, for a fixed $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, the set $\nu((\mathbb{R}^2 \setminus \{0\}) \times \{\theta\})$ is an orbit of the action (1.5). ☐

**Remark 35.** *From now on we will write*

$$\widetilde{\varphi}(w,\theta) = \varphi(\nu(w,\theta)),$$

*for the restriction of a function on $M \setminus \{0\}$. Moreover, for a function $\psi$ on $(\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z})$ and a matrix $A \in \mathrm{GL}_2^+(\mathbb{R})$, we define*

$$\psi_{A^{-1}}(w,\theta) = \psi(A^{-1}w,\theta).$$

For $w \in \mathbb{R}^2 \setminus \{0\}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, in the point

$$M \setminus \{0\} \ni \nu(w,\theta) = \begin{pmatrix} w_1\cos\theta & w_1\sin\theta \\ w_2\cos\theta & w_2\sin\theta \end{pmatrix},$$

the tangent vectors with respect to the manifold $M \setminus \{0\}$ are

$$\overrightarrow{P}(w,\theta) := \begin{pmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{pmatrix},\ \overrightarrow{Q}(w,\theta) := \begin{pmatrix} 0 & 0 \\ \cos\theta & \sin\theta \end{pmatrix},$$

$$\overrightarrow{M}(w,\theta) := \frac{1}{|w|}\begin{pmatrix} -w_1\sin\theta & w_1\cos\theta \\ -w_2\sin\theta & w_2\cos\theta \end{pmatrix}.$$

Therefore the normal direction with respect to the manifold $M \setminus \{0\}$ in $\nu(w,\theta)$ is

$$\overrightarrow{N}(w,\theta) := \frac{1}{|w|}\begin{pmatrix} w_2\sin\theta & -w_2\cos\theta \\ -w_1\sin\theta & w_1\cos\theta \end{pmatrix},$$

in particular

$$\overrightarrow{N}(w,\theta) = \overrightarrow{N}(-w,\theta+\pi). \tag{2.4}$$

**Lemma 36.** *For $A \in \mathrm{GL}_2^+(\mathbb{R})$, we have*

$$\begin{aligned} \overrightarrow{M}_A(w,\theta) &:= A^{-1}\overrightarrow{M}(w,\theta) = \frac{|A^{-1}w|}{|w|}\overrightarrow{M}(A^{-1}w,\theta), \\ \overrightarrow{N}_A(w,\theta) &:= A^{-1}\overrightarrow{N}(w,\theta) = n(w,A)\overrightarrow{N}(A^{-1}w,\theta) + m(w,A)\overrightarrow{M}(A^{-1}w,\theta). \end{aligned} \tag{2.5}$$

*In particular, for $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have*

$$\begin{aligned} n(w,A) &= \frac{|w|}{|A^{-1}w|}\det A^{-1}, \\ m(w,A) &= \frac{(b^2+d^2-a^2-c^2)w_1w_2 + (ab+cd)(w_1^2-w_2^2)}{|w||A^{-1}w|}. \end{aligned}$$

*Proof.* Let $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we have the vectors

$$\begin{aligned} \overrightarrow{M}_A(w,\theta) =& \frac{1}{|w|}\begin{pmatrix} -(aw_1+bw_2)\sin\theta & (aw_1+bw_2)\cos\theta \\ -(cw_1+dw_2)\sin\theta & (cw_1+dw_2)\cos\theta \end{pmatrix}, \\ \overrightarrow{N}_A(w,\theta) =& \frac{1}{|w|}\begin{pmatrix} (aw_2-bw_1)\sin\theta & -(aw_2-bw_1)\cos\theta \\ (cw_2-dw_1)\sin\theta & -(cw_2-dw_1)\cos\theta \end{pmatrix}, \\ \overrightarrow{M}(A^{-1}w,\theta) =& \frac{1}{|A^{-1}w|}\begin{pmatrix} -(aw_1+bw_2)\sin\theta & (aw_1+bw_2)\cos\theta \\ -(cw_1+dw_2)\sin\theta & (cw_1+dw_2)\cos\theta \end{pmatrix}, \\ \overrightarrow{N}(A^{-1}w,\theta) =& \frac{1}{|A^{-1}w|}\begin{pmatrix} (cw_1+dw_2)\sin\theta & -(cw_1+dw_2)\cos\theta \\ -(aw_1+bw_2)\sin\theta & (aw_1+bw_2)\cos\theta \end{pmatrix}. \end{aligned}$$

First, we observe

$$\overrightarrow{M}_A(w,\theta) = \frac{|A^{-1}w|}{|w|}\overrightarrow{M}(A^{-1}w,\theta).$$

We also have

$$\overrightarrow{N}_A(w,\theta) \perp_{\mathrm{HS}} \overrightarrow{P}(A^{-1}w,\theta), \overrightarrow{Q}(A^{-1}w,\theta).$$

Since for every $(w,\theta) \in (\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z})$ we have that

$$\{\overrightarrow{P}(A^{-1}w,\theta), \overrightarrow{Q}(A^{-1}w,\theta), \overrightarrow{M}(A^{-1}w,\theta), \overrightarrow{N}(A^{-1}w,\theta)\}$$

is an orthonormal basis of $\mathbb{R}^{2\times 2}$, then

$$\overrightarrow{N}_A(w,\theta) \in \operatorname{span}\{\overrightarrow{N}(A^{-1}w,\theta), \overrightarrow{M}(A^{-1}w,\theta)\}.$$

In particular,

$$\overrightarrow{N}_A(w,\theta) = n(w,A)\overrightarrow{N}(A^{-1},\theta) + m(w,A)\overrightarrow{M}(A^{-1},\theta),$$

where $n(w,A), m(w,A)$ are independent on $\theta$, since we have the same dependence on $\theta$ in the corresponding entries of the matrices $\overrightarrow{N}_A(w,\theta), \overrightarrow{N}(A^{-1},\theta), \overrightarrow{M}(A^{-1},\theta)$. We have

$$\frac{1}{|A^{-1}w|}\begin{pmatrix} cw_1 + dw_2 & -(aw_1 + bw_2) \\ -(aw_1 + bw_2) & -(cw_1 + dw_2) \end{pmatrix}\begin{pmatrix} n(w,A) \\ m(w,A) \end{pmatrix} = \frac{1}{|w|}\begin{pmatrix} aw_2 - bw_1 \\ cw_2 - dw_1 \end{pmatrix},$$

hence

$$\begin{pmatrix} n(w,A) \\ m(w,A) \end{pmatrix} = \frac{1}{|w||A^{-1}w|}\begin{pmatrix} cw_1 + dw_2 & -(aw_1 + bw_2) \\ -(aw_1 + bw_2) & -(cw_1 + dw_2) \end{pmatrix}\begin{pmatrix} aw_2 - bw_1 \\ cw_2 - dw_1 \end{pmatrix}.$$

To conclude we observe that

$$\begin{aligned}
(cw_1 + dw_2)(aw_2 - bw_1) - (aw_1 + bw_2)(cw_2 - dw_1) &= \\
&= (ad - bc)(w_1^2 + w_2^2) = \det A^{-1}|w|^2, \\
-(aw_1 + bw_2)(aw_2 - bw_1) - (cw_1 + dw_2)(cw_2 - dw_1) &= \\
&= (b^2 + d^2 - a^2 - c^2)w_1w_2 + (ab + cd)(w_1^2 - w_2^2).
\end{aligned}$$

□

In particular:

1. for $A^{-1} = (1+\varepsilon)\mathbb{I}_2$, we have
$$\begin{aligned} n(w,A) &= \frac{|w|}{|A^{-1}w|}(1+\varepsilon)^2 = (1+\varepsilon), \\ m(w,A) &= 0; \end{aligned}$$

2. for $A^{-1} = \mathbb{I}_2 + \varepsilon(E^{1,2} - E^{2,1})$, we have
$$\begin{aligned} n(w,A) &= \frac{|w|}{|A^{-1}w|}(1+\varepsilon^2) = \sqrt{1+\varepsilon^2}, \\ m(w,A) &= 0; \end{aligned}$$

3. for $A^{-1} = \mathbb{I}_2 + \varepsilon E^{1,1}$, we have
$$\begin{aligned} n(w,A) &= \frac{|w|}{|A^{-1}w|}(1+\varepsilon), \\ m(w,A) &= -\frac{(2\varepsilon + \varepsilon^2)w_1w_2}{|w||A^{-1}w|}. \end{aligned}$$

4. for $A^{-1} = \mathbb{I}_2 + \varepsilon E^{1,2}$, we have
$$\begin{aligned} n(w,A) &= \frac{|w|}{|A^{-1}w|}, \\ m(w,A) &= \frac{\varepsilon w_1^2 - \varepsilon w_2^2 + \varepsilon^2 w_1w_2}{|w||A^{-1}w|}. \end{aligned}$$

Let $\partial_N^i \varphi$ denote the $i$-th order normal derivative of $\varphi$ with respect to the manifold $M \setminus \{0\}$, and $\partial_M^i \varphi$ the $i$-th order derivative of $\varphi$ in the direction $\overrightarrow{M}(w, \theta)$.

**Lemma 37.** *For $A \in \mathrm{GL}_2^+(\mathbb{R})$, we have*

$$(\partial_N^i (D_A^\alpha \varphi))^\sim(w,\theta) = \frac{1}{(\det A)^{\alpha+2}} \sum_{j=0}^{i} \binom{i}{j} n(w,A)^{i-j} m(w,A)^j \left( \left( \partial_M^j \partial_N^{i-j} \varphi \right)^\sim \right)(A^{-1}w, \theta).$$

*Proof.* For $V \in \mathbb{R}^{2\times 2}$, let $\nabla^i \varphi(V)$ be the multilinear map

$$\nabla^i \varphi(V) \colon \underbrace{\mathbb{R}^{2\times 2} \times \cdots \times \mathbb{R}^{2\times 2}}_{i \text{ copies}} \to \mathbb{C}$$

such that, for $(l,m)_j \in \{1,2\} \times \{1,2\}$,

$$\left\langle \nabla^i \varphi(V), \left( E^{(l,m)_1}, \ldots, E^{(l,m)_i} \right) \right\rangle = \partial_{(l,m)_1} \ldots \partial_{(l,m)_i} \varphi(V).$$

Then, for $A \in \mathrm{GL}_2^+(\mathbb{R})$, we have

$$\left\langle \nabla^i (\varphi_{A^{-1}})(V), \left( E^{(l,m)_1}, \ldots, E^{(l,m)_i} \right) \right\rangle = \left\langle \nabla^i \varphi(A^{-1}V), \left( A^{-1}E^{(l,m)_1}, \ldots, A^{-1}E^{(l,m)_i} \right) \right\rangle.$$

In fact, for $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\begin{aligned}
(\partial_{1,1}\varphi_{A^{-1}})(V) &= ((a\partial_{1,1} + c\partial_{2,1})\varphi)(A^{-1}V) = \left\langle \nabla\varphi(A^{-1}V), A^{-1}E^{1,1} \right\rangle, \\
(\partial_{1,2}\varphi_{A^{-1}})(V) &= ((a\partial_{1,2} + c\partial_{2,2})\varphi)(A^{-1}V) = \left\langle \nabla\varphi(A^{-1}V), A^{-1}E^{1,2} \right\rangle, \\
(\partial_{2,1}\varphi_{A^{-1}})(V) &= ((b\partial_{1,1} + d\partial_{2,1})\varphi)(A^{-1}V) = \left\langle \nabla\varphi(A^{-1}V), A^{-1}E^{2,1} \right\rangle, \\
(\partial_{2,2}\varphi_{A^{-1}})(V) &= ((b\partial_{1,2} + d\partial_{2,1})\varphi)(A^{-1}V) = \left\langle \nabla\varphi(A^{-1}V), A^{-1}E^{2,2} \right\rangle.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&(\partial_N^i (D_A^\alpha \varphi))^\sim(w,\theta) = \frac{1}{(\det A)^{\alpha+2}} \left\langle (\nabla^i \varphi)^\sim(A^{-1}w,\theta), \left( \overrightarrow{N}_A(w,\theta) \right)^i \right\rangle, \\
&\frac{1}{(\det A)^{\alpha+2}} \left( \left( \partial_M^j \partial_N^{i-j} \varphi \right)^\sim \right)(A^{-1}w,\theta) = \\
&\qquad = \frac{1}{(\det A)^{\alpha+2}} \left\langle (\nabla^i \varphi)^\sim(A^{-1}w,\theta), \left( \left( \overrightarrow{N}(A^{-1}w,\theta) \right)^{i-j}, \left( \overrightarrow{M}(A^{-1}w,\theta) \right)^j \right) \right\rangle.
\end{aligned}$$

Substituting the equality for $\overrightarrow{N}_A(w,\theta)$ from (2.5), and exploiting the linearity properties of $\nabla^i \varphi(V)$, we obtain

$$\begin{aligned}
(\partial_N^i (D_A^\alpha \varphi))^\sim(w,\theta) &= \frac{1}{|\det A|^{\alpha+2}} \left\langle (\nabla^i \varphi)^\sim(A^{-1}w,\theta), \left( \overrightarrow{N}_A(w,\theta) \right)^i \right\rangle = \\
&= \frac{1}{|\det A|^{\alpha+2}} \left\langle (\nabla^i \varphi)^\sim(A^{-1}w,\theta), \left( n(w,A)\overrightarrow{N}(A^{-1}w,\theta) + m(w,A)\overrightarrow{M}(A^{-1}w,\theta) \right)^i \right\rangle \\
&= \frac{1}{(\det A)^{\alpha+2}} \sum_{j=0}^{i} \binom{i}{j} n(w,A)^{i-j} m(w,A)^j \left( \left( \partial_M^j \partial_N^{i-j} \varphi \right)^\sim \right)(A^{-1}w,\theta).
\end{aligned}$$

$\square$

### 2.2.2 The $\alpha$-homogeneous tempered distributions supported on $M$

We start recalling a theorem from [6].

**Theorem 38** (Theorem XXXVII, Ch. III. for $(d-1)$-dimensional submanifolds of $\mathbb{R}^d$)**.** *Let $\mathcal{M}$ be a $(d-1)$-dimensional submanifold of $\mathbb{R}^d$. For a smooth function $\varphi$ on $\mathbb{R}^d$, let $\partial_N^i\varphi$ denote the $i$-th order normal derivative of $\varphi$ with respect to the manifold $\mathcal{M}$. Every distribution $\Gamma$ supported on $\mathcal{M}$ admits a unique decomposition into a locally finite linear combination of the form*

$$\Gamma(\varphi) = \sum_{i=0}^{k} \Gamma_i(\partial_N^i \varphi),$$

*where $\Gamma_i$ are distributions in $\mathcal{D}'(\mathcal{M})$.*

The Theorem is proven locally by a parametrization argument that allows to restrict to the easier case of a $(d-1)$-dimensional hyperplane in $\mathbb{R}^d$. The proof in this case follows the same strategy of the one of Proposition 15. The claim is necessarily "local": a bound on the order $k$ of normal derivatives that may be involved in defining $\Gamma$ can be obtained only from $N$, the index that guarantees the conclusion of Proposition 13, which is local.

**Proposition 39.** *Let $\alpha \in \mathbb{C}$, $\Gamma \in X_\alpha^2$, $\operatorname{supp}(\Gamma) \subset M$. Then $\alpha \in \mathbb{Z}$, $\alpha < 0$.*

*Proof.* We note that $\Gamma$ defines a distribution $\widetilde{\Gamma} \in \mathcal{D}'(\mathbb{R}^{2\times 2} \setminus \{0\})$ as shown in Proposition 11. Moreover, by Theorem 38, the distribution $\widetilde{\Gamma}$ has locally the form

$$\widetilde{\Gamma}(\phi) = \sum_{i=0}^{k} \Gamma_i((\partial_N^i \phi)^{\sim}),$$

where $\Gamma_i \in \mathcal{D}'((\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z}))$. By (2.4), we have that

$$(\partial_N^i \phi)^{\sim}(w,\theta) = (\partial_N^i \phi)^{\sim}(-w, \theta+\pi).$$

Without loss of generality we can assume $\Gamma_i$ to have the same periodicity, namely, for every $\psi \in \mathcal{D}((\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z}))$, we can assume

$$\Gamma_i(\psi) = \Gamma_i(\tau_\pi \psi_{-\mathbb{I}_2}),$$

where

$$\tau_\pi \psi_{-\mathbb{I}_2}(w,\theta) = \psi(-w, \theta+\pi).$$

Therefore all the $\Gamma_i$ are determined by their behaviour against the functions $\psi \in \mathcal{D}((\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z}))$ such that $\psi(w,\theta) = \psi(-w,\theta+\pi)$.

Moreover, we can restrict to the local case. In fact, the arguments and the techniques we use are always local, and the global claim can be proven locally.

The $\alpha$-homogeneity condition for a function $\phi \in \mathcal{D}(\mathbb{R}^{2\times 2} \setminus \{0\})$ is given by

$$\sum_{i=0}^{k} \Gamma_i((\partial_N^i \phi)^{\sim}) = \sum_{i=0}^{k} \Gamma_i((\partial_N^i (D_A^\alpha \phi))^{\sim}).$$

For every $j \geq 0$, $\psi \in \mathcal{D}((\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z}))$ such that $\psi(w, \theta) = \psi(-w, \theta + \pi)$, there exists a function $\phi_j \in \mathcal{D}(\mathbb{R}^{2\times 2} \setminus \{0\})$ such that

$$\begin{cases} (\partial_N^i \phi_j)\tilde{} = 0 & \text{for every } i \geq 0,\ i \neq j, \\ (\partial_N^j \phi_j)\tilde{} = \psi. \end{cases}$$

In fact, let $\Omega \subset (\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z})$ be an open neighbourhood of $\operatorname{supp}(\psi)$ such that $\overline{\Omega} \subset (\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z})$ is compact. Now let $U \subset \mathbb{R}^{2\times 2} \setminus \{0\}$ be an open neighbourhood of $\nu(\overline{\Omega})$ such that on $U$ we can put coordinates $\mathbb{R}^3 \times [-\varepsilon, \varepsilon]$ through the bijection $\sigma$ in a way that $\nu(\overline{\Omega}) \subset \sigma(\mathbb{R}^3 \times \{0\})$, and

$$\lim_{h \to 0} \frac{\sigma(x, h) - \sigma(x, 0)}{|\sigma(x, h) - \sigma(x, 0)|} = \overrightarrow{N}(\sigma(x, 0)).$$

Then, for a function $\eta \in \mathcal{D}(\mathbb{R}^{2\times 2} \setminus \{0\})$ such that $\operatorname{supp}(\eta) \subset U$, $\operatorname{supp}(\eta) \cap (M \setminus \{0\}) \supset \overline{\Omega}$, $\eta \equiv 1$ on $W \subset U$ a open neighbourhood of $\nu(\overline{\Omega})$, we define

$$\begin{cases} \phi_j(V) = 0 & \text{if } V \notin \sigma(\mathbb{R}^3 \times [-\varepsilon, \varepsilon]), \\ \phi_j(\sigma(x, y)) = \psi(\sigma(x, 0))\eta(\sigma(x, y))\frac{y^j}{j!} & \text{otherwise.} \end{cases}$$

Thus the normal derivatives $\partial_N^j$ of $\phi_i$ only depends on the last part. In particular, if $i \neq j$ then $(\partial_N^i \phi_j)\tilde{} = 0$, if $i = j$ then $(\partial_N^j \phi_j)\tilde{} = \psi$.

For every $\psi \in \mathcal{D}((\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z}))$ such that $\psi(w, \theta) = \psi(-w, \theta + \pi)$,

$$\begin{aligned} \Gamma_k(\psi) &= \sum_{i=0}^{k} \Gamma_i((\partial_N^i \phi_k)\tilde{}) = \sum_{i=0}^{k} \Gamma_i((\partial_N^i (D_A^\alpha \phi_k))\tilde{}) = \\ &= \sum_{i=0}^{k} \Gamma_i \left( n(w, A)^i \frac{1}{(\det A)^{\alpha+2}} ((\partial_N^i \phi_k)\tilde{})_{A^{-1}} \right) = \Gamma_k \left( \frac{n(w, A)^k}{(\det A)^{\alpha+2}} \psi_{A^{-1}} \right). \end{aligned}$$

We consider the matrices $A^{-1}$ for which we explicitly computed $n(w, A)$:

1. $A^{-1} = (1 + \varepsilon)\mathbb{I}_2$, then, for every $\psi \in \mathcal{D}((\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z}))$ such that $\psi(w, \theta) = \psi(-w, \theta + \pi)$,
$$\Gamma_k(\psi) = \Gamma_k((1 + \varepsilon)^{k+2(\alpha+2)} \psi_{A^{-1}}).$$
By a limit argument
$$\Gamma_k((k + 2(\alpha + 2))\psi + \partial_1 \psi w_1 + \partial_2 \psi w_2) = 0,$$
which yields
$$(k + 2(\alpha + 2) - 2)\Gamma_k = w_1 \partial_1 \Gamma_k + w_2 \partial_2 \Gamma_k.$$
2. $A^{-1} = \mathbb{I}_2 + \varepsilon(E^{1,2} - E^{2,1})$, then an analogous limit argument yields
$$w_1 \partial_2 \Gamma_k = w_2 \partial_1 \Gamma_k;$$
3. $A^{-1} = \mathbb{I}_2 + \varepsilon E^{1,1}$, then an analogous limit argument yields
$$(k + (\alpha + 2) - 1)\Gamma_k - k\frac{w_1^2}{w_1^2 + w_2^2}\Gamma_k = w_1 \partial_1 \Gamma_k.$$

4. $A^{-1} = \mathbb{I}_2 + \varepsilon E^{1,2}$, then an analogous limit argument yields

$$-k\frac{w_1 w_2}{w_1^2 + w_2^2}\Gamma_k = w_2 \partial_1 \Gamma_k.$$

We obtain locally the following system of equations in the sense of distributions

$$\begin{cases} (k + 2(\alpha+2) - 2)\Gamma_k = w_1\partial_1\Gamma_k + w_2\partial_2\Gamma_k \\ w_1\partial_2\Gamma_k = w_2\partial_1\Gamma_k \\ (k + (\alpha+2) - 1)\Gamma_k - k\frac{w_1^2}{w_1^2+w_2^2}\Gamma_k = w_1\partial_1\Gamma_k \\ -k\frac{w_1w_2}{w_1^2+w_2^2}\Gamma_k = w_2\partial_1\Gamma_k. \end{cases} \tag{2.6}$$

For $k \neq -1-\alpha$, there are 3 linearly independent equations, then by algebraic manipulations we can recover the system

$$\begin{cases} \partial_1\Gamma_k = 0 \\ \partial_2\Gamma_k = 0 \\ \Gamma_k = 0. \end{cases}$$

Let $\Gamma \in X^2_\alpha$, $\operatorname{supp}(\Gamma) \subset M$, $\alpha \neq -1-k$. Then, away from the origin, $\Gamma$ agrees with the zero distribution. Therefore $\operatorname{supp}(\Gamma) \subset \{0\}$, and by Lemma 33 we prove the claim. □

**Lemma 40.** *Let $\Gamma \in X^2_{-1}$, $\operatorname{supp}(\Gamma) \subset M$. Then $\Gamma$ is even and there exists a unique $F \in \mathcal{D}'(\mathbb{R}/\pi\mathbb{Z})$ such that*

$$\Gamma(\varphi) = (\mu(F))(\varphi) \coloneqq F(\psi_\varphi), \tag{2.7}$$

*where, for every $\varphi \in \mathcal{S}(\mathbb{R}^{2\times 2})$,*

$$\psi_\varphi(\theta) \coloneqq \int_{\mathbb{R}^2\setminus\{0\}} \varphi(\nu(w,\theta))\,\mathrm{d}w = \int_{\mathbb{R}^2\setminus\{0\}} \varphi\begin{pmatrix} w\cos\theta & w\sin\theta \end{pmatrix} \mathrm{d}w,$$

*is a smooth $\pi$-periodic function on $\mathbb{R}$.*

*Proof.* From (2.6), we have locally the system of equations in the sense of distributions

$$\begin{cases} w_1\partial_1\Gamma_0 + w_2\partial_2\Gamma_0 = 0 \\ w_2\partial_1\Gamma_0 - w_1\partial_2\Gamma_0 = 0. \end{cases}$$

By multiplying the first equation by $w_1$ and the second by $w_2$ and adding them we obtain the equation

$$(w_1^2 + w_2^2)\partial_1\Gamma_0 = 0,$$

in the sense of distributions. Since the function $w_1^2 + w_2^2$ is nonzero on $(\mathbb{R}^2\setminus\{0\})\times(\mathbb{R}/2\pi\mathbb{Z})$, it yields $\partial_1\Gamma_0 = 0$. In an analogous way we prove $\partial_2\Gamma_0 = 0$.

Then, by Lemma 19, $\Gamma_0$ agrees with a constant function locally on every punctured plane $(\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z})$. Since $M \setminus \{0\}$ is connected, we recover a global result on every punctured plane $(\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z})$, namely, for every $\phi \in \mathcal{D}(\mathbb{R}^{2\times 2} \setminus \{0\})$,

$$\Gamma(\phi) = G\left(\int_{\mathbb{R}^2} \phi\begin{pmatrix} w\cos\theta & w\sin\theta \end{pmatrix} \mathrm{d}w\right),$$

for $G \in \mathcal{D}'(\mathbb{R}/2\pi\mathbb{Z})$.

An easy computation proves that, for $\varphi \in \mathcal{S}(\mathbb{R}^{2\times 2})$, $\psi_\varphi \in \mathcal{C}^\infty(\mathbb{R}/2\pi\mathbb{Z})$. Moreover, $\psi_\varphi$ is $\pi$-periodic, since

$$\begin{aligned}\psi_\varphi(\theta) &= \int_{\mathbb{R}^2} \varphi\left(w\cos\theta \quad w\sin\theta\right) \mathrm{d}w = \int_{\mathbb{R}^2} \varphi\left(-w\cos\theta \quad -w\sin\theta\right) \mathrm{d}w = \\ &= \int_{\mathbb{R}^2} \varphi\left(w\cos(\theta+\pi) \quad w\sin(\theta+\pi)\right) \mathrm{d}w = \psi_\varphi(\theta+\pi).\end{aligned}$$

Therefore, we can choose $G$ such that $G = \tau_\pi G$, where, for a smooth $2\pi$-periodic function $\psi$ on $\mathbb{R}$,

$$(\tau_\pi G)(\psi) = G(\tau_\pi \psi),$$

for $\tau_\pi\psi(\theta) = \psi(\theta - \pi)$. For $\Gamma \in X^2_{-1}$, $\operatorname{supp}(\Gamma) \subset M$, let $\widetilde{G} \in \mathcal{D}'(\mathbb{R}/2\pi\mathbb{Z})$ such that $\Gamma(\phi) = \widetilde{G}(\psi_\phi)$. We define

$$G = \frac{\widetilde{G} + \tau_\pi \widetilde{G}}{2},$$

thus $\tau_\pi G = G$, and, since $\psi_\varphi = \tau_\pi\psi_\varphi$, we have $G(\psi_\varphi) = \widetilde{G}(\psi_\varphi)$. In particular, there exists $F \in \mathcal{D}'(\mathbb{R}/\pi\mathbb{Z})$ such that

$$2F(\psi_\varphi) = G(\psi_\varphi),$$

where as an argument of $F$ the function $\psi_\varphi$ is considered as $\pi$-periodic, as an argument of $G$ as $2\pi$-periodic.

We observe that $\mu(F)$ defined in (2.7) gives an element of $\mathcal{S}'(\mathbb{R}^{2\times 2})$. In fact, by Proposition 13, there exists $N \in \mathbb{N}$ such that, for $\psi \in \mathcal{D}(\mathbb{R}/\pi\mathbb{Z})$, we have

$$|F(\psi)| \leq C \sup_{0\leq i\leq N} \sup_{\theta\in\mathbb{R}/\pi\mathbb{Z}} |\partial_\theta^i \psi(\theta)|.$$

We observe that

$$(\partial_M^i \varphi)^\sim(w,\theta) = \frac{1}{|w|^i}\partial_\theta^i \widetilde{\varphi}(w,\theta),$$

Then, for every $\theta \in \mathbb{R}/\pi\mathbb{Z}$,

$$\begin{aligned}|\partial_\theta^i \psi_\varphi(\theta)| &\leq \int_{\mathbb{R}^2} |\partial_\theta^i \widetilde{\varphi}(w,\theta)| \,\mathrm{d}w \leq \\ &\leq \int_{|w|\leq 1} |w|^i |(\partial_M^i\varphi)^\sim(w,\theta)| \,\mathrm{d}w + \int_{|w|>1} \frac{|w|^i |(\partial_M^i\varphi)^\sim(w,\theta)| |w|^3}{|w|^3} \,\mathrm{d}w \leq \\ &\leq C\|\varphi\|_i + C'\|\varphi\|_{i+3},\end{aligned}$$

since $|w| = \langle \nu(w,\theta), \nu(w,\theta)\rangle_{\mathrm{HS}}$.

Thus, for every $\varphi \in \mathcal{S}(\mathbb{R}^{2\times 2})$,

$$|(\mu(F))(\varphi)| \leq C\|\varphi\|_{N+3}.$$

Therefore $\mu(F) \in \mathcal{S}'(\mathbb{R}^{2\times 2})$ by Proposition 13.

Moreover, $\mu(F)$ is even. Consider the matrix $A^{-1} = E^{1,1} - E^{2,2}$. Then

$$\begin{aligned}\psi_\varphi(\theta) &= \int_{\mathbb{R}^2} \varphi\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\cos\theta \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\sin\theta\right) \mathrm{d}w_1\,\mathrm{d}w_2 = \\ &= \int_{\mathbb{R}^2} \varphi\left(\begin{pmatrix} w_1 \\ -w_2 \end{pmatrix}\cos\theta \quad \begin{pmatrix} w_1 \\ -w_2 \end{pmatrix}\sin\theta\right) \mathrm{d}w_1\,\mathrm{d}w_2 = \psi_{\varphi_{A^{-1}}}(\theta),\end{aligned}$$

which yields

$$(\mu(F))(D_A^{-1}\varphi) = F(\psi_{\varphi_{A^{-1}}}) = F(\psi_\varphi) = (\mu(F))(\varphi).$$

Now let $\Gamma \in X^2_{-1}$, $\operatorname{supp}(\Gamma) \subset M$, then, away from the origin, it agrees with some $\mu(F)$ described above. Therefore the difference $\Gamma - \mu(F)$ is supported at the origin. By Lemma 33 we prove surjectivity.

To conclude the injectivity of the map

$$\mu\colon \mathcal{D}'(\mathbb{R}/\pi\mathbb{Z}) \to \{\Gamma \in X^2_{-1,\text{even}}\colon \operatorname{supp}(\Gamma) \subset M\},$$

we observe that for every $\psi \in \mathcal{D}(\mathbb{R}/\pi\mathbb{Z})$ there exists $\varphi \in \mathcal{S}(\mathbb{R}^{2\times 2})$ such that $\psi = \psi_\varphi$. In fact, let $\eta \in \mathcal{D}(\mathbb{R}^2 \setminus \{0\})$ such that $\operatorname{supp}(\eta) \subset \{w \in \mathbb{R}^2\colon 1 \leq |w| \leq 2,\ w_2 > 0\}$, $\int \eta(w)\,\mathrm{d}w = 1$, $\eta(w) = \eta(-w)$, and define $\phi \in \mathcal{D}((\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z}))$ by

$$\phi(w,\theta) = \eta(w)\psi(\theta).$$

Then let $\varphi = \phi_0$ under the construction described in the proof of Proposition 39. $\square$

**Remark 41.** *In identifying $\mathcal{D}'(\mathbb{R}/2\pi\mathbb{Z})$ with the topological dual of $\mathcal{C}^\infty(\mathbb{R}/2\pi\mathbb{Z})$ there is the subtle step of having a measure on the manifold $\mathbb{R}/2\pi\mathbb{Z}$ (see [4]).*

**Lemma 42.** *The definition*

$$\Gamma(\varphi) := \int_0^{2\pi}\int_{\mathbb{R}^2} \frac{(\partial_N\varphi)^\sim(w,\theta)}{|w|}\,\mathrm{d}w\,\mathrm{d}\theta, \tag{2.8}$$

*gives a nonzero element of $X^2_{-2,\text{odd}}$.*

*Proof.* For every $\varphi \in \mathcal{S}(\mathbb{R}^{2\times 2})$, we have the bound

$$\begin{aligned}
|\Gamma(\varphi)| &= \left|\int_0^{2\pi}\int_{\mathbb{R}^2} \frac{(\partial_N\varphi)^\sim(w,\theta)}{|w|}\,\mathrm{d}w\,\mathrm{d}\theta\right| \leq 2\pi \sup_{\theta\in\mathbb{R}/2\pi\mathbb{Z}}\left|\int_{\mathbb{R}^2}\frac{(\partial_N\varphi)^\sim(w,\theta)}{|w|}\,\mathrm{d}w\right| \leq \\
&\leq 2\pi \sup_{\theta\in\mathbb{R}/2\pi\mathbb{Z}}\left(\int_{|w|\leq 1}\frac{|(\partial_N\varphi)^\sim(w,\theta)|}{|w|}\,\mathrm{d}w + \int_{|w|>1}\frac{|(\partial_N\varphi)^\sim(w,\theta)||w|^2}{|w|^3}\,\mathrm{d}w\right) \leq \\
&\leq C\|\varphi\|_1 + C'\|\varphi\|_3,
\end{aligned}$$

where for $|w| \leq 1$ we use the property of local integrability of $\frac{1}{|w|}$ and the boundedness of $|\partial_N\varphi|$, while for $|w| > 1$ we exploit the fast decay properties of the Schwartz functions. Then, by Proposition 12, $\Gamma \in \mathcal{S}'(\mathbb{R}^{2\times 2})$.

To show it is nonzero, let $\phi_1 \in \mathcal{D}(\mathbb{R}^{2\times 2}\setminus\{0\})$ associated to $\psi \in \mathcal{D}((\mathbb{R}^2\setminus\{0\})\times(\mathbb{R}/2\pi\mathbb{Z}))$ such that $\psi(w,\theta) = \psi(-w,\theta+\pi)$, $\psi \geq 0$, $\psi \neq 0$. Then

$$\Gamma(\phi_1) = \int_0^{2\pi}\int_{\mathbb{R}^2}\frac{\psi(w,\theta)}{|w|}\,\mathrm{d}w\,\mathrm{d}\theta > 0.$$

To show it belongs to $X^2_{-2}$, we observe that, for $A \in \mathrm{GL}_2^+(\mathbb{R})$,

$$\begin{aligned}
\Gamma(D_A^{-2}\varphi) &= \int_0^{2\pi}\int_{\mathbb{R}^2}\frac{(n(w,A)(\partial_N\varphi)^\sim + m(w,A)(\partial_M\varphi)^\sim)(A^{-1}w,\theta)}{|w|}\,\mathrm{d}w\,\mathrm{d}\theta = \\
&= \int_0^{2\pi}\int_{\mathbb{R}^2}\frac{\det A^{-1}(\partial_N\varphi)^\sim(A^{-1}w,\theta)}{|A^{-1}w|}\,\mathrm{d}w\,\mathrm{d}\theta + \\
&\quad + \int_0^{2\pi}\int_{\mathbb{R}^2}\frac{m(w,A)(\partial_M\varphi)^\sim(A^{-1}w,\theta)}{|w|}\,\mathrm{d}w\,\mathrm{d}\theta.
\end{aligned}$$

If we change the variable from $w$ into $Av$ in the first integral, we recover $\Gamma(\varphi)$. To conclude, we observe that

$$\int_0^{2\pi}\int_{\mathbb{R}^2} \frac{m(w,A)(\partial_M\varphi)^{\sim}(A^{-1}w,\theta)}{|w|}\,\mathrm{d}w\,\mathrm{d}\theta = \int_{\mathbb{R}^2}\int_0^{2\pi} \frac{m(w,A)\partial_\theta\widetilde{\varphi}(A^{-1}w,\theta)}{|w||A^{-1}w|}\,\mathrm{d}\theta\,\mathrm{d}w =$$
$$= \int_{\mathbb{R}^2} \frac{m(w,A)}{|w||A^{-1}w|}\int_0^{2\pi} \partial_\theta\widetilde{\varphi}(A^{-1}w,\theta)\,\mathrm{d}\theta\,\mathrm{d}w = 0.$$

The tempered distribution is odd. In fact, $n(w,A) = \frac{|w|}{|A^{-1}w|}\det A^{-1}$, thus for $A^{-1} = E^{1,1} - E^{2,2}$ we have $n(w,A) = -1$. □

**Theorem 43.** *For $\alpha \in \mathbb{C}$, $\alpha \neq -1$, the spaces $X^2_{\alpha,\mathrm{even}}$, $X^2_{\alpha,\mathrm{odd}}$ have dimension 1. The space $X^2_{-1,\mathrm{odd}}$ has dimension 1.*

*Proof.* By Lemma 21, without loss of generality we can assume $\mathrm{Re}(\alpha) \geq -1$.

By Theorem 30, we have that the space $X^2_{\alpha,\mathrm{even}}$ has at least dimension 1. Moreover, for $\mathrm{Re}(\alpha) \geq -1$, $\alpha \neq -1$, we are able to produce an element $\Lambda$ such that $\mathrm{supp}(\Lambda) = R^{2\times 2}$. Suppose to have $\Lambda' \in X^2_{\alpha,\mathrm{even}}$. By Proposition 31, there exists $c \in \mathbb{C}$ such that

$$\Gamma = \Lambda' - c\Lambda \in X^2_{\alpha,\mathrm{even}},\ \mathrm{supp}(\Gamma) \subset M.$$

By Lemma 39, we conclude $\Gamma = 0$, hence that $X^2_{\alpha,\mathrm{even}}$ has dimension 1.

An analogous argument prove the claim for $X^2_{\alpha,\mathrm{odd}}$, for $\mathrm{Re}(\alpha) \geq -1$, $\alpha \neq -1$. For $X^2_{-1,\mathrm{odd}}$ we also need Lemma 40, in particular that if $\Gamma \in X^2_{-1}$, $\mathrm{supp}(\Gamma) \subset M$, then $\Gamma$ is even. □

### 2.2.3 Non extendibility of $|\det V|^{-1}$ to an element of element of $X^2_{-1,\mathrm{even}}$

In this section we prove the following statement.

**Proposition 44.** *There exists no element of $X^2_{-1,\mathrm{even}}$ that, away from the critical variety $M$, agrees with the the function*

$$f(V) := \frac{1}{|\det V|}. \tag{2.9}$$

*Proof.* We first extend (2.9) to a tempered distribution. Then we analyse what it lacks for being homogeneous of degree $-1$. This "lack" has to be taken care by a tempered distributions supported on the critical variety $M$. To conclude we prove that such a tempered distribution cannot exist.

Consider the following linear functional defined, for $\varphi \in \mathcal{S}(\mathbb{R}^{2\times 2})$, by

$$\Lambda(\varphi) := \int_{|\det(x\ y)|\geq 1} \frac{\varphi(x,y)}{|\det(x,y)|}\,\mathrm{d}x\,\mathrm{d}y + \int_{\substack{|\det(x\ y)|<1\\ x\neq 0}} \frac{\varphi(x,y) - \varphi(x,\frac{x\cdot y}{|x|^2}x)}{|\det(x,y)|}\,\mathrm{d}x\,\mathrm{d}y.$$

It is a tempered distribution. In fact we have the following bounds

$$\left|\int_{|\det(x\ y)|\geq 1} \frac{\varphi(x,y)}{|\det(x,y)|}\,\mathrm{d}x\,\mathrm{d}y\right| \leq \int_{|\det(x\ y)|\geq 1} |\varphi(x,y)|\,\mathrm{d}x\,\mathrm{d}y \leq \int_{\mathbb{R}^{2\times 2}} |\varphi(x,y)|\,\mathrm{d}x\,\mathrm{d}y \leq C\|\varphi\|_5,$$

$$\left|\int_{|\det(x\ y)|<1} \frac{\varphi(x,y)-\varphi(x,\frac{x\cdot y}{|x|^2}x)}{|\det(x,y)|}\,\mathrm{d}x\,\mathrm{d}y\right| \leq \int_{\substack{|\det(x\ y)|<1\\ x\neq 0}} \frac{|\nabla\varphi(x,\widetilde{y})\cdot(0,y-\frac{x\cdot y}{|x|^2}x)|}{|\det(x,y)|}\,\mathrm{d}x\,\mathrm{d}y \leq$$
$$\leq \int_{|\det(x\ y)|<1} \frac{|\nabla\varphi(x,\widetilde{y})|}{|x|}\,\mathrm{d}x\,\mathrm{d}y \leq C\|\varphi\|_5.$$

To prove the second inequality we observe

$$\left|\left(0, y-\frac{x\cdot y}{|x|^2}x\right)\right| = \left|y-\frac{x\cdot y}{|x|^2}x\right| = \sqrt{\left(y_1 - x_1\frac{x\cdot y}{|x|^2}\right)^2 + \left(y_2 - x_2\frac{x\cdot y}{|x|^2}\right)^2} =$$
$$= \sqrt{\left(x_2\frac{y_1x_2-x_1y_2}{|x|^2}\right)^2 + \left(x_1\frac{y_2x_1-x_2y_1}{|x|^2}\right)^2} = \frac{|\det(x\ y)|}{|x|}.$$

We observe that

$$\Lambda(D_A^{-1}\varphi) = \frac{1}{\det A}\int_{|\det(x\ y)|\geq 1} \frac{\varphi(A^{-1}x, A^{-1}y)}{|\det(x\ y)|}\,\mathrm{d}x\,\mathrm{d}y +$$
$$+ \frac{1}{\det A}\int_{|\det(x\ y)|<1} \frac{\varphi(A^{-1}x, A^{-1}y) - \varphi(A^{-1}x, \frac{x\cdot y}{|x|^2}A^{-1}x)}{|\det(x\ y)|}\,\mathrm{d}x\,\mathrm{d}y =$$
$$= \int_{|\det(vz)|\geq\frac{1}{\det A}} \frac{\varphi(v,z)}{|\det(vz)|}\,\mathrm{d}v\,\mathrm{d}z +$$
$$+ \int_{|\det(vz)|<\frac{1}{\det A}} \frac{\varphi(v,z)-\varphi(v,\frac{Av\cdot Az}{|Av|^2}v)}{|\det(vz)|}\,\mathrm{d}v\,\mathrm{d}z =$$
$$= \Lambda(\varphi) + \int_{|\det(vz)|<1} \frac{\varphi(v,\frac{v\cdot z}{|v|^2}v)-\varphi(v,\frac{Av\cdot Az}{|Av|^2}v)}{|\det(vz)|}\,\mathrm{d}v\,\mathrm{d}z -$$
$$- \int_{1\leq|\det(vz)|<\frac{1}{\det A}} \frac{\varphi(v,\frac{Av\cdot Az}{|Av|^2}v)}{|\det(vz)|}\,\mathrm{d}v\,\mathrm{d}z.$$

Now we consider the following cases:

1. $A^{-1} = (1+\varepsilon)\mathbb{I}_2$, then

$$\Lambda(D_A^{-1}\varphi) - \Lambda(\varphi) = -\int_{1\leq|\det(vz)|<(1+\varepsilon)^2} \frac{\varphi(v,z)}{|\det(vz)|}\,\mathrm{d}v\,\mathrm{d}z;$$

2. $A^{-1} = \mathbb{I}_2 + \varepsilon(E^{1,2} - E^{2,1})$, then

$$\Lambda(D_A^{-1}\varphi) - \Lambda(\varphi) = -\int_{1\leq|\det(vz)|<(1+\varepsilon^2)} \frac{\varphi(v,z)}{|\det(vz)|}\,\mathrm{d}v\,\mathrm{d}z;$$

3. $A^{-1} = \mathbb{I}_2 + \varepsilon E^{1,2}$, then

$$\Lambda(D_A^{-1}\varphi) - \Lambda(\varphi) = \int_{|\det(vz)|<1} \frac{\varphi(v,\frac{v\cdot z}{|v|^2}v)-\varphi(v,\frac{Av\cdot Az}{|Av|^2}v)}{|\det(vz)|}\,\mathrm{d}v\,\mathrm{d}z;$$

4. $A^{-1} = (1+\varepsilon)E^{1,1} + \frac{1}{1+\varepsilon}E^{2,2}$, then

$$\Lambda(D_A^{-1}\varphi) - \Lambda(\varphi) = \int_{|\det(vz)|<1} \frac{\varphi(v, \frac{v\cdot z}{|v|^2}v) - \varphi(v, \frac{Av\cdot Az}{|Av|^2}v)}{|\det(vz)|}\,\mathrm{d}v\,\mathrm{d}z.$$

If the function in (2.9) was extendible to an element of $X^2_{-1,\text{even}}$, then it would differ from $\Lambda$ for a tempered distribution $\Gamma$ supported on $M$. In particular this last one should take care of the "lack" for being homogeneous of degree $-1$ of $\Lambda$, i.e.

$$\Lambda(D_A^{-1}\varphi) - \Lambda(\varphi) = \Gamma(\varphi) - \Gamma(D_A^{-1}\varphi).$$

Therefore we want to differentiate the equation in the four cases:

1. in the first case,

$$\begin{aligned}
-\int_{1\le|\det(vz)|<(1+\varepsilon)^2} \frac{\varphi(v,z)}{|\det(vz)|}\,\mathrm{d}v\,\mathrm{d}z = \\
&= -\int_{v\in\mathbb{R}^2\setminus\{0\}}\int_{a\in\mathbb{R}}\int_{1\le|b|<(1+\varepsilon)^2} \frac{\varphi(v,av)}{|b|}\,\mathrm{d}b\,\mathrm{d}a\,\mathrm{d}v = \\
&= -\int_{v\in\mathbb{R}^2\setminus\{0\}}\int_{a\in\mathbb{R}} 2\log(1+\varepsilon)^2\varphi(v,av)\,\mathrm{d}a\,\mathrm{d}v = \\
&= -\int_{w\in\mathbb{R}^2\setminus\{0\}}\int_{\theta\in\mathbb{R}/2\pi\mathbb{Z}} \log(1+\varepsilon)^2\varphi\left(w\cos\theta \quad w\sin\theta\right)\mathrm{d}\theta\,\mathrm{d}w,
\end{aligned}$$

where we performed the following two changes of variables:

- for $v \in \mathbb{R}^2 \setminus \{0\}$ fixed,

$$z = av + b\frac{\widetilde{v}}{|v|^2},\ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} v_1 & -\frac{v_2}{|v|^2} \\ v_2 & -\frac{v_1}{|v|^2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix},\ a, b \in \mathbb{R}; \tag{2.10}$$

- $v = w\cos\theta,\ av = w\sin\theta,\ w \in \mathbb{R}^2 \setminus \{0\}, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, for which

$$J_{v,a}(w,\theta) = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ -w_1\sin\theta & -w_2\sin\theta & \frac{1}{\cos^2\theta} \end{pmatrix}.$$

Therefore

$$\Gamma(\widetilde{\varphi}) - \Gamma((1+\varepsilon)^2\widetilde{\varphi}_{A^{-1}}) = -\iint \log(1+\varepsilon)^2\widetilde{\varphi}\,\mathrm{d}w\,\mathrm{d}\theta.$$

Dividing both terms of the inequality by $\varepsilon$ and letting $\varepsilon$ go to 0 yields

$$\Gamma(2\widetilde{\varphi} + \partial_1\widetilde{\varphi}w_1 + \partial_2\widetilde{\varphi}w_2) = -2\iint \widetilde{\varphi}\,\mathrm{d}w\,\mathrm{d}\theta,$$

in particular we have the following equation in the sense of distributions on $(\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z})$

$$-w_1\partial_1\Gamma - w_2\partial_2\Gamma = 2;$$

2. in the second case, the same argument as in the first case provides

$$\Gamma(\widetilde{\varphi}) - \Gamma((1+\varepsilon^2)\widetilde{\varphi}_{A^{-1}}) = -\iint \log(1+\varepsilon^2)\widetilde{\varphi}\,\mathrm{d}w\,\mathrm{d}\theta,$$

   and, by differentiating in $\varepsilon$, we have the following equation in the sense of distributions on $(\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z})$

$$w_2\partial_1\Gamma - w_1\partial_2\Gamma = 0;$$

3. in the third case we perform the same change of coordinates as in (2.10)

$$\int_{|\det(vz)|<1} \frac{\varphi\left(v, \frac{v\cdot z}{|v|^2}v\right) - \varphi\left(v, \frac{Av\cdot Az}{|Av|^2}v\right)}{|\det(vz)|}\,\mathrm{d}v\,\mathrm{d}z =$$
$$= \int_{v\in\mathbb{R}^2\setminus\{0\}}\int_{a\in\mathbb{R}}\int_{|b|<1} \frac{\varphi\left(v, \frac{v\cdot z}{|v|^2}v\right) - \varphi\left(v, \left(a + b\frac{\varepsilon v_1^2 - \varepsilon v_2^2 + \varepsilon^2 v_1 v_2}{|v|^2((v_1+\varepsilon v_2)^2 + v_2^2)}\right)v\right)}{|b|}\,\mathrm{d}b\,\mathrm{d}a\,\mathrm{d}v.$$

   Differentiating inside the integral yields

$$\int_{v\in\mathbb{R}^2\setminus\{0\}}\int_{a\in\mathbb{R}}\int_{|b|<1} -\left(\frac{\partial_3\varphi(v,av)b\frac{v_1^2-v_2^2}{|v|^4}v_1}{|b|} + \frac{\partial_4\varphi(v,av)b\frac{v_1^2-v_2^2}{|v|^4}v_2}{|b|}\right)\mathrm{d}b\,\mathrm{d}a\,\mathrm{d}v.$$

   We finally observe that in both of summands we have the dependence on $b$ given by $\frac{b}{|b|} = \mathrm{sgn}(b)$ and we are integrating over the interval $|b| < 1$ symmetric with respect to the origin.
   Therefore we have the following equation in the sense of distributions on $(\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z})$

$$w_2\partial_1\Gamma = 0;$$

4. in the fourth case, the same argument as in the third case provides the following equation in the sense of distributions on $(\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z})$

$$w_1\partial_1\Gamma - w_2\partial_2\Gamma = 0.$$

Therefore, away from the origin, $\Gamma$ has to satisfy the following system of equations in the sense of distributions,

$$\begin{cases} -w_1\partial_1\Gamma - w_2\partial_2\Gamma = 2 \\ w_2\partial_1\Gamma + w_1\partial_2\Gamma = 0 \\ w_2\partial_1\Gamma = 0 \\ w_1\partial_1\Gamma - w_2\partial_2\Gamma = 0, \end{cases}$$

which is equivalent, through algebraic manipulations, to

$$\begin{cases} w_1\partial_1\Gamma = -1 \\ w_2\partial_2\Gamma = -1 \\ w_2\partial_1\Gamma = 0 \\ w_1\partial_2\Gamma = 0. \end{cases}$$

It has no solutions. In fact, for example for $\psi \in \mathcal{D}(\{w \in \mathbb{R}^2 \colon w_1 w_2 > 0\} \times (\mathbb{R}/2\pi\mathbb{Z}))$, in general we have

$$0 = w_2 \partial_1 \Gamma(\psi) = -w_1 \partial_1 \Gamma\left(-\frac{w_2}{w_1}\psi\right) = -\int_0^{2\pi}\int_{\mathbb{R}^2} \frac{w_2}{w_1}\psi(w,\theta)\,\mathrm{d}w\,\mathrm{d}\theta \neq 0.$$

A priori our $\Gamma$ only depends on the values of $\varphi$ and its tangential derivatives with respect to $M$. However we can easily observe that the lack of $\Lambda$ from being in $X^2_{-1}$ only depends on the values of the function on $M$. Therefore assuming a distribution locally of the form

$$\Gamma(\varphi) = \sum_{i=0}^{n} \Gamma_i(\partial_N^i \varphi),$$

the computations we made in studying $\alpha$-homogeneous distributions supported on $M$ implies that $n = -1 - \alpha = 0$. In fact, for $\psi \in \mathcal{D}((\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z}))$ such that $\psi(w,\theta) = \psi(-w,\theta+\pi)$, for $i > 0$, let $\phi_i$ such that $(\partial_N^j \phi_i)^\sim = 0$ for $j \neq i$, $(\partial_N^i \phi_i)^\sim = \psi$. Then, for every matrix $A \in \mathrm{GL}_2^+(\mathbb{R})$,

$$\Lambda(\phi_i) = \Lambda(D_A^{-1}\phi_i),$$

so that $\Gamma_i(\psi) = 0$ for every $\psi \in \mathcal{D}((\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}/2\pi\mathbb{Z}))$ such that $\psi(w,\theta) = \psi(-w,\theta+\pi)$. □

We are ready to prove the classification theorem for $n = 2$.

*Proof of Thm. 5.* By Lemma 21, without loss of generality we can assume $\mathrm{Re}(\alpha) \geq -1$.
We distinguish two cases:

- $X^2_{\alpha,\mathrm{even}}$, $X^2_{\alpha,\mathrm{odd}}$ for $\mathrm{Re}(\alpha) \geq -1$, $\alpha \neq -1$, and $X^2_{-1,\mathrm{odd}}$;
- $X^2_{-1,\mathrm{even}}$.

In the first case, the spaces have dimension at least 1 by Theorem 30, which a posteriori provides the explicit generators, and they have dimension 1 by Theorem 43.

In the second one, let $\Lambda \in X^2_{-1,\mathrm{even}}$. By Proposition 31, away from $M$, $\Lambda$ agrees with the function $c|\det V|^{-1}$, for some $c \in \mathbb{C}$. By Proposition 44, we have $c = 0$. Hence $\mathrm{supp}(\Lambda) \subset M$, and by Lemma 39 we conclude the classification result. □

### 2.2.4 Corollaries

**Corollary 45.** *Let $\alpha \in \mathbb{C}$, $\Lambda \in X^2_\alpha$, $\Lambda \neq 0$. Then* $\mathrm{supp}(\Lambda) \subset M$ *if and only if one of the following conditions holds:*

- $\alpha \in \mathbb{Z}$, $\alpha \leq -2$;
- $\alpha = -1$, *and* $\Lambda \in X^2_{-1,\mathrm{even}}$.

*Proof.* The necessary condition follows from Lemma 39 and Lemma 40.
The sufficient condition is proven by the following argument.
By Theorem 43, the spaces $X^2_{\alpha,\mathrm{even}}$, $X^2_{\alpha,\mathrm{odd}}$, for $\alpha \in \mathbb{Z}$, $\alpha \leq -2$, have dimension 1.

For $\alpha = -2$, we have two explicit generators, namely the Dirac delta function $\delta$ for $X^2_{\alpha,\text{even}}$, the tempered distribution $\Gamma$ defined in (2.8) for $X^2_{\alpha,\text{odd}}$. They both have support inside $M$.

By Lemma 21 and Theorem 43, $\widehat{\delta} = 1 \in X^2_{0,\text{even}}$ and $\widehat{\Gamma} = c\,\text{sgn}(\det V) \in X^2_{0,\text{odd}}$, for $c \in \mathbb{C} \setminus \{0\}$. For $k \in \mathbb{N}$, the generators of $X^2_{k,\text{even}}$, $X^2_{k,\text{odd}}$ are defined by applying $k$ times the map described in Lemma 22 to the tempered distributions associated to the functions 1 and $\text{sgn}(\det V)$. By (1.6), generators of $X^2_{-2-k,\text{even}}$, $X^2_{-2-k,\text{odd}}$ are obtained by applying $k$ times the map described in Lemma 23 to $\delta$ and $\Gamma$. To conclude, we observe that $\text{supp}(\det(\partial_{i,j})\Lambda) \subset \text{supp}(\Lambda)$.

The claim about $\Lambda \in X^2_{-1,\text{even}}$ follows by Theorem 5. □

**Corollary 46.** *The Fourier transform of the tempered distribution $\Lambda$ defined in* (2) *is $-\Lambda$. For $F \in \mathcal{D}'(\mathbb{R}/\pi\mathbb{Z})$, the Fourier transform of $\mu(F)$ is*

$$\widehat{\mu(F)}(\varphi) = F(\tau\psi_\varphi),$$

*where $\tau\psi_\varphi(\theta) = \psi_\varphi(\theta - \frac{\pi}{2})$.*

*Proof.* The first claim follows by the fact that $\Lambda \in X^2_{-1,\text{odd}}$, and this space has dimension 1 by Theorem 43. Since we have $\widehat{\Lambda} \in X^2_{-1,\text{odd}}$ by Lemma 21, thus there exists $c \in \mathbb{C} \setminus \{0\}$ such that $\widehat{\Lambda} = c\Lambda$. Let $\varphi(x, y) = \det(x\ y)e^{-\pi|x,y|^2}$, for which $\widehat{\varphi} = -\varphi$. Then

$$c\Lambda(\varphi) = \widehat{\Lambda}(\varphi) = \Lambda(\widehat{\varphi}) = -\Lambda(\varphi),$$

and $c = -1$.

For the second claim, we use the well known result that integrating a function on a subspace is equal to integrating the Fourier transform of the function on the perpendicular subspace. Since

$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta,\ \sin\left(\theta - \frac{\pi}{2}\right) = -\cos\theta,$$

we have

$$\nu\left((\mathbb{R}^2 \setminus \{0\}) \times \{\theta\}\right) \perp_{\text{HS}} \nu\left((\mathbb{R}^2 \setminus \{0\}) \times \left\{\theta - \frac{\pi}{2}\right\}\right).$$

Thus

$$\begin{aligned}\widehat{\mu(F)}(\varphi) &= F\left(\int_{\mathbb{R}^2} \widehat{\varphi}\left(w\cos\theta \quad w\sin\theta\right) \mathrm{d}w\right) = \\ &= F\left(\int_{\mathbb{R}^2} \varphi\left(w\cos\left(\theta - \tfrac{\pi}{2}\right) \quad w\sin\left(\theta - \tfrac{\pi}{2}\right)\right) \mathrm{d}w\right) = F(\tau\psi_\varphi).\end{aligned}$$

□

**Corollary 47.** *For $\alpha \in \mathbb{C}$, the map*

$$\det(\partial_{i,j})\colon X^2_\alpha \to X^2_{\alpha-1}$$

*defines a bijection with inverse*

$$\frac{\det V}{\alpha(\alpha+1)}\colon X^2_{\alpha-1} \to X^2_\alpha,$$

*except for the cases:*

- $\alpha = 0$, *then*
$$\det(\partial_{i,j})(1) = 0,\ \det(\partial_{i,j})(\operatorname{sgn}(\det V)) = 8\pi^2 c\mu(1),$$
*where* $c \in \mathbb{C} \setminus \{0\}$, *but the map is not surjective;*

- $\alpha = -1$, *then*
$$\det(\partial_{i,j})\left(\text{p. v.}\,\frac{1}{\det V}\right) = 4\pi^2\delta,\ \det(\partial_{i,j})(\mu(F)) = 0$$
*for every* $F \in \mathcal{D}'(\mathbb{R}/\pi\mathbb{Z})$.

*Proof.* By Lemma 21 and (1.6), it is enough to consider the case $\operatorname{Re}(\alpha) \geq -1$.

Let $\alpha \notin \{-1, 0\}$. By Theorem 5, the spaces $X^2_\alpha$ have dimension 2, and we show generators whose support is $\mathbb{R}^{2\times 2}$. Away from $M$, as seen in the proof of Lemma 26,

$$\det(\partial_{i,j})|\det V|^\alpha = \alpha(\alpha+1)\operatorname{sgn}(\det V)|\det V|^{\alpha-1}.$$

Multiplying by $\det V$ we obtain $\alpha(\alpha+1)|\det V|^\alpha$. Thus the maps are bijections, and one is the inverse of the other. An analogous argument holds for $\operatorname{sgn}(\det V)|\det V|^\alpha$.

For $\alpha = 0$, we have trivially $\det(\partial_{i,j})(1) = 0$. On the other hand, let $\Gamma$ be the tempered distribution defined in (2.8). By Corollary 46 and (1.6) we have

$$\det(\partial_{i,j})(\operatorname{sgn}(\det V)) = \det(\partial_{i,j})(c\widehat{\Gamma}) = -4\pi^2 c(\det V\Gamma)^\frown = 8\pi^2 c\widehat{\mu(1)} = 8\pi^2 c\mu(1),$$

where $c \in \mathbb{C}\setminus\{0\}$ since the Fourier transform of $\operatorname{sgn}(\det V)$ is a nonzero element of $X^2_{-2,\text{odd}}$, which is generated by $\Gamma$. To prove the second to last equality, we observe

$$(\partial_N\overline{\varphi})^\sim(w,\theta) = (\overline{\partial_N\varphi})^\sim(w,\theta) + \lim_{\varepsilon\to 0}\frac{\det\left(\nu(w,\theta)+\varepsilon\overrightarrow{N}(w,\theta)\right)}{\varepsilon}\widetilde{\varphi}(w,\theta) =$$
$$= 0 + \lim_{\varepsilon\to 0}\frac{\det\begin{pmatrix} w_1\cos\theta - \varepsilon\frac{w_2}{|w|}\sin\theta & w_1\sin\theta + \varepsilon\frac{w_2}{|w|}\cos\theta \\ w_2\cos\theta + \varepsilon\frac{w_1}{|w|}\sin\theta & w_2\sin\theta - \varepsilon\frac{w_1}{|w|}\cos\theta\end{pmatrix}}{\varepsilon}\widetilde{\varphi}(w,\theta) = -|w|\widetilde{\varphi}(w,\theta),$$

thus

$$(\det V\Gamma)^\frown(\varphi) = \Gamma(\overline{\widehat{\varphi}}) = \int_0^{2\pi}\int_{\mathbb{R}^2}\frac{(\partial_N\overline{\widehat{\varphi}})^\sim(w,\theta)}{|w|}\,\mathrm{d}w\,\mathrm{d}\theta =$$
$$= \int_0^{2\pi}\int_{\mathbb{R}^2}\frac{-|w|\widetilde{\widehat{\varphi}}(w,\theta)}{|w|}\,\mathrm{d}w\,\mathrm{d}\theta = -2(\mu(1))(\widehat{\varphi}) = -2\widehat{\mu(1)}(\varphi).$$

The map is clearly not surjective because it goes from a 1-dimensional space to an infinite-dimensional one.

For $\alpha = -1$, let $\Lambda$ be the tempered distribution defined in (2). By Corollary 46 and (1.6) we have

$$\det(\partial_{i,j})\Lambda = -\det(\partial_{i,j})\widehat{\Lambda} = 4\pi^2\,(\det V\Lambda)^\frown = 4\pi^2\widehat{1} = 4\pi^2\delta.$$

On the other hand,

$$\det(\partial_{i,j})(\mu(F)) = \det(\partial_{i,j})(\widehat{\mu(F)})^\frown = -4\pi^2(\det V\widehat{\mu(F)})^\frown = 0,$$

since for every $\varphi \in \mathcal{S}(\mathbb{R}^{2\times 2})$, then $\overline{\varphi}$ is zero on $M$. $\square$

## 2.3 Proof of Lemma 6

*Proof.* To compute the constant in the first case, we choose the function

$$\varphi(x,y) = e^{-\pi|x,y|^2},$$

for which

$$\widehat{\varphi}(\xi,\eta) = e^{-\pi|\xi,\eta|^2},$$
$$\det(\partial_{i,j})\varphi(x,y) = 4\pi^2 \det(x\ y) e^{-\pi|x,y|^2}.$$

To compute the constant in the second case, we choose the function

$$\phi(x,y) = \det(x\ y) e^{-\pi|x,y|^2},$$

for which

$$\widehat{\phi}(\xi,\eta) = -\det(\xi\ \eta) e^{-\pi|\xi,\eta|^2},$$
$$\det(\partial_{i,j})\phi(x,y) = 4\pi^2 (\det(x\ y))^2 e^{-\pi|x,y|^2} - 2\pi|x,y|^2 e^{-\pi|x,y|^2} + 2e^{-\pi|x,y|^2}.$$

We need to compute the value of some integrals, namely

**Lemma 48.** *For $\beta \in \mathbb{C}$, $\mathrm{Re}(\beta) > -1$, then*

$$\int_{\mathbb{R}^{2\times 2}} |\det(x\ y)|^\beta e^{-\pi|x,y|^2}\,\mathrm{d}x\,\mathrm{d}y = \pi^{-1-\beta}\Gamma\left(\frac{\beta+2}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right),$$
$$\int_{\mathbb{R}^{2\times 2}} |\det(x\ y)|^\beta |x,y|^2 e^{-\pi|x,y|^2}\,\mathrm{d}x\,\mathrm{d}y = \pi^{-2-\beta}(\beta+2)\Gamma\left(\frac{\beta+2}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right).$$

Then, for $-2 < \mathrm{Re}\,(\alpha) < -1$, in the first case we have

$$(I^2_{-\alpha-2,\mathrm{even}})\widehat{\ }(\varphi) = \int_{\mathbb{R}^{2\times 2}} |\det(\xi\ \eta)|^{-\alpha-2}\widehat{\varphi}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta =$$
$$= \int_{\mathbb{R}^{2\times 2}} |\det(\xi\ \eta)|^{-\alpha-2} e^{-\pi|\xi,\eta|^2}\,\mathrm{d}\xi\,\mathrm{d}\eta = \pi^{\alpha+1}\ \Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{\alpha+1}{2}\right),$$

$$(\det(\partial_{i,j}) I^2_{\alpha+1,\mathrm{odd}})(\varphi) = \int_{\mathbb{R}^{2\times 2}} |\det(x\ y)|^{\alpha+1}\operatorname{sgn}(\det(x\ y))\det(\partial_{i,j})\varphi(x,y)\,\mathrm{d}x\,\mathrm{d}y =$$
$$= 4\pi^2 \int_{\mathbb{R}^{2\times 2}} |\det(x\ y)|^{\alpha+2} e^{-\pi|x,y|^2}\,\mathrm{d}x\,\mathrm{d}y = 4\pi^{-\alpha-1}\ \Gamma\left(\frac{\alpha+4}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\alpha+3}{2}\right).$$

In the second case we have

$$(I^2_{-\alpha-2,\mathrm{odd}})\widehat{\ }(\phi) = \int_{\mathbb{R}^{2\times 2}} \operatorname{sgn}(\det(\xi\ \eta))|\det(\xi\ \eta)|^{-\alpha-2}\widehat{\phi}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta =$$
$$= -\int_{\mathbb{R}^{2\times 2}} |\det(\xi\ \eta)|^{-\alpha-1} e^{-\pi|\xi,\eta|^2}\,\mathrm{d}\xi\,\mathrm{d}\eta = -\pi^{\alpha}\ \Gamma\left(-\frac{\alpha-1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{\alpha}{2}\right) =$$
$$= \frac{\alpha+1}{2}\pi^{\alpha}\ \Gamma\left(-\frac{\alpha+1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{\alpha}{2}\right),$$

$$
\begin{aligned}
&(\det(\partial_{i,j}) I^2_{\alpha+1,\mathrm{even}})(\phi) = \int_{\mathbb{R}^{2\times 2}} |\det(x\ y)|^{\alpha+1} \det(\partial_{i,j})\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y = \\
&= \int_{\mathbb{R}^{2\times 2}} |\det(x\ y)|^{\alpha+1} (4\pi^2 (\det(x\ y))^2 e^{-\pi|x,y|^2} - 2\pi |x,y|^2 e^{-\pi|x,y|^2} + 2e^{-\pi|x,y|^2})\,\mathrm{d}x\,\mathrm{d}y = \\
&= \pi^{-\alpha-2}\Bigg[4\Gamma\left(\frac{\alpha+5}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\alpha+4}{2}\right) - 2(\alpha+3)\Gamma\left(\frac{\alpha+3}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\alpha+2}{2}\right) + \\
&\qquad\qquad + 2\pi^{-\alpha-2}\Gamma\left(\frac{\alpha+3}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\alpha+2}{2}\right)\Bigg] = \\
&= 2(\alpha+1)\pi^{-\alpha-2}\,\Gamma\left(\frac{\alpha+4}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\alpha+3}{2}\right),
\end{aligned}
$$

where we used the property, for $\mathrm{Re}(\beta) > -1$,

$$\beta\Gamma(\beta) = \Gamma(\beta+1). \tag{2.11}$$

The functions defined on the two open half planes are holomorphic because:

- the exponential function is holomorphic on $\mathbb{C}$;
- the function $\Gamma$ is holomorphic and nonzero on $\{\beta \in \mathbb{C} \colon \mathrm{Re}(\beta) > 0\}$;
- for a fixed function $\varphi \in \mathcal{S}(\mathbb{R}^{2\times 2})$, the function

  $$f \colon \{\mathrm{Re}(\alpha) < -1\} \to \mathbb{C},\ f(\alpha) \coloneqq \int_{\mathbb{R}^{2\times 2}} |\det \Upsilon|^{-\alpha-2} \widehat{\varphi}(\Upsilon)\,\mathrm{d}\Upsilon,$$

  is continuous and complex-differentiable. Let

  $$g \colon \{\mathrm{Re}(\alpha) < -1\} \to \mathbb{C},\ g(\alpha) \coloneqq \int_{\mathbb{R}^{2\times 2}} |\det \Upsilon|^{-\alpha-2} \lg(|\det \Upsilon|^{-1}) \widehat{\varphi}(\Upsilon)\,\mathrm{d}\Upsilon.$$

  For every $\alpha$ in the domain, $g(\alpha)$ is finite. In fact, for $\alpha \in \{\mathrm{Re}(\alpha) < -1\}$ there exists $\varepsilon > 0$ such that $\alpha + \varepsilon \in \{\mathrm{Re}(\alpha) < -1\}$. Then, $|\det \Upsilon|^{\varepsilon} \lg(|\det \Upsilon|^{-1})$ is bounded when $|\det \Upsilon| \leq 1$ and $|\det \Upsilon|^{\varepsilon} \lg(|\det \Upsilon|) \leq C |\det \Upsilon|^{2\varepsilon}$ when $|\det \Upsilon| \geq 1$. To conclude we observe that

  $$f(\alpha) - f(\alpha_0) = g(\alpha_0)(\alpha - \alpha_0) + o(|\alpha - \alpha_0|),$$

  by Lebesgue Dominated Convergence Theorem.
  The same argument can be used to prove holomorphicity of the function

  $$h \colon \{\mathrm{Re}(\alpha) > -2\} \to \mathbb{C},\ h(\alpha) \coloneqq \int_{\mathbb{R}^{2\times 2}} \mathrm{sgn}(\det V) |\det V|^{\alpha+1} \det(\partial_{i,j})\varphi(V)\,\mathrm{d}V.$$

The two functions coincide on the intersection, therefore they define a holomorphic function on $\mathbb{C}$. An analogous argument can be used for the other claim about holomorphicity. □

*Proof of Lemma 48.* For $\mathrm{Re}(\beta) > -1$ the integral is well-defined by Lemma 24.

$$
\begin{aligned}
&\int_{\mathbb{R}^{2\times 2}} |\det(x\ y)|^{\beta} e^{-\pi|x,y|^2}\,\mathrm{d}x\,\mathrm{d}y = 2\int_{M_+} (\det(x\ y))^{\beta} e^{-\pi|x,y|^2}\,\mathrm{d}x\,\mathrm{d}y = \\
&\qquad = 2\int_{x \in \mathbb{R}^2 \setminus \{0\}} \int_{\{y \in \mathbb{R}^2 \colon (x,y) \in M_+\}} (\det(x\ y))^{\beta} e^{-\pi|y|^2}\,\mathrm{d}y\ e^{-\pi|x|^2}\,\mathrm{d}x.
\end{aligned}
$$

For a fixed $x \in \mathbb{R}^2 \setminus \{0\}$, we change the variable $y$ into

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{|x|} \begin{pmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

We have $|y|^2 = y_1^2 + y_2^2 = a^2 + b^2$, $\det(x\ y) = b|x|$, $\mathrm{d}y_1\,\mathrm{d}y_2 = \mathrm{d}a\,\mathrm{d}b$. Moreover, $(x, y) \in M_+$ if and only if $\{(a, b) \in \mathbb{R} \times (0, \infty)\}$. Then,

$$\int_{\mathbb{R}^{2\times 2}} |\det(x\ y)|^\beta e^{-\pi|x,y|^2}\,\mathrm{d}x\,\mathrm{d}y = 2\int_{\mathbb{R}^2\setminus\{0\}} \int_{\mathbb{R}} \int_0^\infty (b|x|)^\beta e^{-\pi(a^2+b^2)}\,\mathrm{d}b\,\mathrm{d}a\ e^{-\pi|x|^2}\,\mathrm{d}x.$$

We change the variable $x$ into spherical coordinates. Thus,

$$\begin{aligned}
\int_{\mathbb{R}^{2\times 2}} |\det(x\ y)|^\beta e^{-\pi|x,y|^2}\,\mathrm{d}x\,\mathrm{d}y &= 8\pi \int_0^\infty \int_0^\infty \int_0^\infty (br)^\beta e^{-\pi b^2}\,\mathrm{d}b\ e^{-\pi a^2}\,\mathrm{d}a\ e^{-\pi r^2} r\,\mathrm{d}r = \\
&= 8\pi \int_0^\infty r^{\beta+1} e^{-\pi r^2}\,\mathrm{d}r \int_0^\infty e^{-\pi a^2}\,\mathrm{d}a \int_0^\infty b^\beta e^{-\pi b^2}\,\mathrm{d}b = \\
&= 8\pi\pi^{-\frac{\beta}{2}} \frac{1}{2\pi} \int_0^\infty t^{\frac{\beta}{2}} e^{-t}\,\mathrm{d}t\ \pi^{\frac{1}{2}} \frac{1}{2\pi} \int_0^\infty u^{-\frac{1}{2}} e^{-u}\,\mathrm{d}u\ \pi^{-\frac{\beta-1}{2}} \frac{1}{2\pi} \int_0^\infty s^{\frac{\beta-1}{2}} e^{-s}\,\mathrm{d}s = \\
&= \pi^{-1-\beta} \Gamma\left(\frac{\beta+2}{2}\right) \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{\beta+1}{2}\right).
\end{aligned}$$

For the second claim we proceed with the same changes of variables, obtaining

$$\begin{aligned}
\int_{\mathbb{R}^{2\times 2}} &|\det(x\ y)|^\beta |x,y|^2 e^{-\pi|x,y|^2}\,\mathrm{d}x\,\mathrm{d}y = \\
&= 8\pi \int_0^\infty \int_0^\infty \int_0^\infty (br)^\beta (r^2 + a^2 + b^2) e^{-\pi b^2}\,\mathrm{d}b\ e^{-\pi a^2}\,\mathrm{d}a\ e^{-\pi r^2} r\,\mathrm{d}r = \\
&= \pi^{-2-\beta} \Bigg[ \Gamma\left(\frac{\beta+4}{2}\right) \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{\beta+1}{2}\right) + \Gamma\left(\frac{\beta+2}{2}\right) \Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{\beta+1}{2}\right) + \\
&\qquad + \Gamma\left(\frac{\beta+2}{2}\right) \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{\beta+3}{2}\right) \Bigg] = \\
&= \pi^{-2-\beta} (\beta+2) \Gamma\left(\frac{\beta+2}{2}\right) \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{\beta+1}{2}\right).
\end{aligned}$$

In the last equality we used the property (2.11). □

## 2.4 Proof of Corollary 8

*Proof.* We prove the uniqueness of the Fourier transform of the tempered distribution $\Lambda \in \mathcal{S}'(\mathbb{R}^6)$ defined in (3) in the Introduction. First of all, we study the properties of $\widehat{\Lambda}$ corresponding to the ones of $\Lambda$:

- the modulation invariance is equivalent to the translation invariance in the space of frequencies. For a vector $b \in \mathbb{R}^2$, a function $\varphi \in \mathcal{S}(\mathbb{R}^6)$, define

$$T_b \widehat{\varphi}(\xi, \eta, \zeta) = \widehat{\varphi}(\xi - b, \eta - b, \zeta - b) = \widehat{M_b \varphi}(\xi, \eta, \zeta).$$

Then

$$\widehat{\Lambda}(T_b\varphi) = \Lambda(\widehat{T_b\varphi}) = \Lambda(M_{-b}\widehat{\varphi}) = \Lambda(\widehat{\varphi}) = \widehat{\Lambda}(\varphi),$$

for every vector $b \in \mathbb{R}^2$ and every function $\varphi \in \mathcal{S}(\mathbb{R}^6)$,

- the invariance (4) for $A \in \mathrm{GL}_2(\mathbb{R})$ is equivalent to the property

$$\widehat{\Lambda}\left(\frac{1}{|\det A|}B_A\varphi\right) = \Lambda\left(\frac{1}{|\det A|}\widehat{B_A\varphi}\right) = \Lambda(B_{A^{-T}}\widehat{\varphi}) = \Lambda(\widehat{\varphi}) = \widehat{\Lambda}(\varphi),$$

for every matrix $A \in \mathrm{GL}_2(\mathbb{R})$ and every function $\varphi \in \mathcal{S}(\mathbb{R}^6)$.

We differentiate the equality associated to the translation invariance in the directions

$$\begin{aligned} v_1 &= (w_1, w_1, w_1), \\ v_2 &= (w_2, w_2, w_2), \end{aligned}$$

where $w_1, w_2 \in \mathbb{R}^2$, $w_1 = (1, 0)$, $w_2 = (0, 1)$. For $\varepsilon > 0$, we consider the equalities

$$\widehat{\Lambda}(T_{\varepsilon v_i}\varphi) = \widehat{\Lambda}(\varphi), \qquad \text{for } i \in \{1, 2\},$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^6)$. Dividing by $\varepsilon$ and taking the limit as $\varepsilon$ goes to 0 we obtain

$$\partial_{v_i}\widehat{\Lambda} = 0, \qquad \text{for } i \in \{1, 2\}.$$

Therefore, by Lemma 18, we can assume $\widehat{\Lambda}$ to be of the form

$$\widehat{\Lambda}(\varphi) = F(\widetilde{\varphi}),$$

where $F \in \mathcal{S}'(\mathbb{R}^{2\times 2})$, and $\widetilde{\varphi} \in \mathcal{S}(\mathbb{R}^{2\times 2})$ is defined by

$$\widetilde{\varphi}\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} := \int_{\mathbb{R}^2} \varphi\left(\begin{pmatrix} a + x_1 + x_2 \\ b + y_1 + y_2 \end{pmatrix}, \begin{pmatrix} a - x_1 \\ b - y_1 \end{pmatrix}, \begin{pmatrix} a - x_2 \\ b - y_2 \end{pmatrix}\right) \mathrm{d}a\, \mathrm{d}b,$$

Now we observe that for a matrix $A \in \mathrm{GL}_2(\mathbb{R})$ and a function $\varphi \in \mathcal{S}'(\mathbb{R}^6)$,

$$\begin{aligned}
&\frac{1}{|\det A|}(B_A\varphi)^{\sim}\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \\
&= \frac{1}{\det A}\frac{1}{|\det A|}\int_{\mathbb{R}^2} \varphi\left(A^{-1}\begin{pmatrix} a + x_1 + x_2 \\ b + y_1 + y_2 \end{pmatrix}, A^{-1}\begin{pmatrix} a - x_1 \\ b - y_1 \end{pmatrix}, A^{-1}\begin{pmatrix} a - x_2 \\ b - y_2 \end{pmatrix}\right) \mathrm{d}a\, \mathrm{d}b = \\
&= \frac{1}{\det A}\int_{\mathbb{R}^2} \varphi\left(\begin{pmatrix} s \\ t \end{pmatrix} + A^{-1}\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}, \begin{pmatrix} s \\ t \end{pmatrix} - A^{-1}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} s \\ t \end{pmatrix} - A^{-1}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) \mathrm{d}s\, \mathrm{d}t = \\
&= \frac{1}{\det A}\widetilde{\varphi}\left(A^{-1}\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\right) = D_A^{-1}\widetilde{\varphi}\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}.
\end{aligned}$$

Hence, we have

$$F(D_A^{-1}\widetilde{\varphi}) = F\left(\frac{1}{|\det A|}(B_A\varphi)^{\sim}\right) = \widehat{\Lambda}\left(\frac{1}{|\det A|}B_A\varphi\right) = \widehat{\Lambda}(\varphi) = F(\widetilde{\varphi}).$$

Now let $\eta \in \mathcal{D}(\mathbb{R}^2)$ so that $\operatorname{supp}(\eta) \subset B_1(0)$, $\int_{\mathbb{R}^2} \eta = 1$. Then, for every $\phi \in \mathcal{S}(\mathbb{R}^{2\times 2})$, there exists $\varphi \in \mathcal{S}(\mathbb{R}^6)$ such that

$$\phi \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \widetilde{\varphi} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix},$$

namely

$$\varphi(x + y + av_1 + bv_2) = \phi \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \eta(a, b).$$

The function is clearly smooth, while for the boundedness of the Schwartz norms is enough to observe that, on the support of $\phi$,

$$|x + y + av_1 + bv_2|^2 \leq 2(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_1x_2 + y_1y_2 + 1).$$

Thus the equality above implies that

$$F(D_A^{-1}\phi) = F(\phi),$$

for every matrix $A \in \mathrm{GL}_2(\mathbb{R})$ and every function $\phi \in \mathcal{S}(\mathbb{R}^{2\times 2})$.
Therefore $F \in X^2_{-1,\mathrm{odd}}$, and we conclude uniqueness up to multiplication by a constant by Theorem 5. $\square$

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