Exercise Session 12

① (a) Let A be a 2-dim. normal noeth. local integral domain and $a, b \in A$ s.t. $A/(a,b)$ is artinian. Then

$$\operatorname{len}_A A/(a,b) = \sum_{\substack{\mathfrak{p} \subseteq A \\ ht\, 1}} \operatorname{len}_{A_\mathfrak{p}}(A_\mathfrak{p}/aA_\mathfrak{p}) \cdot \operatorname{len}_{A/\mathfrak{p}}(A/\mathfrak{p}+bA)$$

In other words, show that $(A/a)[b] = 0$.

- Need to show:
  $\{x \in A \mid xb \in (a)\} \subseteq (a)$

- For all $\mathfrak{p} \in \operatorname{Spec} A$ of height 1, we must have $a \notin \mathfrak{p}$ or $b \notin \mathfrak{p}$.
  (otherwise $\dim A/(a,b) \geq 1$)

- Let $\mathfrak{p} \in \operatorname{Spec} A$ of height 1, $x \in A$ s.t. $xb \in aA$. In $A_\mathfrak{p}$ either a or b is a unit. In both cases $x \in aA_\mathfrak{p}$

- A normal $\Rightarrow A = \bigcap_\mathfrak{p} A_\mathfrak{p} \rightsquigarrow aA = \bigcap_\mathfrak{p} aA_\mathfrak{p}$

  $\rightsquigarrow$ If $x \in A$ is s.t. $x \in aA_\mathfrak{p}$ then $x \in aA$.

(b) k field, X 2-dim. proper normal variety$/k$, $Z_1, Z_2 \subseteq X$ effective Cartier divisors whose intersection has dimension 0. Then

$$\mathcal{O}(Z_1) \cdot \mathcal{O}(Z_2) = \operatorname{len}(Z_1 \cap Z_2).$$

$Z_1 \cap Z_2 = \operatorname{Spec} A$ affine. Take $\dim_k A$.

$\mathrm{len}_{\mathcal{O}_{X,x}} \kappa(x) = 1 \neq \dim_k \kappa(x)$ in general

- By def., $\mathcal{O}(Z_i) = \mathcal{I}(Z_i)^\vee$. By linearity of the intersection product, we have

$$\mathcal{O}(Z_1)\cdot\mathcal{O}(Z_2) = (-1)^2\cdot \mathcal{I}(Z_1)\cdot\mathcal{I}(Z_2) = \mathcal{I}(Z_1)\cdot\mathcal{I}(Z_2).$$

- Pick $s_i \in \mathcal{I}(Z_i)^x_{\eta_X}$. Then

$$\mathcal{I}(Z_2)\cdot[X] = \mathrm{div}(s_2) = \sum_{\substack{x\in X\\ \mathrm{ht}\,1}} n_x\cdot[x]$$

$$\mathcal{I}(Z_1)\,\mathcal{I}(Z_2)\cdot[X] = \sum_{\substack{x\in X\\ \mathrm{ht}\,1}} n_x\cdot \mathrm{div}(s_1|_x)$$

For a careful choice of s_1, assuming X proj

- Reason why it should work: Choose locally generators $\mathcal{I}(Z_1) = (a)$, $\mathcal{I}(Z_2) = (b)$. Then for every $x\in X$, the coefficient of $\mathcal{O}(Z_1)\cdot\mathcal{O}(Z_2)\cdot[X]$ of x is

$$n_x \sum_{\substack{\mathfrak{p}\subseteq\mathcal{O}_{X,x}\\ \mathrm{ht}\,1}} \mathrm{len}_{\mathcal{O}_{X,\mathfrak{p}}}(\mathcal{O}_{X,\mathfrak{p}}/a)\cdot \mathrm{len}_{\mathcal{O}_{X,x}/\mathfrak{p}}(\mathcal{O}_{X,x}/(\mathfrak{p}+b\,\mathcal{O}_{X,x}))$$

$$\overset{(a)}{=} \mathrm{len}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(a,b)) = \mathrm{len}_{\mathcal{O}_{X,x}}(\mathcal{O}(Z_1\cap Z_2)_x)$$

$$\rightsquigarrow \mathcal{O}(Z_1)\cdot\mathcal{O}(Z_2) = \sum_{x\in X} n_x\cdot[\kappa(x):k].$$

(c) $F_1, F_2 \in k[x,y,z]$ homogeneous, $Z_i := V_+(F_i) \subseteq \mathbb{P}^2_k$. If $Z_1\cap Z_2$ has dim 0 then

$$\mathrm{len}(Z_1 \cap Z_2) = \deg F_1 \cdot \deg F_2$$

- Recall $\mathcal{O}(Z_i) = \mathcal{O}(\deg F_i)$.
- $\mathrm{len}(Z_1 \cap Z_2) \overset{(6)}{=} \mathcal{O}(\deg F_1) \cdot \mathcal{O}(\deg F_2)$

$$\overset{\text{linearity}}{=} \deg F_1 \cdot \deg F_2 \cdot \underbrace{\mathcal{O}(1) \cdot \mathcal{O}(1)}_{=1}$$

$\rightsquigarrow$ # of intersections of 2 generic lines

② Let $k = \bar{k}$.

(a) Let $f: X \to Y$ be a hom. of AV's$/k$. Endow $f(X)$ with the reduced scheme structure. Then f factors over $f(X)$ and $f(X)$ is an AV.

Lemma: Let X be a scheme, $Z \subseteq X$ closed subscheme, Y reduced scheme. Then $f: Y \to X$ factors over $Z \iff f(Y) \subseteq Z$ set-theoretically.

Proof: W.l.o.g. all schemes are affine. Rest is exercise. □

$\rightsquigarrow X \to Y$ factors over $f(X)$.

$f(X)$ is AV:

- $f(X)$ is proper (closed subscheme of proper scheme)
  connected (image of connected X)
- $f(X)$ is group scheme with restricted group structure of Y.

$\rightsquigarrow$ Need to see

$$\begin{tikzcd}
f(X) \times f(X) \arrow[r] \arrow[dr, dashed, "\exists!"'] & Y \times Y \arrow[r, "m"] & Y \\
& & f(X) \arrow[u]
\end{tikzcd}$$

By above lemma, enough to see $m(f(X) \times f(X)) \subseteq f(X)$ set-theoretically.

$\rightsquigarrow$ Can be checked on k-points, i.e. want to show that

$$\begin{tikzcd}
f(X)(k) \times f(X)(k) \arrow[r] \arrow[dr, dashed] & Y(k) \\
& f(X)(k) \arrow[u]
\end{tikzcd}$$

Follows from $f(k): X(k) \to Y(k)$ being grp hom with image $f(X)(k)$.

- $f(X)$ is smooth, as it is reduced + grp sch.

(b) $G \to H$ morphism of fin. grp sch. $/k$, $K := \ker(G \to H)$. Then

$$H = G/K \iff \deg G = \deg K \cdot \deg H \iff G \to H \text{ flat + surjective.}$$

- G/K is fin. grp. sch and $G \to G/K$ is flat + surj. of degree $\deg K$. (lecture on quotients). Also $G \to H$ factors over $G/K \to H$.

$\deg G = \deg K \cdot \deg G/K$

Check explicitly: $G/K \hookrightarrow H$ is closed immersion.

$\rightsquigarrow$ $G/K = H \iff \deg G/K = \deg H$

$\rightsquigarrow$ $H = G/K \iff \deg G = \deg K \cdot \deg G/K$ ✓

- Clear: $H = G/K \Rightarrow G \to H$ is flat + surjective (lecture).

• Assume $G \to H$ flat + surjective. Then $G/K \to H$ is flat, surj, closed immersion $\Rightarrow$ isom.

$\Rightarrow$ flat

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend right, "\text{flat}"'] & B \arrow[r, "\text{f.flat}"] & C \end{tikzcd}$$

(Alternatively, use

$$G \times_H G = K \times G$$

$(\deg G)^2 / \deg H$ $\qquad$ $\deg K \cdot \deg G$)