Paraproducts and stochastic integration

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Young integral

Differential equation driven by rough signal

Consider the equation

$$\frac{\mathrm{d}Z_t}{\mathrm{d}t} = F(Z_t)\frac{\mathrm{d}X_t}{\mathrm{d}t}. \tag{1}$$

We want to solve this equation
with input X_t that is not differentiable.
Formally (1) can be written as

$$\mathrm{d}Z_t = F(Z_t)\,\mathrm{d}X_t, \tag{2}$$

or more precisely as

$$Z_t = Z_0 + \int_0^t F(Z_t)\,\mathrm{d}X_t. \tag{ODE}$$

The integral above is a Riemann–Stieltjes integral:

$$\int_0^t F(Z_t)\,\mathrm{d}X_t = \lim_{\substack{0=t_0<\cdots<t_J=t \\ |t_{j+1}-t_j|\to 0}} \sum_{j=1}^{J} F(Z_{t_{j-1}})(X_j - X_{j-1}).$$

Fixed point argument

Existence and uniqueness of solutions are frequently proved using the following iterative procedure.
Start with a guess $Z^{(0)}$ for the solution.
Given $Z^{(k)}$, let $Y^{(k)} := F(Z^{(k)})$, and

$$Z_t^{(k+1)} = Z_0 + \int_0^t Y^{(k)} \, \mathrm{d}X_t.$$

This iteration should stay in some function space for it to be useful.
If X is continuous and has bounded variation:

$$V^1(X) := \sup_{t_0 < \cdots < t_J} \sum_{j=1}^{J} |X_{t_j} - X_{t_{j-1}}| < \infty,$$

then one suitable space are bounded continuous functions (if F is Lipschitz).

Bounded r-variation

We are interested in inputs X that are not of bounded variation (e.g. sample paths of Brownian motion).
How should we measure their regularity?
Since our ODE is parametrization-invariant, it is natural to use a parametrization-invariant space.

Definition

For $0 < r < \infty$ the *r-variation* of a sequence (X_t) is given by

$$V^r(X) := \sup_{t_0 < \cdots < t_J} \Big(\sum_{j=1}^{J} |X_{t_j} - X_{t_{j-1}}|^r \Big)^{1/r}. \qquad (V^r)$$

Basic properties of bounded r-variation

Example

Bounded r-variation is a parametrization-invariant version of $1/r$-Hölder continuity. Indeed, if X is defined on a bounded interval $[0, T]$ and $|X_s - X_t| \le C|s-t|^{1/r}$ for all s, t, then

$$\begin{aligned} V^r(X) &\le \sup_{t_0 < \cdots < t_J} \Big(\sum_{j=1}^{J} |C|t_j - t_{j-1}|^{1/r}|^r \Big)^{1/r} \\ &\le C \sup_{t_0 < \cdots < t_J} \Big(\sum_{j=1}^{J} |t_j - t_{j-1}| \Big)^{1/r} \\ &= CT^{1/r}. \end{aligned}$$

Lemma

$V^r(F \circ X) \le \|F\|_{Lip} V^r(X)$.

Discrete version

To avoid technical difficulties, we consider a difference equation that is a discrete analogue of our ODE:

$$Z_j - Z_{j-1} = F(Z_{j-1})(X_j - X_{j-1}). \qquad (\Delta\mathrm{E})$$

Setting $Y_j := F(Z_j)$, we obtain

$$Z_J = Z_0 + \sum_{0<j\leq J} Y_{j-1}(X_j - X_{j-1}).$$

We will ignore Z_0 and try to obtain estimates for the map $(X, Y) \mapsto Z$ given by

$$Z_J = \sum_{0<j\leq J} Y_{j-1}(X_j - X_{j-1}). \qquad (\Delta 1)$$

All estimates should be independent of the number of j's, so they can be transferred to the ODE.
The spaces should be invariant under composition with suitable F.

First paraproduct estimate

Lemma (E.R. Love and L.C. Young, 1936)

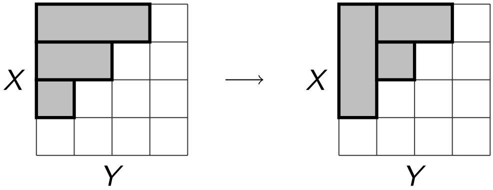
For $r < 2$ we have

$$\Big|\sum_{0<j\leq J}(Y_{j-1}-Y_0)(X_j-X_{j-1})\Big| \leq \zeta(2/r)V^r(Y)V^r(X). \qquad \text{(LY)}$$

The basic idea is that

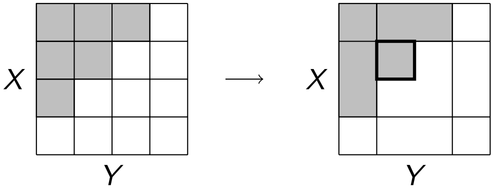
$$\sum_{0<j\leq J}(Y_{j-1}-Y_0)(X_j-X_{j-1}) = \sum_{0<i<j\leq J}(Y_i-Y_{i-1})(X_j-X_{j-1})$$

is a two-dimensional sum. But it can be much better to arrange this sum in a different collection of rectangles:



Inductive splitting of the paraproduct

The new partition is chosen inductively. First, choose a small square near the diagonal with the smallest contribution. After removing this square, the remaining summation region has a similar shape as before, but with J decreased by 1:



It remains to understand how small the contribution of a small square near the diagonal can be. Estimating the minimum by an average and using Hölder's inequality we obtain

$$
\begin{aligned}
&\inf_{0<k<J} |(Y_k - Y_{k-1})(X_{k+1} - X_k)| \\
&\qquad \le \Big((J-1)^{-1} \sum_{0<k<J} |(Y_k - Y_{k-1})(X_{k+1} - X_k)|^{r/2}\Big)^{2/r} \\
&\qquad \le (J-1)^{-2/r}\Big(\sum_{0<k<J} |Y_k - Y_{k-1}|^r\Big)^{1/r}\Big(\sum_{0<k<J} |X_{k+1} - X_k|^r\Big)^{1/r} \\
&\qquad \le (J-1)^{-2/r} V^r(Y) V^r(X).
\end{aligned}
$$

The hypothesis $r < 2$ is needed to ensure summability of the coefficients $(J-1)^{-2/r}$.

Mapping properties of the discrete Stieltjes integral

Corollary

Let Z_J be given by (Δ1). Then for $r < 2$ we have

$$V^r(Z) \leq (\|Y\|_\infty + C_r V^r(Y))V^r(X).$$

Proof: For any $J < J'$ we have

$$\begin{aligned} |Z_{J'} - Z_J| &= \Big| \sum_{J<j\leq J'} Y_{j-1}(X_j - X_{j-1}) \Big| \\ &= \Big| Y_J(X_{J'} - X_J) + \sum_{J<j\leq J'} (Y_{j-1} - Y_J)(X_j - X_{j-1}) \Big| \\ &\leq \|Y\|_\infty |X_{J'} - X_J| + C_r V^r(Y, [J, J'])V^r(X, [J, J']). \end{aligned}$$

Hence for any increasing sequence (J_l) we have

$$\Big(\sum_l |Z_{J_l} - Z_{J_{l-1}}|^r\Big)^{1/r} \leq \Big(\sum_l |Y_{J_{l-1}}(X_{J_l} - X_{J_{l-1}})|^r\Big)^{1/r} + C_r\Big(\sum_l |V^r(Y,[J_{l-1},J_l])V^r(X,[J_{l-1},J_l])|^r\Big)^{1/r}.$$

The first term is crealy bounded by $\|Y\|_\infty V^r(X)$.
In the second term we can actually bound the larger quantity

$$\Big(\sum_l |V^r(Y,[J_{l-1},J_l])V^r(X,[J_{l-1},J_l])|^{r/2}\Big)^{2/r} \leq \Big(\sum_l |V^r(Y,[J_{l-1},J_l])|^r\Big)^{1/r}\Big(\sum_l |V^r(X,[J_{l-1},J_l])|^r\Big)^{1/r} \leq V^r(Y)V^r(X).$$

Rough integral

Controlled paths

We want a theory that works for $X \in V^r$ with $r \geq 2$.

Definition

Let X, Y' be functions with bounded r-variation.
We say that a function Y is *controlled by X*
with *Gubinelli derivative Y'* if the error term

$$R_{s,t} := (Y_t - Y_s) - Y'_s(X_t - X_s), \quad s \leq t,$$

has bounded $r/2$-variation in the sense that

$$V^{r/2}(R) := \sup_{t_0 < \cdots < t_J} \Big(\sum_{j=1}^{J} |R_{t_j, t_{j-1}}|^{r/2} \Big)^{2/r} < \infty.$$

The space of controlled paths turns out to be robust
under a version of (Δ1).

Controlled paths have bounded r-variation

Lemma

If Y is controlled by X with Gubinelli derivative Y' and error term R, then

$$V^r Y \leq V^{r/2} R + \|Y'\|_\infty V^r X.$$

Proof.

$$|Y_t - Y_s| \leq |R_{s,t}| + |Y'_s||X_t - X_s|.$$

Insert this into the definition of r-variation:

$$V^r(Y) = \sup_{t_0 < \cdots < t_J} \Big(\sum_{j=1}^{J} |Y_{t_j} - Y_{t_{j-1}}|^r \Big)^{1/r}. \qquad \square$$

Composition of controlled paths with C^2 functions

Unlike bounded r-variation, controlled rough path property is *not* preserved under composition with Lipschitz functions. We need more regularity:

Lemma

If (Y, Y') is controlled by X, then for every C^2 function F also $F \circ Y$ is controlled by X, with Gubinelli derivative $F'(Y) \cdot Y'$.

Proof

For $s < t$ by Taylor's formula we have

$$F(Y_t) - F(Y_s) = F'(Y_s)(Y_t - Y_s) + O((Y_t - Y_s)^2).$$

Since Y is V^r, the second summand above is $V^{r/2}$.
The first summand equals

$$F'(Y_s)Y'_s(X_t - X_s) + F'(Y_s)R_{s,t},$$

where R is the error term of rough path (Y, Y').

Proof continued.

Just seen: $F'(Y_s)Y'_s$ is a Gubinelli derivative.
It remains to check that it is V^r.

- Y' is V^r by hypothesis.
- Since Y is a controlled path, it is V^r.
- Since $F \in C^2$, F' is Lipschitz, hence $F \circ Y$ is V^r.
- Product of V^r paths Y' and $F' \circ Y$ is again V^r.

$\square$

Rough path

Want: define $Z_t := \int_0^t Y_s \, \mathrm{d}X_s$ for controlled Y's
(and hope that the result will still be controlled).
If we can take $Y = 1$, we should get $Z = X$.
Then we should be able to take $Y = Z$.
But there is no way to make sense of $\int X \, \mathrm{d}X$ if X is too irregular.
Solution: we *postulate* the value of this integral.

Definition (Lyons)

For $2 \leq r < 3$, an r-rough path is a pair of functions $(X_t, \mathbb{X}_{s,t})$ such that $V^r(X) < \infty$, $V^{r/2}(\mathbb{X}) < \infty$, and *Chen's relation*

$$\mathbb{X}_{s,u} = \mathbb{X}_{s,t} + \mathbb{X}_{t,u} + (X_t - X_s)(X_u - X_t) \qquad \text{(Chen)}$$

holds for all $s \leq t \leq u$.

- One should imagine (picture!) $\mathbb{X}_{s,t}$ "=" $\int_s^t (X_{w-} - X_s) \, \mathrm{d}X_w = \int_{s<u<w<t} \mathrm{d}X_u \, \mathrm{d}X_w$.
- A rough path can be interpreted as a function of one variable.

Why postulate the integral?

If (X_j) is a discrete sequence, there is a canonical choice of $\mathbb{X}$ that satisfies Chen's relation, namely

$$\mathbb{X}_{s,t} := \sum_{s<j\le t} (X_{j-1} - X_s)(X_j - X_{j-1}). \qquad (\Delta\text{area})$$

The quantitative content of the definition of rough path is that we assume a bound on $V^{r/2}(\mathbb{X})$.
No such bound (independent of the length of the sequence) can be deduced from a bound on $V^r(X)$ if $r \ge 2$.

Modified Riemann sums

Given a rough parth $(X, \mathbb{X})$ and a controlled path (Y, Y'), we define modified Riemann sums for $\int Y_{u-}\,\mathrm{d}X_u$ by

$$Z_J := \sum_{j=1}^{J} \Big(Y_{j-1}(X_j - X_{j-1}) + Y'_{j-1}\mathbb{X}_{j-1,j} \Big). \qquad (\Delta 2)$$

Why does this modification work?
Consider $Y = X$, it is controlled by X with derivative $Y' \equiv 1$. By Chen's relation

$$\begin{aligned}
&\sum_{j=J}^{J+1} \Big(X_{j-1}(X_j - X_{j-1}) + \mathbb{X}_{j-1,j} \Big) \\
&= X_{J-1}(X_J - X_{J-1}) + \mathbb{X}_{J-1,J} + X_J(X_{J+1} - X_J) + \mathbb{X}_{J,J+1} \\
&= X_{J-1}(X_{J+1} - X_{J-1}) + \mathbb{X}_{J-1,J} + \mathbb{X}_{J,J+1} + (X_J - X_{J-1})(X_{J+1} - X_J) \\
&= X_{J-1}(X_{J+1} - X_{J-1}) + \mathbb{X}_{J-1,J+1}
\end{aligned}$$

Hence $(\Delta 2)$ telescopes to $X_0(X_J - X_0) + \mathbb{X}_{0,J}$.

Estimate for modified Riemann sums

Lemma

Let $2 \le r < 3$. Let $(X, \mathbb{X})$ be a rough path indexed by $0, \dots, J$, and let Y be controlled by X with Gubinelli derivative Y' and remainder R. Then

$$\Big|\sum_{j=1}^{J}\Big((Y_{j-1} - Y_0)(X_j - X_{j-1}) + Y'_{j-1}\mathbb{X}_{j-1,j}\Big)\Big| \lesssim V^{r/2}(R)V^r(X) + V^r(Y')V^{r/2}(\mathbb{X}) + |Y'_0||\mathbb{X}_{0,J}|.$$

Induction base

In the case $J = 1$ LHS equals $\mathbb{X}_{0,1}$.

Proof of estimate for modified Riemann sums

Inductive step: $J \to J+1$. Wlog $Y_0 = 0$. For any $1 \leq k \leq J$ have

$$\begin{aligned}
&\sum_j \Big(Y_{j-1}(X_j - X_{j-1}) + Y'_{j-1}\mathbb{X}_{j-1,j} \Big) \\
&= \sum_{j \notin \{k,k+1\}} \Big(Y_{j-1}(X_j - X_{j-1}) + Y'_{j-1}\mathbb{X}_{j-1,j} \Big) \\
&\quad + Y_{k-1}(X_k - X_{k-1}) + Y_{k-1}(X_{k+1} - X_k) + (Y_k - Y_{k-1})(X_{k+1} - X_k) \\
&\quad + Y'_{k-1}\mathbb{X}_{k-1,k} + Y'_{k-1}\mathbb{X}_{k,k+1} + (Y'_k - Y'_{k-1})\mathbb{X}_{k,k+1} \\
&= \sum_{j \notin \{k,k+1\}} \Big(Y_{j-1}(X_j - X_{j-1}) + Y'_{j-1}\mathbb{X}_{j-1,j} \Big) \\
&\quad + Y_{k-1}(X_{k+1} - X_{k-1}) + Y'_{k-1}\mathbb{X}_{k-1,k+1} \\
&\quad + (Y_k - Y_{k-1})(X_{k+1} - X_k) - Y'_{k-1}(X_k - X_{k-1})(X_{k+1} - X_k) \\
&\quad + (Y'_k - Y'_{k-1})\mathbb{X}_{k,k+1}
\end{aligned}$$

$$\text{last 2 lines } = R_{k-1,k}(X_{k+1} - X_k) + (Y'_k - Y'_{k-1})\mathbb{X}_{k,k+1}.$$

Proof continued.

We choose k that minimizes the error term and estimate

$$
\begin{aligned}
&\min_{1\le k\le J}|R_{k-1,k}(X_{k+1}-X_k)+(Y'_k-Y'_{k-1})\mathbb{X}_{k,k+1}| \\
&\le \left(J^{-1}\sum_{k=1}^{J}|R_{k-1,k}(X_{k+1}-X_k)+(Y'_k-Y'_{k-1})\mathbb{X}_{k,k+1}|^{r/3}\right)^{3/r} \\
&\lesssim J^{-3/r}\left(\sum|R_{k-1,k}(X_{k+1}-X_k)|^{r/3}\right)^{3/r} \\
&\quad + J^{-3/r}\left(\sum|(Y'_k-Y'_{k-1})\mathbb{X}_{k,k+1}|^{r/3}\right)^{3/r} \\
&\le J^{-3/r}\left(\sum|R_{k-1,k}|^{r/2}\right)^{2/r}\left(\sum|X_{k+1}-X_k|^{r}\right)^{1/r} \\
&\quad + J^{-3/r}\left(\sum|Y'_k-Y'_{k-1}|^{r}\right)^{1/r}\left(\sum|\mathbb{X}_{k,k+1}|^{r/2}\right)^{2/r} \\
&\le J^{-3/r}V^{r/2}(R)V^{r}(X)+J^{-3/r}V^{r}(Y')V^{r/2}(\mathbb{X}).
\end{aligned}
$$

The factors $J^{-3/r}$ are summable by hypothesis $r<3$. □

Modified Riemann sums are again controlled

Theorem

Let $2 \le r < 3$ and let $(X, \mathbb{X})$ be an r-rough path. Suppose that (Y, Y') is controlled by X. Then Z, given by (Δ2), is also controlled by X with Gubinelli derivative Y.

Proof

For $J < J'$ we have

$$
\begin{aligned}
Z_{J'} - Z_J &= \sum_{J<j\le J'} \Big(Y_{j-1}(X_j - X_{j-1}) + Y'_{j-1}\mathbb{X}_{j-1,j} \Big) \\
&= Y_J(X_{J'} - X_J) \\
&\quad + \sum_{J<j\le J'} \Big((Y_{j-1} - Y_J)(X_j - X_{j-1}) + Y'_{j-1}\mathbb{X}_{j-1,j} \Big)
\end{aligned}
$$

To see that Y is a Gubinelli derivative we need an $\ell^{r/2}$ bound for the latter sum.

Proof continued.

By Lemma

$$\sum_{J<j\leq J'} \Big((Y_{j-1} - Y_J)(X_j - X_{j-1}) + Y'_{j-1}\mathbb{X}_{j-1,j} \Big)$$
$$\lesssim V^{r/2}(R,[J,J'])V^r(X,[J,J'])$$
$$+ V^r(Y',[J,J'])V^{r/2}(\mathbb{X},[J,J']) + \|Y'\|_\infty |\mathbb{X}_{J,J'}|.$$

This is $\ell^{r/2}$ summable over any sequence of disjoint intervals $[J,J']$.
Let us look for example at the first term.
For $J_0 < J_1 < J_2 < \cdots$ consider the larger quantity

$$\Big(\sum_j \big(V^{r/2}(R,[J_{j-1},J_j]) V^r(X,[J_{j-1},J_j]) \big)^{r/3} \Big)^{3/r}$$
$$\leq \Big(\sum_j \big(V^{r/2}(R,[J_{j-1},J_j]) \big)^{r/2} \Big)^{2/r} \Big(\sum_j \big(V^r(X,[J_{j-1},J_j]) \big)^r \Big)^{1/r}$$
$$\leq V^{r/2}(R)V^r(X). \quad \square$$

Sample paths of martingales

Sample paths have bounded r-variation

Theorem (Lépingle, 1976)

Let $X = (X_t)$ be a martingale. For $1 < p < \infty$ and $2 < r$ we have

$$\|V_t^r X_t\|_p \leq C_{p,r} \|X\|_p.$$

- refines martingale maximal inequality: $Mf \leq X_0 + V_t^r X_t$
- quantifies martingale convergence:
  $V^r X_t$ finite $\implies$ X_t converges

Tools from probability

Lemma

Let $(X_n)_n$ be a martingale and $(\tau_j)_j$ an increasing sequence of stopping times. Then the sequence $(X_{\tau_j})_j$ is a martingale with respect to the filtration $(\mathcal{F}_{\tau_j})_j$.

Recall

$$\begin{aligned}\mathcal{F}_\tau &= \{A \in \mathcal{F}_\infty \mid A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\} \\ &= \{A \in \mathcal{F}_\infty \mid A \cap \{\tau = t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.\end{aligned}$$

Theorem (Martingale square function estimate/BDG)

Let $(X_n)_n$ be a martingale and

$$SX := \Big(\sum_{j \geq 1} |X_j - X_{j-1}|^2\Big)^{1/2}.$$

Then for $1 < p < \infty$ we have

$$\|SX\|_p \lesssim \|X\|_p.$$

Proof of Lépingle's inequality

$(\Omega, \mu, (\mathcal{F}_n)_n)$ filtered probability space,
$(X_n)_n$ adapted process with values in a metric space,
$V_n^\infty := \sup_{n'' \le n' \le n} d(X_{n''}, X_{n'})$.
Stopping times with $m \in \mathbb{N}$:

$$\tau_0^{(m)} := 0, \quad \tau_{j+1}^{(m)} := \inf\{t > \tau_j^{(m)} \mid d(X_t, X_{\tau_j^{(m)}}) > 2^{-m} V_t^\infty / 10\}.$$

$$\text{Claim:} \quad \big(V^r X\big)^r \le C \sum_{m=0}^\infty (2^{-m} V_\infty^\infty)^{r-2} \sum_{j=1}^\infty d(X_{\tau_j^{(m)}}, X_{\tau_{j-1}^{(m)}})^2.$$

Since $V^\infty \le V^r$, and assuming $V^r < \infty$, this implies

$$\big(V^r X\big)^2 \le C \sum_{m=0}^\infty (2^{-m})^{r-2} \sum_{j=1}^\infty d(X_{\tau_j^{(m)}}, X_{\tau_{j-1}^{(m)}})^2.$$

If (X_n) is a martingale, then by optional sampling
also the sampled process $(X_{\tau_j^{(m)}})_j$ is a martingale.

The red sum $=: S_{(m)}^2$ is the square function of the sampled process,
hence by BDG inequality $\|S_{(m)}\|_p \lesssim \|X\|_p$, $1 < p < \infty$.

Proof of claim

Claim: $\big(V^r(X_n)\big)^r \le C \sum_{m=0}^{\infty} (2^{-m} V_\infty^\infty)^{r-2} \sum_{j=1}^{\infty} d(X_{\tau_j^{(m)}}, X_{\tau_{j-1}^{(m)}})^2.$

Let $0 \le t' < t < \infty$ and $m \ge 2$. Suppose that

$$2 < \frac{d(X_{t'}, X_t)}{2^{-m} V_t^\infty} \le 4.$$

It suffices to find j with $t' < \tau_j^{(m)} \le t$ and

$$d(X_{t'}, X_t) \le 8 d(X_{\tau_{j-1}^{(m)}}, X_{\tau_j^{(m)}}).$$

Enhanced martingales

Rough paths in nilpotent groups

In order to apply the stopping time estimate,
we interpret a rough path $(X, \mathbb{X})$ as a path in the 3-dimensional Heisenberg group $\mathbb{H} \cong \mathbb{R}^3$ with the group operation

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy').$$

by setting $\mathbf{X}_t := (X_t, X_t, \mathbb{X}_{0,t})$.
From Chen's relation for $s < t$ we obtain

$$\mathbf{X}_s^{-1}\mathbf{X}_t = (X_t - X_s, X_t - X_s, \mathbb{X}_{s,t}).$$

With box norm on $\mathbb{H}$:

$$\|(x, y, z)\| := \max(|x|, |y|, |z|^{1/2})$$

and the corresponding distance $d(H, H') := \|H^{-1}H'\|$ we have

$$V^r X + (V^{r/2}\mathbb{X})^{1/2} \sim V^r \mathbf{X}.$$

Square function of enhanced martingale

Let X be a martingale and $\mathbb{X}$ be given by (Δarea).

Theorem

For $1 < p < \infty$ and $r > 2$ we have

$$\|V^{r/2}\mathbb{X}\|_p \lesssim \|X\|_p.$$

The stopping time argument applied to $\mathbf{X}$ shows that it suffices to bound

$$\sum_j \sum_{j=1}^{\infty} d(\mathbf{X}_{\tau_j}, \mathbf{X}_{\tau_{j-1}})^2$$

in $L^{p/2}$, where $(\tau_j)_j$ is an increasing sequence of stopping times.

Proposition

For every $1 < p < \infty$ we have

$$\|\sum_{j=1}^{\infty} |\mathbb{X}_{\tau_{j-1},\tau_j}|\|_{p/2} \lesssim \|X\|_p^2.$$

Paraproduct formulation

Proposition (diagonal case)

For $1 < p < \infty$ *and every increasing sequence of stopping times* (τ_j) *we have*

$$\|\sum_{j=1}^{\infty} |\mathbb{X}_{\tau_{j-1},\tau_j}|\|_{p/2} \lesssim \|X\|_p^2.$$

Proposition (off-diagonal case)

For every $1 \leq p_1, p_2 < \infty$ *and every increasing sequence of stopping times* (τ_j) *we have*

$$\|\sum_{j=1}^{\infty} |\Pi_{\tau_{j-1},\tau_j}(f,g)|\|_{1/(1/p_1+1/p_2)} \lesssim \|Sf\|_{p_1} \|Sg\|_{p_2},$$

$$\textit{where} \quad \Pi_{s,t}(f,g) := \sum_{s<j\leq t} (f_{j-1} - f_s) dg_j, \quad dg_j = g_j - g_{j-1}.$$

Tools from probability 2

Theorem (Reverse martingale square function/BDG)

If SX is the square function of a martingale X, then for $1 \leq p < \infty$ we have

$$\|X\|_p \lesssim \|SX\|_p.$$

Theorem (Martingale maximal inequality)

If $(X_n)_n$ is a martingale, then for $1 \leq p < \infty$ we have

$$\|\sup_n |X_n|\|_p \lesssim \|X\|_p.$$

Preliminary remarks

The paraproduct is given by

$$\Pi_{\tau_{j-1},\tau_j} = \sum_{\tau_{j-1}<k\leq\tau_j} (f_{k-1} - f_{\tau_{j-1}})(X_k - X_{k-1})$$
$$= \sum_{k=1}^{\infty} f_{k-1}^{(j)}(X_k^{(j)} - X_{k-1}^{(j)}),$$

where

$$f_k^{(j)} = f_k^{\tau_j} - f_k^{\tau_{j-1}} = f_{k\wedge\tau_j} - f_{k\wedge\tau_{j-1}}. \qquad \text{(stopped)}$$

Truncating the summation to $k \leq K$ we obtain a martingale.

Proof of the paraproduct estimate for $p_1 = p_2 = 2$

$$
\begin{aligned}
&\Big\| \sum_{j=1}^{\infty} |\Pi_{\tau_{j-1},\tau_j}| \Big\|_1 \\
&= \sum_{j=1}^{\infty} \|\Pi_{\tau_{j-1},\tau_j}\|_1 \\
&\lesssim \sum_{j=1}^{\infty} \|S\Pi_{\tau_{j-1},\tau_j}\|_1 \quad \text{by reverse square function estimate} \\
&= \mathbb{E} \sum_{j=1}^{\infty} \Big(\sum_k |f_{k-1}^{(j)}|^2 |X_k^{(j)} - X_{k-1}^{(j)}|^2 \Big)^{1/2} \\
&\le \mathbb{E} \sum_{j=1}^{\infty} M(f^{(j)}) \Big(\sum_k |X_k^{(j)} - X_{k-1}^{(j)}|^2 \Big)^{1/2} \\
&\le \Big(\mathbb{E} \sum_{j=1}^{\infty} M(f^{(j)})^2 \Big)^{1/2} \Big(\mathbb{E} \sum_{j=1}^{\infty} \sum_k |X_k^{(j)} - X_{k-1}^{(j)}|^2 \Big)^{1/2}
\end{aligned}
$$

Proof of the paraproduct estimate continued

$$
\begin{aligned}
&\Big(\mathbb{E}\sum_{j=1}^{\infty} M(f^{(j)})^2\Big)^{1/2}\Big(\mathbb{E}\sum_{j=1}^{\infty}\sum_{k}|X_k^{(j)} - X_{k-1}^{(j)}|^2\Big)^{1/2} \\
&= \Big(\sum_{j=1}^{\infty}\|M(f^{(j)})\|_2^2\Big)^{1/2}\Big(\mathbb{E}\sum_{k}|X_k - X_{k-1}|^2\Big)^{1/2} \\
&\lesssim \Big(\sum_{j=1}^{\infty}\|f^{(j)}\|_2^2\Big)^{1/2}\|SX\|_2 \\
&= \Big(\mathbb{E}\sum_{j=1}^{\infty}|f^{(j)}|^2\Big)^{1/2}\|SX\|_2 \\
&= \|Sf\|_2\|SX\|_2. \quad \square (p_1 = p_2 = 1)
\end{aligned}
$$

Tools from probability 3

Lemma (Vector-valued BDG inequality)

Let $h^{(k)}$ be martingales with respect to some fixed filtration. Let $1 \leq q, r < \infty$. Then we have

$$\left\| Mh^{(k)} \right\|_{L^q(\ell^r_k)} \lesssim_{q,r} \left\| Sh^{(k)} \right\|_{L^q(\ell^r_k)}.$$

This is different from vector-valued estimates in Martikainen's lecture because

- the maximal function is inside the ℓ^r norm, and
- ℓ^1 is not UMD.

We postpone the proof and look at how this vector-valued inequality is applied.

Proof of the paraproduct estimate for $1/p_1 + 1/p_2 \leq 1$

$$
\begin{aligned}
&\|\sum_{j=1}^{\infty} |\Pi_{\tau_{j-1},\tau_j}|\|_{1/(1/p_1+1/p_2)} \\
&\lesssim \|\sum_{j=1}^{\infty} S\Pi_{\tau_{j-1},\tau_j}\|_{1/(1/p_1+1/p_2)} \quad \text{by vector-valued BDG} \\
&= \|\sum_{j=1}^{\infty} \big(\sum_{k} |f_{k-1}^{(j)}|^2 |X_k^{(j)} - X_{k-1}^{(j)}|^2\big)^{1/2}\|_{1/(1/p_1+1/p_2)} \\
&\leq \|\sum_{j=1}^{\infty} Mf^{(j)} \big(\sum_{k} |X_k^{(j)} - X_{k-1}^{(j)}|^2\big)^{1/2}\|_{1/(1/p_1+1/p_2)} \\
&\leq \|\big(\sum_{j=1}^{\infty} (Mf^{(j)})^2\big)^{1/2} \big(\sum_{j=1}^{\infty} \sum_{k} |X_k^{(j)} - X_{k-1}^{(j)}|^2\big)^{1/2}\|_{1/(1/p_1+1/p_2)} \\
&= \|\big(\sum_{j=1}^{\infty} (Mf^{(j)})^2\big)^{1/2} Sg\|_{1/(1/p_1+1/p_2)}
\end{aligned}
$$

Proof of the paraproduct estimate continued

$$\|\big(\sum_{j=1}^{\infty}(Mf^{(j)})^2\big)^{1/2}Sg\|_{1/(1/p_1+1/p_2)}$$

$$\leq \|\big(\sum_{j=1}^{\infty}(Mf^{(j)})^2\big)^{1/2}\|_{p_1}\|Sg\|_{p_2}$$

$$\leq \|\big(\sum_{j=1}^{\infty}(Sf^{(j)})^2\big)^{1/2}\|_{p_1}\|Sg\|_{p_2} \quad \text{by vector-valued BDG}$$

$$= \|Sf\|_{p_1}\|Sg\|_{p_2}. \quad \Box(1/p_1+1/p_2 \geq 1)$$

We used BDG inequality with exponent $1/(1/p_1+1/p_2) \geq 1$.
How to handle smaller p_1, p_2?
For singular integrals one uses the Calderón–Zygmund decomposition.
The CZ decomposition uses the doulbing property of cubes in $\mathbb{R}^n$, so we need a different decomposition for martingales.