ARITHMETISCHE GEOMETRIE OBERSEMINAR

Patching and the p-adic local Langlands correspondence

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In this ARGOS we want to study the paper [CEGGPS] by A. Caraiani, M. Emerton, T. Gee, D. Geraghty, V. Paskunas and S. W. Shin, where many cases of the Breuil-Schneider conjecture are proven. The Breuil-Schneider conjecture is a first approximation to a conjectural p-adic local Langlands correspondence and asserts the existence of certain p-adic Banach space representations of $\mathrm{GL}_n(K)$, where K is a finite extension of $\mathbb{Q}_p$. More precisely, given an n-dimensional (Frobenius semi-simple) Weil-Deligne representation r of the Weil-Deligne group of K the (slightly modified) classical local Langlands correspondence assigns to r a smooth representation $\pi(r)$ of $\mathrm{GL}_n(K)$ and given a dominant weight ξ of $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_n$ we can associate an algebraic representation V_ξ of $\mathrm{GL}_n(K)$ to ξ. The Breuil-Schneider conjecture asserts that there exists an invariant norm on $\pi(r)\otimes V_\xi$ if and only if there is a potentially semi-stable representation ρ of the absolute Galois group of K whose Hodge-Tate weights are described by ξ and whose associated (Frobenius semi-simplified) Weil-Deligne representation is r.
The main idea is to embed the locally algebraic representation $\pi(r)\otimes V_\xi$ into the completed cohomology of some Shimura variety. As not every local Galois representation can be realized globally one uses a patching construction instead of the completed cohomology itself.

1) The local Langlands correspondence

Recall the Weil-Deligne group and Weil-Deligne representations, the classical local Langlands correspondence and the Bernstein-Zelevinsky classification, see [We] for example. Further recall the realization of the local Langlands correspondence in the cohomology of Shimura varieties.

2) The Breuil-Schneider conjecture

Recall the notions of crystalline, semi-stable and de Rham representations from p-adic Hodge theory and the corresponding p-adic Hodge structures. Explain the construction of a Weil-Deligne representation from a potentially semi-stable representation, see [BS, 4] and [Fo].
State the Breuil-Schneider conjecture [BS, Conjecture 4.3] and outline the proof of the easy direction (at least in the crystalline case [BS, 3]) and prove the conjecture in the case of an irreducible Weil-Deligne representation [BS, Theorem 5.2]).
Give a survey of the p-adic local Langlands correspondence (following [Be] for example) and comment on the Breuil-Schneider conjecture in this case [Be, Theorem 4.2.2].

3) Completed cohomology

Define completed cohomology and homology and discuss a few properties. See [CE] for example. Explain the duality between homology and cohomology as a special case of the duality between finitely generated modules over a completed group ring (the so-called Iwasawa algebra), and admissible representations on a Banach space, as in [ST]. Whenever it simplifies the discussion, discuss only the case of a 0-dimensional spaces. Moreover, show that classical automorphic forms embed into completed cohomology, and form precisely the subset of locally algebraic vectors.

Prove the Breuil-Schneider conjecture for automorphic representations, cf. e.g. [So].

4) Patching with conditions at p, Part I
Introduce automorphic forms on quaternion algebras [Gee, 4.8], their associated Galois representations [Gee, 4.19] and the Jacquet-Langlands correspondence [Gee, 4.17]. Discuss the integral theory of automorphic forms on quaternion algebras following [Gee, 5.2].

5) Patching with conditions at p, Part II
Recall some background on deformations of Galois representations. Recall Kisin's potentially crystalline deformation rings [Ki, 1] and [Gee, 3.20]. Discuss deformation conditions, see [Gee, 3.15] and the references cited there. Then start explaining the patching construction following [Gee, 5.5, p.36-p.38] and explain how it is used to prove [Gee, Theorem 5.1].

6) Patching with conditions at p, Part III
Finish [Gee, 5.5] and prove [Gee, Theorem 5.1].

7) Patching without conditions at p, Part I
Explain the setup of [CEGGPS, 2.1-2.5].

8) Patching without conditions at p, Part II
Explain the construction of the patching module M_∞ in [CEGGPS, 2.6].

9) Type theory
Prove the results on Bushnell-Kutzko types in [CEGGPS, 3.2-3.11]. Here, the most important result needed later is Corollary 3.11, saying that under a genericity assumption, the local Langlands correspondence exists in families over the Bernstein variety. It may be helpful to illustrate the statements in the case of the spherical Bernstein component of GL_2.

10) Interpolating the classical local Langlands correspondence, Part I
Prove [CEGGPS, Prop. 4.2] which says that one can interpolate the classical local Langlands correspondence p-adically on the rigid generic fiber of a potentially semi-stable deformation ring.

11) Interpolating the classical local Langlands correspondence, Part II
Prove [CEGGPS, Thm. 4.1] sharpening the result of the previous talk.

12) Proof of certain cases of the Breuil-Schneider conjecture
In this talk we deduce the main theorem of [CEGGPS] from the patching construction and the interpolation of the classical Langlands correspondence. This follows [CEGGPS, 4.13-5.5].

References

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