Dr. E. Hellmann WS 2013/14

**Algebraic Geometry I**

**Exercise Sheet 6**

**Due Date: 28.11.2013**

**Exercise 1:**

Let $\mathscr{F}$ be a presheaf in sets on a topological space $X$. We define the *espace étale* of $\mathscr{F}$ as follows. Set

$$\mathrm{Spe}(\mathscr{F}) = \coprod_{x\in X} \mathscr{F}_x$$

as a set and endow it with the strongest topology such that for all $U \subset X$ open and $s \in \mathscr{F}(u)$ the induced maps

$$U \longrightarrow \mathrm{Spe}(\mathscr{F})$$
$$x \longmapsto s_x \in \mathscr{F}_x$$

are continuous.

(i) Show that the canonical projection map $\pi : \mathrm{Spe}(\mathscr{F}) \to X$ defined by $\mathscr{F}_x \ni s \mapsto x$ is continuous.

(ii) Show that the assignment

$$X \supset U \longmapsto \{s : U \to \mathrm{Spe}(\mathscr{F}) \text{ continuous} \mid \pi \circ s = \mathrm{id}_U\}$$

defines a sheaf which agrees with the sheafification $\mathscr{F}^+$ of $\mathscr{F}$.

**Exercise 2:**

Let $(X, \mathcal{O}_X)$ be a locally ringed space.

(i) Let $U$ be an open and closed subset of $X$. Then there is a unique section $e_U \in \mathcal{O}_X(X)$ such that $e_U|_U = 1 \in \mathcal{O}_X(U)$ and $e_U|_{X\setminus U} = 0 \in \mathcal{O}_X(X\setminus U)$. Show that $U \mapsto e_U$ yields a bijection

$$\{\text{open and closed subsets } U \subset X\} \Longleftrightarrow \{\text{idempotent elements } e \in \mathcal{O}_X(X)\}.$$

(ii) Show that $X$ is not connected if and only if there exists a decomposition $\mathcal{O}_X(X) \cong A_1 \times A_2$ with rings $A_1, A_2 \neq 0$.

**Exercise 3:**

Let $A$ be a commutative ring and $\mathfrak{a} \subset A$ be an ideal and write $f : \mathrm{Spec}\, A/\mathfrak{a} \to \mathrm{Spec}\, A$ for the map induced by the projection $A \to A/\mathfrak{a}$.

(i) Show that the map $f$ identifies $\mathrm{Spec}\, A/\mathfrak{a}$ with the subspace $V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}\, A | \mathfrak{a} \subset \mathfrak{p}\}$ equipped with the subspace topology of $\mathrm{Spec}\, A$.

(ii) Show that the map $f$ is a homeomorphism if and only if $\mathfrak{a} \subset \mathrm{Nil}(A)$, where

$$\mathrm{Nil}(A) = \sqrt{(0)} = \{a \in A \mid a^n = 0 \text{ for some } n \gg 0\}.$$

(iii) Show that $\mathrm{Spec}\, A$ is irreducible if and only if $A/\mathrm{Nil}(A)$ is a domain if and only if $\mathrm{Nil}(A)$ is a prime ideal.

(iv) Deduce that every closed irreducible subset $X \subset \mathrm{Spec}\, A$ has a unique generic point.

**Exercise 4:**

(i) Let $k$ be a field and consider the canonical inclusion $\varphi : B = k[X] \to A = k[X,Y]$. We denote by $f : \operatorname{Spec} A \to \operatorname{Spec} B$ the induced map. Show that

$$f^{-1}(x) = V((g)) \cong \operatorname{Spec}(\kappa(x)[Y])$$

where $x = (g) \in \operatorname{Spec} k[X]$ for some irreducible $g \in k[X]$.

(ii) Show that

$$f^{-1}(\eta) = \operatorname{Spec}(S^{-1}k[X,Y]) \cong \operatorname{Spec}(\kappa(\eta)[Y])$$

where $\eta = (0) \in \operatorname{Spec} k[X]$ is the generic point and $S = k[X]\backslash\{0\}$.

(iii) Use (i) and (ii) to describe all points of $\operatorname{Spec} k[X,Y]$ and their closure in the case where $k$ is algebraically closed.

(iii) Use a similar method to describe $\operatorname{Spec} \mathbb{Z}[X]$.

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeom`