Dr. E. Hellmann SS 2014

Algebraic Geometry II

Exercise Sheet 1

Due Date: 24.04.2014

Exercise 1:

Let Y be a scheme and let $\mathscr{A}$ be a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras. Let $X = \underline{\mathrm{Spec}}_Y \mathscr{A}$ and let $f : X \to Y$ be the corresponding affine morphism.

(i) Let $\mathscr{F}$ be a sheaf of $\mathscr{A}$-modules on Y. Show that $\mathscr{F}$ is quasi-coherent as an $\mathscr{A}$-module if and only if it is quasi-coherent as an $\mathcal{O}_Y$-module.

(ii) Show that $\mathscr{G} \mapsto f_*\mathscr{G}$ induces an equivalence of categories between quasi-coherent $\mathcal{O}_X$-modules on X and quasi-coherent $\mathscr{A}$-modules on Y.

(iii) Show that the functor f_* is exact on the category of quasi-coherent $\mathcal{O}_X$-modules.

Exercise 2:

Let k be a field and $X = \mathbb{A}^2_k = \mathrm{Spec}\, k[T_1, T_2]$. Further let $Z = V(T_1, T_2)$ and let $\tilde{X} = \mathrm{Bl}_Z X$ denote the blow-up of X at the origin. Let $i : \tilde{X} \hookrightarrow \mathbb{A}^2_k \times \mathbb{P}^1_k$ be the embedding induced by the graded surjection

$$k[T_1, T_2][S_1, S_2] \longrightarrow \bigoplus_{i \geq 0} (T_1, T_2)^i$$

mapping S_i to T_i. Let $f : \tilde{X} \to \mathbb{P}^1_k$ be the composition of i with the projection $\mathbb{A}^2_k \times \mathbb{P}^1_k \to \mathbb{P}^1_k$. Show that this makes $\tilde{X}$ into a geometric vector bundle over $\mathbb{P}^1_k$ and compute its sheaf of sections.

Exercise 3:

Let k be a field and let $X = V(T_1T_2 - T_3^2) \subset \mathbb{A}^3_k$ and $Z = \{(0, 0, 0)\} \subset X$ viewed as a closed subscheme with the reduced scheme struture. Further let $\tilde{X} = \mathrm{Bl}_Z X$ denote the blow up of X in Z.

(i) Show that there is a morphism $f : X \backslash Z \to \mathbb{P}^1_k$ that is given by $(t_1, t_2, t_3) \mapsto (t_1 : t_3) = (t_3 : t_2)$ on k-valued points.

(ii) Show that f extends to a morphism $\tilde{f} : \tilde{X} \to \mathbb{P}^1_k$.

(iii) Show that f makes $\tilde{X}$ into the geometric vector bundle $\mathbb{V}(\mathcal{O}(2))$ on $\mathbb{P}^1_k$.

Exercise 4:

Let k be a field $\mathrm{Gr}_{2,4}$ be the Grassmannian of 2-dimensional subspaces of $k^4 = \bigoplus_{i=1}^4 ke_i$, that is $\mathrm{Gr}_{2,4}$ represents the functor

$$T \longmapsto \left\{ \begin{array}{c} \text{2-dimensional locally free sub-modules } \mathscr{E} \subset \mathcal{O}_T^4 \\ \text{such that } \mathcal{O}_T^4/\mathscr{E} \text{ is locally free} \end{array} \right\}$$

on the category of k-schemes with universal subspace $\mathscr{F} \subset \mathcal{O}^4_{\mathrm{Gr}_{2,4}}$.
Further let $N : k^4 \to k^4$ denote the nilpotent morphism given by

$$\begin{aligned} e_1, e_3 &\longmapsto 0 \\ e_2 &\longmapsto e_1 \\ e_4 &\longmapsto e_3. \end{aligned}$$

We identify the image V of N with k^2 via the choice of basis e_1, e_3.

(i) Show that $T \mapsto \{\mathscr{E} \in \mathrm{Gr}_{2,4}(T) \mid N(\mathscr{E}) \subset \mathscr{E}\}$ cuts out a closed subscheme X of $\mathrm{Gr}_{2,4}$.

(ii) Show that the functor $T \mapsto \{\mathscr{E} \in X(T) \mid N(\mathscr{E}) \text{ is locally free of rank } 1\}$ defines an open subscheme $U \subset X$ such that $X \backslash U$ is a single point.

(iii) Let us still write $\mathscr{F}$ for the restriction of the universal subspace $\mathscr{F} \subset \mathcal{O}^4_{\mathrm{Gr}_{2,4}}$ to U. Note that $N(\mathscr{F}) \subset \mathrm{Im}\, N \otimes_k \mathcal{O}_U = \mathcal{O}_U^2$ via the identification $\mathrm{Im}\, N = k^2$. Show that the quotient $\mathcal{O}_U^2 \to \mathcal{O}_U^2/N(\mathscr{F})$ defines a morphism $U \to \mathbb{P}^1_k$ which is a geometric vector bundle of rank 1.

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeomII`