A glimpse at symplectic geometry and pseudo-holomorphic curves

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Three motivating questions

Question 1: dynamical systems

What can we say about periodic orbits of a mechanical system (e.g. double pendulum, the solar system)?

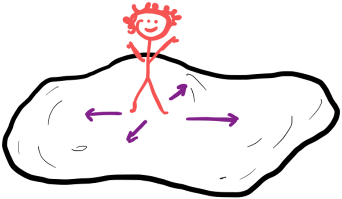
Question 2: symplectic fillings

When is a smooth manifold the boundary of a compact manifold?

Question 3: moduli spaces

What does the solution space to an elliptic PDE look like?

Manifolds



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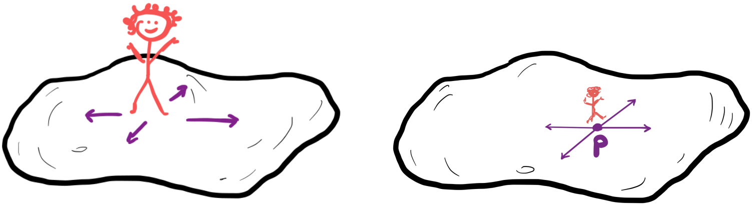
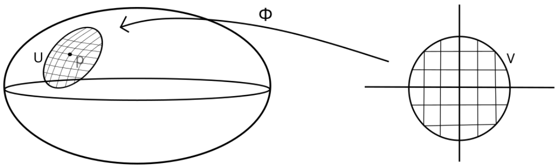


surface of a potato is a manifold: locally looks like a disk

Smooth manifolds

- manifold: second countable Hausdorff topological space M locally homeomorphic to open ball in $\mathbb{R}^n$
- every $p \in M$ has a coordinate chart: $p \in U \subset M$ open, homeomorphism $\phi\colon V \to U$ for $V \subset \mathbb{R}^n$ open ball
- smooth manifold: all coordinate transformations from overlapping charts are smooth
- boundary: looks like upper half of $\mathbb{R}^n$



Picture courtesy of Dominik Gutwein.

Examples of smooth n-dimensional manifolds

- $n = 0$: isolated points
- $n = 1$: $\mathbb{R}$, $\mathbb{S}^1$
- $n = 2$: $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{T}^2$, Σ_g for $g \geq 1$

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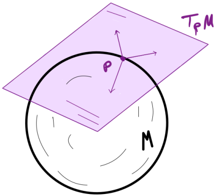
- $n \geq 3$: complicated; classification for $n \geq 4$ impossible
- $n \geq 3$: $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{T}^n$, $\mathbb{RP}^n$, $\mathbb{CP}^n$,
  $\{[z_0 : z_1 : z_2 : z_3 : z_4] \in \mathbb{CP}^4 \mid z_0^5 + \cdots + z_4^5 = 0\}$
  configuration spaces in physics and engineering

How to measure area on a 2-manifold?

- locally: integrate density function
- globally: use a differential 2-form

- each $p \in M$ has **tangent space** T_pM, n-dimensional $\mathbb{R}$-vector space
- 2-form $\omega = \{\omega_p \colon T_pM \times T_pM \to \mathbb{R}\}_{p \in M}$ ω_p anti-symmetric bilinear, smoothly varying with p
- **area form**: each ω_p is non-degenerate
- **symplectic 2-manifold**: M plus choice of area form

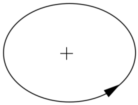


Symplectic manifolds

Definition

A **symplectic manifold** (M, ω) is a smooth manifold M together with a closed non-degenerate 2-form ω.

- equivalently: atlas of **Darboux charts** $(x_1, y_1, \ldots, x_n, y_n)$ in which ω looks like $\omega_0 = \sum_{i=1}^{n} dx^i \wedge dy^i$
- geometrically: symp. structure = signed area of closed curves
- γ embedded closed curve in $\mathbb{R}^2$
  $\rightarrow A(\gamma)$ signed area of enclosed disc



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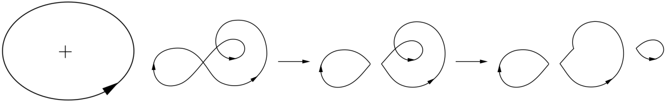

- γ any oriented closed piece-wise smooth curve: decompose into closed embedded pieces

Pictures taken from Schlenk, *Symplectic embedding problems old and new* (2017).

Symplectic manifolds (cont.)

- **standard symplectic structure** on $\mathbb{R}^{2n}$:
  map $\gamma \to A(\gamma) = A(\gamma_1) + \cdots + A(\gamma_n)$,
  where $\gamma = (\gamma_1, \ldots, \gamma_n)$ any closed oriented curve
- symplectic structure on M is an atlas whose transition functions preserve signed area

Motivation: Hamiltonian mechanics

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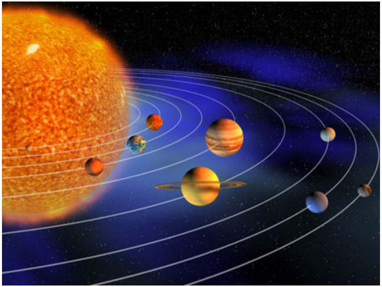
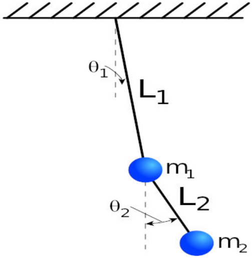


The solar system (simplified).

Source: http://www.scienceclarified.com/photos/solar-system-2865.jpg



A double pendulum.

Source: By JabberWok, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=1601029

Hamiltonian systems: from Newton's to Hamilton's equations

- system of particles moving with n degrees of freedom

$$q(t) = (q_1(t), \dots q_n(t))$$

- forces are derived from a **potential** $V(q)$ by $F(q) = -\nabla V(q)$
- Newton's second law states $m_i \ddot{q}_j = -\frac{\partial V}{\partial q_j}$

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- Newton's second law states $m_i \ddot{q}_j = -\frac{\partial V}{\partial q_j}$
- Hamilton: consider momenta $p_j := m_j \dot{q}_j$
- total energy defines the Hamiltonian function

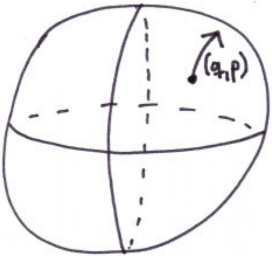
$$H \colon \mathbb{R}^{2n} \to \mathbb{R}, \quad (q,p) \mapsto \underbrace{\sum_{j=1}^{n} \frac{p_j^2}{2m_j}}_{\text{kinetic energy}} + \underbrace{V(q)}_{\text{potential forces}}$$

- Newton's equations become **Hamilton's equations**

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \quad \text{and} \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \qquad \text{for } j = 1, \dots n \qquad \text{(H)}$$

Hamilton's equations on a manifold: symplectic manifolds

- key insight: regard $(q(t), p(t))$ as trajectory in **phase space** $\mathbb{R}^{2n} = T^*\mathbb{R}^n$
- double pendulum: rigid arms mean $q(t) = (q_1(t), q_2(t)) \in \mathbb{T}^2$, phase space is cotangent bundle $T^*\mathbb{T}^2$
- for systems with constraints, treat (q, p) as **local coordinates** of a point moving in a manifold



Fact

A smooth $2n$-dimensional manifold it is covered by coordinate charts $(q_1, p_1, \ldots, q_n, p_n)$ such that for all smooth $H\colon M \to \mathbb{R}$, all coordinate changes preserve the form of (H) iff it is symplectic.

Hamilton's equation in symplectic manifolds

Definition

For (M, ω) symplectic, $H\colon \mathbb{R} \times M \to \mathbb{R}$ smooth, the **Hamiltonian vector field** X_{H_t} of H is defined by $\omega(X_{H_t}, \cdot) = -dH(t, \cdot)$.

Exercise

Solutions (q, p) of (H) are the integral curves of X_{H_t}.

Sample theorems I: periodic orbits of dynamical systems

Arnold conjecture

If M is a closed* symplectic manifold and $H\colon \mathbb{S}^1 \times M \to \mathbb{R}$ smooth and non-degenerate, then

$$\#\ \text{1-periodic orbits of } X_H \geq \sum_{i=1}^{n} b_i(M),$$

where $b_i(M) := \operatorname{rk} H_i(M)$ is the i-th Betti number of M.

(Almost the) Conley conjecture

M is a closed symplectic manifold with e.g. $\pi_2(M) = 0$.
$H\colon \mathbb{S}^1 \times M \to \mathbb{R}$ is smooth and non-degenerate, X_H has infinitely many simple orbits of integer period.

Sample theorems II: symplectic fillings

Definition

A **smooth filling** of a smooth manifold M is a compact manifold N with $\partial N \cong M$.

not always possible ($\mathbb{CP}^2$ has no smooth filling),
but understood (bordism theory, 1960s)

Sample theorems II: symplectic fillings

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Definition

A **contact manifold** $(M^{2n-1}, \xi = \ker \alpha)$ is a smooth manifold M together with a choice of 1-form α s.t. $\alpha \wedge d\alpha^{n-1} \neq 0$.

Template definition

A **symplectic filling** of (M, ξ) is a compact symplectic manifold (W, ω) with $\partial W \cong (M, \xi)$.

Sample theorem II: symplectic fillings (cont.)

Template definition

A symplectic filling of (M, ξ) is a compact symplectic manifold (W, ω) with $\partial W \cong (M, \xi)$.

Definition

An **exact symplectic filling** of (M, ξ) is a compact symplectic manifold $(W, \omega = d\lambda)$ s.t. $\partial W \cong (M, \xi)$ and the vector field X induced by $\iota_X \omega = \lambda$ points outwards along ∂W.

Theorem (Zhou '20,'22)

If $n \geq 3$ and $n \neq 4$, $(\mathbb{RP}^{2n-1}, \xi_{std})$ has no exact symplectic filling.

Underlying paradigm: symplectic invariants

- Arnold, Conley conjecture: use Hamiltonian Floer homology
- (M, ω) symplectic $\to$ homology groups $HF_*(M)$, generated by 1-periodic Hamiltonian orbits
- Arnold conjecture: bound # orbits via rk $HF_*(M)$
- Conley conjecture: pass to higher iterates
- Zhou's theorem: use more advanced invariant to exclude hypothetical filling (action-filtered positive symplectic homology)

Pseudo-holomorphic curves

Definition

An **almost complex structure** on a smooth manifold M is a collection of maps $J_p \colon T_pM \to T_pM$ with $J_p^2 = -\,\mathrm{id}$, smoothly varying in p.

Theorem

Every symplectic manifold admits an almost complex structure.

intuition: J is an auxiliary object

Pseudo-holomorphic curves

Definition

A **Riemann surface** is a smooth surface with a choice of almost complex structure.

Fact

If (Σ, j) is a Riemann surface and Σ is closed, then $(\Sigma, j) \cong (\Sigma_g, j')$ for some $g \geq 0$. We call g the ***genus*** *of Σ.*

Definition

A closed **pseudo-holomorphic curve** is a smooth map $u\colon (\Sigma, j) \to (M, J)$ with $J \circ du = du \circ j$, where (Σ, j) is a closed Riemann surface and (M, J) an almost complex manifold.

Moduli space of holomorphic curves

given: (M, ω) symplectic, almost complex structure J on M for $g \geq 0$ and $A \in H_2(M)$, consider the **moduli space** of holomorphic curves

$$\mathcal{M}_g(A, J) := \{u\colon (\Sigma, j) \to (M, J) \mid \text{u ps.-holo; } \Sigma \cong \Sigma_g, u_*[\Sigma] = A\}/_\sim$$

Wishful thinking

$\mathcal{M}_g(A, J)$ is a compact smooth manifold (and finite-dimensional).

Understanding the moduli space of holomorphic curves

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$\mathcal{M}_g(A, J)$ is a compact smooth manifold (and finite-dimensional).

- rephrase: $u\colon (\Sigma, j) \to (M, J)$ is J-holomorphic iff $J \circ du \circ j = -du$ iff $du + J \circ du \circ j = 0$
- so: $\mathcal{M}_g(A, J)$ is the zero set of $\Phi\colon (u, J) \mapsto du + J \circ du \circ j$

Understanding the moduli space of holomorphic curves

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Finite-dimensional Implicit function theorem

$E \to B$ smooth vector bundle, $s\colon B \to E$ smooth section transverse to the zero section. Then $s^{-1}(0) \subset B$ is a smooth submanifold.

domain of Φ is $C^\infty(\Sigma, M) \times \mathcal{J}(M, \omega)$, where $\mathcal{J}(M, \omega)$ is the space of all compatible almost complex structures

Infinite-dimensional complications

$\mathcal{M}_g(A, J)$ is the zero set of $\Phi\colon C^\infty(\Sigma, M) \times \mathcal{J}(M, \omega) \to \ldots$, $(u, J) \mapsto du + J \circ du \circ j$

- linearisation of section has a bounded inverse: ok, $d\Phi$ is a **Fredholm operator**

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- linearisation of section has a bounded inverse:
  ok, $d\Phi$ is a **Fredholm operator**
- domain must be a **Banach manifold**:
  but $C^\infty(\Sigma, M)$ is not complete!
- solution: extend Φ to a larger domain,
  e.g. **Sobolev spaces** $W^{k,p}(\Sigma, M)$ for $kp > 2$
- **elliptic regularity**: extension has same zero set

Bad news: transversality and compactness

- $\mathcal{M}_g(A, J)$ is not compact, but compactifiable: require compatible J (i.e. $\omega(\cdot, J\cdot)$ Riemannian metric)

Bad news: transversality and compactness

- $\mathcal{M}_g(A, J)$ is not compact, but compactifiable: require compatible J (i.e. $\omega(\cdot, J\cdot)$ Riemannian metric)
- transversality failure: for some J, $\mathcal{M}_g(A, J)$ is not a manifold best case: holds for "generic" J
- more generally: transversality doesn't like symmetry e.g. multiply covered curves (or external group action)

Theorem

For "almost all" compatible J, $\mathcal{M}_g^*(A, J)$ is a compactifiable smooth manifold of dimension $(\frac{\dim M}{2} - 3)(2 - 2g) + 2\langle c_1(TM), A\rangle$.

Summary

1. Symplectic manifolds arise when describing mechanical systems.
2. Periodic orbits of Hamiltonian systems can be understood using symplectic invariants.
3. These invariants are defined using moduli spaces of pseudo-holomorphic curves.

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Thanks for listening! Any questions?

Which manifolds are symplectic?

- no full answer known!
- necessary conditions
  - even dimension, orientable
  - $\exists$ (compatible) almost complex structure
  - if compact: $H^{2i}(M) \neq 0$ for $0 < 2i < \dim(M)$
  - additional conditions on dimension 4

Example

Sphere $\mathbb{S}^n$ is **not** symplectic for $n > 2$.

Proof sketch of Arnold conjecture

given (M, ω) closed*; $H \colon \mathbb{S}^1 \times M \to \mathbb{R}$ smooth non-degenerate

- $CF_k(M)$ is generated by 1-periodic orbits with index k
- in particular: $\#$1-periodic orbits $\geq \sum_k \operatorname{rk} HF_k(M)$
- Morse theory: $\sum_k \operatorname{rk} H_k(M) \geq \sum_{i=0}^{2n} \operatorname{rk} H_k(M)$

Theorem

For each k, there is an isomorphism $HF_k(M) \cong H_{2n-k}(M)$.

Details: Hamiltonian Floer homology

given: (M, ω) closed* symplectic manifold; $H\colon \mathbb{S}^1 \times M \to \mathbb{R}$ smooth, non-degenerate

- **Floer chain complex** $(CF_*(M,\omega), \partial)$,
  Hamiltonian Floer homology $HF(M,\omega) = H_*(CF_*(M,\omega), \partial)$
- $CF_*(M)$ generated by 1-periodic orbits of X_H
- grading by Conley-Zehnder index
- differential counts finite energy **Floer cylinders** connecting two 1-periodic orbits
- show: well-defined; independent of H