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Exercises for **Topology I**
Sheet 3

You can obtain up to 10 points per exercise.

Exercise 1. Let (X, A) be a pair of spaces enjoying the homotopy extension property, and write $i\colon A \hookrightarrow X$ and $p\colon X \to X/A$ for the inclusion and collapse map, respectively.

1. Show: if the inclusion i is homotopic to a constant map, then there exists a map $r\colon X/A \to X$ together with a homotopy $rp \simeq \mathrm{id}_X$.
2. Show: if A is contractible (i.e. homotopy equivalent to a 1-point space), then p is a homotopy equivalence.

Exercise 2. Let X be a CW-complex with skeleta $X_k \subseteq X$, let A be a subcomplex, and write $p\colon X \to X/A$ for the collapse map again. Show that

$$\emptyset \subseteq p(X_0) \subseteq p(X_1) \subseteq \cdots$$

defines a CW-structure on X/A.

Exercise 3. Let $(X, x_0), (Y, y_0)$ be pointed topological spaces. We write $[X, Y]_*$ and $[X, Y]$ for the set of pointed and unpointed homotopy classes of continuous maps $X \to Y$, respectively. There is a tautological 'forgetful map' $\alpha\colon [X, Y]_* \to [X, Y], [f] \mapsto [f]$.

1. Show that α is surjective if X admits the strucure of a CW-complex with $x_0 \in X_0$ and Y is path-connected.
2. Show that α is injective if Y is in addition simply connected.
3. Give examples showing that α is neither injective nor surjective in general.

$*$ **Exercise 4** (*10 bonus points*)**.** Let G be a (discrete) group. The goal of this bonus exercise is to construct a specific connected CW-complex X such that $\pi_1(X, x_0) \cong G$ for some (hence any) basepoint x_0.

1. Construct a CW-complex $Y \neq \emptyset$ together with a continuous G-action satisfying the following properties:
   (a) For every $n \geq 0$ and every choice of basepoints $[S^n, Y]_* = 0$.
   (b) 'G freely permutes open cells,' i.e. if $1 \neq g \in G$ and e is any open cell, then $ge \cap e = \emptyset$.

   Hint. Start with $Y_0 = G$ and then construct the skeleta $Y_1, Y_2, \ldots$ inductively.
2. Show that the quotient map $Y \to Y/G =: X$ is a covering map with deck transformation group G.

   Hint. By an exercise from the course *Einführung in die Geometrie und Topologie* last term you 'only' have to show that the G-action on Y is properly discontinuous.
3. Conclude that $\pi_1(X, x_0) \cong G$ for every $x_0 \in X$. Compute $[S^n, X]_*$ more generally for all $n \geq 0$.

   Remark. One can show that the resulting space X admits the structure of a CW-complex (this is easy), and that it is up to homotopy uniquely determined by the properties established in Subtask 3 (this is not easy). In particular, X is up to homotopy independent of the concrete construction you came up with in Subtask 1.

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Exercise 5. Let X be a connected CW-complex, $x_0 \in X_0$. The goal of this exercise is to give an algebraic description of the fundamental group $\pi_1(X, x_0)$. For this we proceed as follows:

1. Show that there is a contractible subcomplex $X_0 \subseteq Y \subseteq X_1$, and construct an isomorphism $\pi_1(X, x_0) \cong \pi_1(X_2/Y, [x_0])$.

From now on, we may therefore assume that X is a 2-dimensional CW-complex with precisely one 0-cell x_0.

2. Show that X is homotopy equivalent, relative to X_1, to a 2-dimensional CW-complex Y for which 'the' attaching maps $S^1 \to Y$ of the 2-cells are based maps.

We will assume for the rest of this exercise, that X is of this form. As an upshot, the characteristic map $\chi\colon D^1 \to X$ of any closed 1-cell factors through a pointed map $\bar{\chi}\colon (S^1, 1) \cong D^1/\partial D^1 \to X$, defining a class $[\bar{\chi}] \in \pi_1(X_1, x_0)$, and likewise the attaching map $f\colon S^1 = \partial D^2 \to X_1$ of every 2-cell defines a class in $\pi_1(X_1, x_0)$.

3. Show that $\pi_1(X_1, x_0)$ is freely generated by the classes $[\bar{\chi}]$ for the characteristic maps $\chi\colon D^1 \to X_1$ of 1-cells.
4. Show that the homomorphism $\pi_1(X_1, x_0) \to \pi_1(X, x_0)$ induced by $X_1 \hookrightarrow X$ is surjective, and its kernel contains all the classes $[f]$ for attaching maps f.
5. Let Y be any pointed space. Show that a based map $g\colon X_1 \to Y$ extends to $\bar{g}\colon X \to Y$ if and only if $[gf]$ is trivial in $\pi_1(Y)$ for every attaching map $f\colon S^1 \to X_1$ of a 2-cell of X.
6. Use the bonus exercise to conclude that the kernel of the surjection $\pi_1(X_1, x_0) \to \pi_1(X, x_0)$ is the *smallest* normal subgroup containing the classes $[f]$.

Remark. Altogether, we have constructed a surjective homomorphism from the free group generated by the 1-cells of X to $\pi_1(X, x_0)$, and we have identified a set of elements generating the kernel of this homomorphism (as a normal subgroup). In the language of abstract group theory, this means we have given a *presentation* of $\pi_1(X, x_0)$. In favorable cases, one can use this presentation to identify $\pi_1(X, x_0)$ with some known group. But don't get your hopes up too high: in general, it is provably impossible to even decide whether a given finite presentation describes the trivial group (and hence, by the above exercise, there is no algorithm that takes a 'nice' description of a finite CW-complex and decides whether it is simply connected).