WS 2024/25

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Exercises for **Topology I**
Sheet 6

You can obtain up to 10 points per exercise (plus bonus points, where applicable).

Definition. A category $\mathcal{C}$ is called *small* if the collection $\mathrm{Ob}(\mathcal{C})$ of objects of $\mathcal{C}$ forms a set. We write **Cat** for the category whose objects are small categories, with the hom set $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$ given by the set of functors $\mathcal{C} \to \mathcal{D}$ (the smallness of $\mathcal{C}$ and $\mathcal{D}$ guarantees that this is indeed a set). Composition in **Cat** is given by composition of functors.

Exercise 1.
1. Extend the assignment $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$ to an isomorphism $\mathbf{Cat} \to \mathbf{Cat}$.
2. A small category $\mathcal{C}$ is called a *groupoid* if every morphism in $\mathcal{C}$ is invertible. Show that for every groupoid $\mathcal{C}$ there is an isomorphism $\mathcal{C} \cong \mathcal{C}^{\mathrm{op}}$.
3. Give an example of a category $\mathcal{C}$ with $\mathcal{C} \cong \mathcal{C}^{\mathrm{op}}$, but such that $\mathcal{C}$ is *not* a groupoid.

Definition. Let X, Y be simplicial sets. A *morphism of simplicial sets* $X \to Y$ consists of maps $X_n \to Y_n$ for all $n \geq 0$, such that for every $\alpha\colon [m] \to [n]$ in Δ the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle \alpha^*}\downarrow & & \downarrow{\scriptstyle \alpha^*} \\ X_m & \xrightarrow[f_m]{} & Y_m \end{array}$$

commutes. (In the language of category theory, this means that morphisms of simplicial sets are precisely the *natural transformations* of functors $\Delta^{\mathrm{op}} \to \mathbf{Set}$.) You can convince yourself that the simplicial sets and morphisms of simplicial sets form a category, with composition given degreewise.

Definition. Let X be a simplicial set. A *subsimplicial set* of X consists of sets $A_n \subseteq X_n$ for all $n \geq 0$ such that for every $\alpha\colon [m] \to [n]$ in Δ the map $\alpha^*\colon X_n \to X_m$ satisfies $\alpha^*(A_n) \subseteq A_m$.

Exercise 2.
1. Let X be a simplicial sets and let $(A_n \subseteq X_n)_{n \geq 0}$ be a subsimplicial set. Show that the A_n's can be made into a simplicial set A in a unique way such that the inclusions define a morphism of simplicial sets $A \to X$.
2. For $m, n \geq 0$ and $k \in [n]$ we define
$$(\Lambda^n_k)_m \subseteq (\Delta^n)_m = \{f\colon [m] \to [n] \text{ weakly monotone}\}$$
as the subset of those maps f whose image does not contain $[n] \smallsetminus \{k\}$. Show that the subsets $(\Lambda^n_k)_m$ define a subsimplicial set of Δ^n.

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Definition. Let X be a set. A *partial order* on X is a relation $\leq$ satisfying the following conditions:

1. Reflexivity: $x \leq x$ for every $x \in X$.
2. Transitivity: If $x \leq y$ and $y \leq z$, then also $x \leq z$.
3. Antisymmetry: If $x \leq y$ and $y \leq x$, then $x = y$.

The pair $(X, \leq)$ is called a *partially ordered set*, or *poset* for short. There is a category **Poset** of posets, with morphisms given by the weakly monotone maps.

Exercise 3. 1. Let $(X, \leq)$ be a poset. We define $\mathrm{Ob}(\mathcal{C}_X) = X$ and for all $x, y \in X$

$$\mathrm{Hom}_{\mathcal{C}_X}(x, y) = \begin{cases} \{*\} & \text{if } x \leq y \\ \emptyset & \text{otherwise.} \end{cases}$$

Show that there is a unique composition law that turns $\mathcal{C}_X$ into a category.

2. Extend the assignment $X \mapsto \mathcal{C}_X$ to a functor $\mathbf{Poset} \to \mathbf{Cat}$ inducing bijections

$$\mathrm{Hom}_{\mathbf{Poset}}(X, Y) \to \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}_X, \mathcal{C}_Y).$$

Remark. A functor inducing bijections on hom sets is called *fully faithful*.

3. Let $\mathcal{C}$ be a small category such that $|\mathrm{Hom}_{\mathcal{C}}(x, y)| \leq 1$ for all $x, y \in \mathrm{Ob}(\mathcal{C})$. We define a relation $\leq$ on X by declaring that $x \leq y$ iff there exists a map $x \to y$ in $\mathcal{C}$. Show that this relation is not a partial order in general, but that it descends to a partial order on the set $\pi_0(\mathcal{C})$ of isomorphism classes of objects.

Exercise 4. Let X be a simplicial set. Given $x, y \in X_0$, we write $x \sim y$ if there exists an element $e \in X_1$ such that $d_1^*(e) = x$ and $d_0^*(e) = y$.

1. Show that $\sim$ is reflexive, but in general neither symmetric nor transitive.
2. Show that $\sim$ is an equivalence relation provided that every morphism of simplicial sets $f\colon \Lambda^2_k \to X$, $0 \leq k \leq 2$ extends to Δ^2, i.e. there exists a morphism $F\colon \Delta^2 \to X$ making the following diagram of simplicial sets commute:

$$\begin{array}{ccc} \Lambda^2_k & \xrightarrow{f} & X. \\ \big\downarrow & \nearrow_{F} & \\ \Delta^2 & & \end{array}$$

Does the converse also hold?

Hint. First show that morphisms $f\colon \Delta^n \to X$ of simplicial sets are in bijection with n-simplices of X via the assignment $f \mapsto f_n(\mathrm{id}_{[n]})$.

3. Let Y be a topological space. Show that $\sim$ defines an equivalence relation on $\mathcal{S}(Y)_0$, and construct a bijection $\mathcal{S}(Y)_0/_\sim \cong \pi_0(Y)$.

$*$4. (*10 bonus points*) Let Y be a topological space and let $0 \leq k \leq n$, $n \neq 0$. Show that any morphism of simplicial sets $\Lambda^n_k \to \mathcal{S}(Y)$ extends to Δ^n.

Remark. Simplicial sets with this property are called *Kan complexes* (in honor of Daniel Kan).