Construction of tame supercuspidal representations in arbitrary residue characteristic

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Abstract

Let F be a non-archimedean local field whose residue field has at least four elements. Let G be a connected reductive group over F that splits over a tamely ramified extension of F. We provide a construction of supercuspidal representations of $G(F)$ that contains all the supercuspidal representations constructed by Yu in 2001 ([Yu01]), but that also works in residual characteristic two. The input for our construction is described uniformly for all residual characteristics and is analogous to Yu's input except that we do not require our characters to satisfy the genericity condition (GE2) that Yu imposes.

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> Remark. *It is precisely here that our assumption of p odd makes its impact. We are dealing with the representation theory of a 2-step nilpotent p-group. The extra complications in this theory that arise when $p = 2$ could be handled, but at the expense of a long digression.*
>
> — Roger Howe, 1977*

1 Introduction

The construction of supercuspidal representations of p-adic groups plays a central role in the representation theory of p-adic groups and beyond. While for GL_n and its inner forms such a construction is known in full generality ([BK93, SS08]), for other families of reductive groups, including classical groups, the existing general constructions [Adl98, Yu01, Ste08] assume that $p \neq 2$. In the present paper we provide a construction of supercuspidal representations of general tame p-adic groups for all p. Our construction generalizes Yu's construction ([Yu01]) by allowing $p = 2$ and, in addition, relaxing a genericity condition imposed by Yu on the input for the construction. In particular, we recover as a special case the supercuspidal representations constructed by Yu, which are all supercuspidal representations if p does not divide the order of the absolute Weyl group of G. Even in this already known setting, our proof contains new elements. In particular, we do not rely on Gérardin's delicate analysis of the Weil representation in [Gér77, Theorem 2.4(b)].

The reasons that previous authors required $p \neq 2$ are subtle and depend on the setting. Stevens' work for classical groups assumed $p \neq 2$ because the Glauberman correspondence (see [Ste01, (2.1) Theorem]) does not apply to involutions of pro-2-groups. Yu's work for more general tame p-adic groups assumed $p \neq 2$ since he crucially relied on the theory of Heisenberg–Weil representations (see [Yu01, Section 10]). At the simplest level, this theory does not immediately extend to the case $p = 2$ because a factor of $1/2$ appears in Yu's definition of the Heisenberg group over $\mathbb{F}_p$. But as we will see below, there are much more serious obstructions, which we address in this paper.

To describe our results in more detail, let F be a non-archimedean local field whose residue field has characteristic p and cardinality q, and let G be a connected reductive group over F that splits over a tamely ramified extension of F. Let $\Upsilon = ((G_i)_{1 \le i \le n+1}, x, (r_i)_{1 \le i \le n}, \rho, (\phi_i)_{1 \le i \le n})$ be an input for Yu's construction, but where we allow $p = 2$ and relax the assumption that each ϕ_i satisfies the genericity condition (GE2) of [Yu01]; see Section 3.1 for more details.

Theorem A (cf. Theorem 3.6.9)**.** *If $q > 2$, then the input Υ gives rise to a finite set of supercuspidal representations, each of which is a compact induction $\text{c-ind}_{\widetilde{K}}^{G(F)}(\sigma)$ of an irreducible representation σ from an open, compact-mod-center subgroup $\widetilde{K}$ of $G(F)$.*

*[How77, p. 447], in a paper constructing supercuspidal representations of $\mathrm{GL}_n(F)$.

See Theorem 3.6.9 for a more precise statement[2]. For the reader familiar with Yu's construction, let us mention that the open, compact-mod-center subgroup $\widetilde{K}$ lies between the following two subgroups that are built out of Moy–Prasad filtration subgroups of the reductive groups appearing in the input Υ (the notation is explained in Section 1.1.):

$$\begin{aligned} K^+ &:= G_1(F)_{x,r_1/2} \cdot G_2(F)_{x,r_2/2} \cdots G_n(F)_{x,r_n/2} \cdot N_G(G_1, G_2, \dots, G_n, G_{n+1})(F)_{[x]}\,, \\ K &:= G_1(F)_{x,r_1/2} \cdot G_2(F)_{x,r_2/2} \cdots G_n(F)_{x,r_n/2} \cdot G_{n+1}(F)_{[x]}\,. \end{aligned}$$

The construction of the representation σ proceeds in two steps. The first step produces from the given input Υ a unique representation, called $\rho \otimes \kappa^-$, of a subgroup K^- of K with K/K^- being a finite abelian 2-group; see (3.2.2) for the precise definition of K^- and Section 3.2 for an overview of the construction of the representation. The second step produces a representation σ of $\widetilde{K}$ using Clifford theory, which allows the reader to make choices that we discuss and parameterize in Section 3.3. If $G = \mathrm{GL}_n$, or if G is a classical group and $p \neq 2$, or if we are in Yu's setting, i.e., if $p \neq 2$ and each ϕ_i satisfies the additional genericity condition (GE2) of [Yu01], then $\widetilde{K} = K^- = K$ and no choices are required.

We provide a few more details on the two steps. The first step generalizes Yu's construction using the theory of Heisenberg–Weil representations. The key challenge is that the theory of Heisenberg–Weil representations as used by Yu is not available if $p = 2$.

Already the Heisenberg $\mathbb{F}_p$-group itself shows an exceptional feature for $p = 2$. While for $p > 2$ all Heisenberg $\mathbb{F}_p$-groups of the same cardinality are isomorphic, for $p = 2$ there are two isomorphisms classes of Heisenberg groups of cardinality 2^{1+2n} for any $n > 1$. Both classes of groups arise in the construction of supercuspidal representations (see Example 3.4.2). However, this quirk is not an obstacle for the construction of supercuspidal representations as the theory of Heisenberg representations carries over to the setting of $p = 2$. That theory has also been already used in the case of the the general linear group; see, for example, [Wal86, Section II] and [BK93, (7.2.4) Proposition].

On the other hand, the theory of Weil representations, which is crucial in the construction of supercuspidal representations, does not work for $p = 2$ in the same way as for $p > 2$. Let H be a Heisenberg $\mathbb{F}_p$-group of order p^{1+2n}. A key difference consists in the structure of the group of automorphisms $\mathrm{Aut}_{Z\text{-fix}}(H)$ of H that act trivially on the center of H. When $p \neq 2$, the group $\mathrm{Aut}_{Z\text{-fix}}(H)$ decomposes as a semi-direct product $\mathrm{Aut}_{Z\text{-fix}}(H) \simeq \mathbb{F}_p^{2n} \rtimes \mathrm{Sp}_{2n}(\mathbb{F}_p)$, and the projective Weil representation of $\mathrm{Sp}_{2n}(\mathbb{F}_p)$ admits a linearization, which Yu used to construct supercuspidal representations. When $p = 2$, the group $\mathrm{Aut}_{Z\text{-fix}}(H)$, which we call the pseudosymplectic group $\mathrm{Ps}_{2n}(\mathbb{F}_2)$, following Weil, does not factor as a semidirect product: rather, there is a short exact sequence $1 \to \mathbb{F}_2^{2n} \to \mathrm{Ps}_{2n}(\mathbb{F}_2) \to \mathrm{O}_{2n}(\mathbb{F}_2) \to 1$, which is nonsplit if $n \geq 3$. Because $\mathrm{Ps}_{2n}(\mathbb{F}_2)$ contains the group $\mathbb{F}_2^{2n}$ of inner automorphisms, its projective Weil representation does not linearize (see Remark 2.5.1).

[2] In fact, Theorem 3.6.9 assumes $q > 3$, ultimately because of Lemma 3.6.7. When $q = 3$, although our analysis of the Heisenberg–Weil representation is insufficient to treat this case, one can combine [Yu01] and [Fin21] with our analysis of the failure of (GE2) to construct supercuspidal representations from Υ.

So when $p = 2$, there is no natural ambient group, like $\mathrm{Sp}_{2n}(\mathbb{F}_p)$ when $p \neq 2$, whose Weil representation we can use to construct supercuspidal representations. Instead, we prove in Lemma 2.5.5 that the projective Weil representation linearizes over certain large subgroups of the automorphism group $\mathrm{Aut}_{Z\text{-fix}}(H)$, and we show that we can arrange for all the relevant groups that appear in the construction of supercuspidal representations to map to such linearizing subgroups (Lemma 3.5.2 and Corollary 3.5.3).

A priori two possible linearizations of the (restriction of the) projective Weil representations differ by a character of the linearizing subgroup. Contrary to $\mathrm{Sp}_{2n}(\mathbb{F}_p)$ with $p \neq 2$, whose characters are trivial unless $(n, p) = (1, 3)$, our linearizing subgroups do admit non-trivial characters in general. In order to obtain a unique representation κ^- of K^- using our theory of Heisenberg–Weil representations, we observe that if $p = 2$, the Heisenberg representation is self-dual and thus carries an additional real or quaternionic structure. Requiring the Weil representation to preserve this structure pins down the Weil representation up to a character of order two (Remark 2.5.3), and pin down κ^- uniquely if $q > 2$ (Proposition 3.5.6).

The second step in the construction of σ involves Clifford theory (see Section 3 for details). This allows the reader to make a choice and the finite set of supercuspidal representations mentioned in Theorem A correspond to different choices. These additional choices can only be described after κ^- is constructed and we therefore found it unnatural to record them as part of the input Υ, as we explain in more detail in Remark 3.3.3. Note that the quotient $\widetilde{K}/K$ (and hence also $\widetilde{K}/K^-$) is not always abelian (see Appendix D). However, we prove that $\widetilde{K}/K^-$ is a p-group ((3.2.2) and Theorem 3.6.8(b)).

Once the construction of $(\widetilde{K}, \sigma)$ is achieved, the proof that the resulting representation $\text{c-ind}_{\widetilde{K}}^{G(F)}(\sigma)$ is irreducible supercuspidal follows in rough terms [Yu01] and [Fin21], though the details require some new key ideas.

First, we can no longer rely on Gérardin's analysis of the Weil representation, in particular [Gér77, Theorem 2.4(b)], as he only covers the Weil representations of $\mathrm{Sp}_{2n}(\mathbb{F}_p)$ that appear for $p \neq 2$. Consequently, our arguments reprove without using [Gér77, Theorem 2.4(b)] that Yu's original construction yields supercuspidal representations.

Second, once we remove the condition that the characters in the input Υ satisfy Yu's condition (GE2), the proof of supercuspidality requires more complicated arguments. The inducing group $\widetilde{K}$ is now larger than K in general, and at certain steps in the arguments we must replace the reductive groups G_i of the input Υ by disconnected groups with identity component G_i. The problem is that the character ϕ_{i-1} of $G_i(F)$ in the input Υ need not extend to this disconnected group, invalidating one key step in the old arguments of supercuspidality. To compensate, we carefully pass certain results about intertwiners to simply-connected covers. We refer the reader to the proof of Theorem 3.6.8 for details.

Note that not requiring (GE2) produces supercuspidal representations not covered by Yu's construction not only if $p = 2$, but also if p is odd and small[3], e.g., if $G = \mathrm{SL}_n$ and p divides n.

[3](GE2) can fail only if p is a torsion prime for the dual root datum of G. The torsion primes p of a simple adjoint group are $p \mid n$ for A_n; 2 for B_n, C_n, D_n, G_2; 2, 3 for F_4, E_6 E_7; and 2, 3, 5 for E_8. See [Ste75].

1.1 Notation and conventions

We let F be a nonarchimedean local field of residue characteristic p with discrete valuation $\mathrm{val}\colon F \twoheadrightarrow \mathbb{Z} \cup \{\infty\}$. We denote by k_F the residue field of F and by $q = |k_F|$ the cardinality of k_F. We fix a separable closure F^{sep} of F and take all finite separable field extensions of F to lie inside F^{sep}.

All reductive groups in this paper are required to be connected unless explicitly stated otherwise. Let G be a reductive group over F. We write G^{der} for the derived subgroup of G and G^{sc} for the simply connected cover of G^{der}. We denote the image of $G^{\mathrm{sc}}(F)$ in $G(F)$ by $G(F)^{\natural}$. We denote the Lie algebra of G either by $\mathrm{Lie}(G)$ or by using lowercase Fraktur letters, so that $\mathfrak{g}$ is the Lie algebra of G, for example. Let $\mathrm{Lie}^*(G)$ denote the dual Lie algebra. We write $\widehat{G}$ for the Langlands dual group of G.

Given a linear algebraic group H, we write H° for the connected component of H containing the identity and $\pi_0(H) := H/H^\circ$ for the component group of H, a finite algebraic group.

Given a torus T, we denote by $X^*(T)$ the set of characters of $T_{F^{\mathrm{sep}}} := T \times_F F^{\mathrm{sep}}$ with action of the absolute Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$. For an algebraic group P containing T we write $\Phi(P, T)$ for the set of non-zero weights of T acting on the Lie algebra of P, equipped with the action of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$. In particular, if T is a maximal torus of G, then $\Phi(G, T) \subset X^*(T)$ is the absolute root system of G with respect to T, equipped with Galois action. In this case, given $\alpha \in \Phi(G, T)$, we write $H_\alpha := d\alpha^\vee(1) \in \mathrm{Lie}(T)(F^{\mathrm{sep}})$. When T is a maximal split torus of G, so that $\Phi(G, T)$ is the relative root system, we denote by U_α the root group for the set of positive-integer multiples of $\alpha \in \Phi(G, T)$. So if $2\alpha \in \Phi(G, T)$, then U_α is nonabelian.

Let $\widetilde{\mathbb{R}} := \mathbb{R} \cup \{r+ \mid r \in \mathbb{R}\} \cup \{\infty\}$ with its usual order, as in [KP23, Section 1.6]. We write $\mathscr{B}(G, F)$ for the enlarged Bruhat–Tits building of G over F. For $r \in \widetilde{\mathbb{R}}$ and $x \in \mathscr{B}(G, F)$, we denote by $\mathfrak{g}(F)_{x,r}$, $\mathfrak{g}(F)^*_{x,r}$, $U_\alpha(F)_{x,r}$, and $G(F)_{x,r}$ the respective depth-r Moy–Prasad subgroups at x of the F-points of the Lie algebra $\mathfrak{g}$, its linear dual $\mathfrak{g}^*$, the root group U_α, and the group G, where in the last case we assume $r \geq 0$. If F is clear from the context, we might omit it from the notation, e.g., we write $\mathfrak{g}_{x,r}$ instead of $\mathfrak{g}(F)_{x,r}$ and $G_{x,r}$ instead of $G(F)_{x,r}$. If G^{der} is anisotropic, for example, if $G = T$ is a torus, then we may suppress x from the notation and write $\mathfrak{g}_r$, $\mathfrak{g}^*_r$, and $G(F)_r$. Let $G(F)^{\natural}_{x,r} := G(F)^{\natural} \cap G(F)_{x,r}$. Given $x \in \mathscr{B}(G, F)$, we write $[x]$ for the image of x in the reduced building of G. If a group H acts on the reduced building of G, then we denote by $H_{[x]}$ the stabilizer of $[x]$ in $H(F)$.

A subgroup H of G is a twisted Levi subgroup if $H_E := H \times_F E$ is a Levi subgroup of a parabolic subgroup of G for some (finite, separable) field extension E/F. If a twisted Levi subgroup H splits over a tame extension E, then there is an admissible embedding of buildings $\mathscr{B}(H, F) \to \mathscr{B}(G, F)$ ([KP23, Section 14.2]). In general, this embedding is only well-defined up to translation, but all translations have the same image. In this paper we will identify $\mathscr{B}(H, F)$ with its image in $\mathscr{B}(G, F)$ for some fixed choice of embedding, and all constructions are independent of this choice.

In this paper, the word "representation" with no additional modifiers refers to a complex representation. However, we will sometimes work with "R-representations" or "R-linear representations" for $R \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, referring to R-linear representations on R-modules. See page 11 for a discussion of these notions, which can also be viewed as extra structure on an underlying complex representation. We write c-ind for compact induction. Given an irreducible representation π of $G(F)$, we denote by $\operatorname{depth}(\pi)$ the depth of π.

Given a group A, we denote by $Z(A)$ the center of A and by $\operatorname{Irr}(A)$ the set of isomorphism classes of irreducible representations of A. We write $[a, b] := aba^{-1}b^{-1}$ for the commutator of a and b. Given a subgroup B of A, let $N_A(B)$ be the normalizer of B in A. More generally, given subgroups $B_1, \ldots, B_n$, let $N_A(B_1, \ldots, B_n) := \bigcap_{i=1}^{n} N_A(B)$. We write ${}^aB := aBa^{-1}$ and given a representation π of B, we write ${}^a\pi$ for the representation $x \mapsto \pi(a^{-1}xa)$ of ${}^aB$. If in addition B is normal in A, then we denote by $N_A(\pi)$ the set of $a \in A$ such that ${}^a\pi \simeq \pi$ and by $\operatorname{Irr}(A, B, \pi)$ the set of $\sigma \in \operatorname{Irr}(A)$ such that $\sigma|_B$ contains π. Given a set X and an action of A on X, we write X^A for the set of elements of X fixed by A, and given in addition an element $x \in X$, we write $Z_A(x)$ for the set of $a \in A$ such that $a(x) = x$.

Suppose k is an arbitrary field and ℓ/k is a field extension. If ℓ/k is finite and T is a k-torus, then we denote by $\operatorname{Nm}_{\ell/k} \colon T(\ell) \to T(k)$ the corresponding norm map. In particular, when $T = \mathbb{G}_{\mathrm{m}}$, this map is the usual norm $\ell^\times \to k^\times$. If ℓ/k is Galois, we write $\operatorname{Gal}(\ell/k)$ for the Galois group of the extension.

Let V be a vector space over a field k. Given a quadratic form Q on V, we write $\mathrm{O}(\mathsf{V}, Q)$ for the usual orthogonal group, the elements of $\mathrm{GL}(\mathsf{V})$ stabilizing Q. We define the subgroup $\mathrm{SO}(\mathsf{V}, Q)$ of $\mathrm{O}(\mathsf{V}, Q)$ as the kernel of the determinant if $\operatorname{char}(k) \neq 2$ or the kernel of the Dickson invariant if $\operatorname{char}(k) = 2$, so that $[\mathrm{O}(\mathsf{V}, Q) : \mathrm{SO}(\mathsf{V}, Q)] = 2$ if $\mathsf{V} \neq 0$. Similarly, given an alternating form ω on V, we write $\mathrm{Sp}(\mathsf{V}, \omega)$ for the usual symplectic group, the elements of $\mathrm{GL}(\mathsf{V})$ stabilizing ω. We will often drop Q or ω from the notation in $\mathrm{O}(\mathsf{V}, Q)$, $\mathrm{SO}(\mathsf{V}, Q)$ and $\mathrm{Sp}(\mathsf{V}, \omega)$ when their presence is clear from context.

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2 Heisenberg–Weil representations

Let k be a field. The reader is welcome to take $k = \mathbb{F}_p$ since this is the only case that will be needed in the construction of supercuspidal representations. Nonetheless, we allow k to be a general field, or sometimes, a finite field, because it is no harder to state the results in that setting. Recall that p is a prime number, including possibly $p = 2$, and q is a positive integer power of p.

2.1 Heisenberg groups

In this subsection we explain how to extend the definition of the Heisenberg group over $\mathbb{F}_p$ for odd p (cf. Example 2.1.7) to the case $p = 2$. The resulting group has extremely explicit models (Construction 2.1.3 and Examples 2.1.7 and 2.1.8), but to find this group within a p-adic group, we will also characterize it by intrinsic properties (Definition 2.1.2).

Recall that the exponent of a finite group is the least common multiple of the orders of its elements. The following class of finite p-groups already appeared in the work of Hall and Higman [HH56, Section 2.3], and has been extensively studied and used as a tool by finite group theorists; see [Gor80, p. 183 and Chapter 5.5] for a textbook treatment.

Definition 2.1.1. A finite p-group P is an extraspecial p-group if its center $Z(P)$ has order p and $P/Z(P)$ is abelian of exponent p.

Specializing the definition of extraspecial p-group very slightly and allowing the degenerate case $\mathbb{Z}/p\mathbb{Z}$ yields the version of the Heisenberg group relevant to the construction of supercuspidal representations.

Definition 2.1.2. A Heisenberg $\mathbb{F}_p$-group is a finite group P whose center $Z(P)$ has order p and for which $P/Z(P)$ is abelian of exponent at most p and if $p \neq 2$ then also P is of exponent at most p.

In other words, a Heisenberg $\mathbb{F}_p$-group is either $\mathbb{Z}/p\mathbb{Z}$ or an extraspecial p-group, which is in addition required to have exponent at most p when $p \neq 2$. The case $p = 2$ requires special care: In this case we cannot require P to have exponent 2 because that would force P to be abelian, and hence we would only obtain the group $P = Z(P) \simeq \mathbb{Z}/2\mathbb{Z}$.

The "$\mathbb{F}_p$" appearing in our terminology reflects the fact that there is a general construction of the Heisenberg group over a field k, specializing to Definition 2.1.2 when $k = \mathbb{F}_p$. We recall this construction to help with computations and comparison with the literature, though we will more often take the intrinsic viewpoint of Definition 2.1.2.

Construction 2.1.3 (Heisenberg k-groups)**.** Let k be a field and V a finite-dimensional k-vector space. Given a bilinear form $B\colon \mathsf{V} \otimes_k \mathsf{V} \to k$, we can interpret B as an element of $Z^2(\mathsf{V}, k)$ and define the resulting extension $\mathsf{V}^\sharp_B$ of V by k. In other words, $\mathsf{V}^\sharp_B$ is the group with underlying set $k \times \mathsf{V}$ and multiplication

$$(a, v) \cdot (b, w) = (a + b + B(v, w), v + w).$$

The group $\mathsf{V}^\sharp_B$ is a Heisenberg k-group if $Z(\mathsf{V}^\sharp_B) = k$, where we identify k with $k \times \{0\}$ from now on, or equivalently, if the associated alternating form below is nondegenerate:

$$\omega_B(v, w) := B(v, w) - B(w, v). \tag{2.1.4}$$

If $q = p^d$ with $d \geq 2$, then a Heisenberg $\mathbb{F}_q$-group is a p-group but not a Heisenberg $\mathbb{F}_p$-group because the center is too large. Relatedly, our construction of supercuspidal representations will ultimately use Heisenberg $\mathbb{F}_p$-groups, even though the residue field of F may be larger than $\mathbb{F}_p$. However, the two notions agree for $k = \mathbb{F}_p$.

Lemma 2.1.5. *Every Heisenberg $\mathbb{F}_p$-group is obtained from Construction 2.1.3 with $k = \mathbb{F}_p$.*

Proof. Let P be a Heisenberg $\mathbb{F}_p$-group. If P has order p, then $P \simeq \mathsf{V}_B^\sharp$ for V being a zero-dimensional $\mathbb{F}_P$-vector space. Hence, we assume the order of P is p^{2n+1} with $n \geq 1$, which implies that P is an extraspecial p-group. Therefore P has the following explicit presentation (see, e.g. [Win72, p. 160]): P has generators $x_1, x_2, \dots x_{2n}$ and relations

$$x_i x_j x_i^{-1} x_j^{-1} = \begin{cases} z & \text{if } (i,j) = (2d-1, 2d) \text{ for } 1 \leq d \leq n, \\ z^{-1} & \text{if } (i,j) = (2d, 2d-1) \text{ for } 1 \leq d \leq n, \\ 1 & \text{otherwise,} \end{cases}$$

$zx_i = x_i z$ for $1 \leq i \leq 2n$, $z^p = 1$, and $x_i^p = 1$ for $1 \leq i \leq 2n-2$, and
if $p \neq 2$, then $x_{2n-1}^p = x_{2n}^p = 1$, and
if $p = 2$, then either (case a) $x_{2n-1}^2 = x_{2n}^2 = 1$ or (case b) $x_{2n-1}^2 = x_{2n}^2 = z$.

Let $\mathsf{V} = \mathbb{F}_p^{2n}$ with standard basis $\{e_i : 1 \leq i \leq 2n\}$. When $p \neq 2$ or $p = 2$ and P satisfies the relations in case a above, then we define B by

$$B(e_i, e_j) = \begin{cases} 1 & \text{if } (i,j) = (2d-1, 2d) \text{ for some } d \\ 0 & \text{otherwise.} \end{cases}$$

When $p = 2$ and P satisfies the relations in case b above, then define B by setting $B(e_{2d-1}, e_{2d}) = 1$ for $1 \leq d < n$,

$$B(e_{2n-1}, e_{2n-1}) = B(e_{2n}, e_{2n}) = B(e_{2n-1}, e_{2n}) = 1,$$

and $B(e_i, e_j) = 0$ otherwise. Sending x_i to $(0, e_i)$ and z to $(1, 0)$ defines a surjective group homomorphism $P \twoheadrightarrow V_B^\sharp$, which is an isomorphism since both groups have the same order. □

Although there are many possible bilinear forms B for which ω_B is nondegenerate, the classification of Heisenberg k-groups is rather simple.

Lemma 2.1.6. *In the setting of Construction 2.1.3, let $B, B' \colon \mathsf{V} \otimes_k \mathsf{V} \to k$ be two bilinear forms whose associated alternating forms ω_B and $\omega_{B'}$ are nondegenerate.*

(a) Suppose $\operatorname{char}(k) \neq 2$. *Then $\mathsf{V}_B^\sharp \simeq \mathsf{V}_{\omega_B}^\sharp \simeq \mathsf{V}_{B'}^\sharp$.*

(b) Suppose $\operatorname{char}(k) = 2$. *If the quadratic forms $B(v,v)$ and $B'(v,v)$ are* $\operatorname{GL}(\mathsf{V})$*-conjugate, then $\mathsf{V}_B^\sharp \simeq \mathsf{V}_{B'}^\sharp$. The converse holds if $k = \mathbb{F}_2$.*

We refer the reader to Appendix A for a review of the definition and properties of alternating and quadratic forms in characteristic 2.

Proof. Given a function $f \colon \mathsf{V} \to k$ and a linear automorphism $\sigma \in \operatorname{GL}(\mathsf{V})$, the map

$$(a, v) \mapsto (a + f(v), \sigma v)$$

defines an isomorphism $\mathsf{V}_B^\sharp \to \mathsf{V}_{B'}^\sharp$ as long as the following identity holds:

$$f(v+w) - f(v) - f(w) = B'(\sigma v, \sigma w) - B(v,w), \qquad v, w \in \mathsf{V}.$$

If in addition $f \in \mathrm{Sym}^2(\mathsf{V}^*)$ is a quadratic form, then the lefthand side of this expression is a general symmetric bilinear form when $\mathrm{char}(k) \neq 2$ and a general alternating form when $\mathrm{char}(k) = 2$ (see Appendix A).

When $\mathrm{char}(k) \neq 2$, since the form

$$(2B - \omega_B)(v,w) = B(v,w) + B(w,v)$$

is symmetric, $\mathsf{V}_B^\sharp \simeq \mathsf{V}_{2B}^\sharp \simeq \mathsf{V}_{\omega_B}^\sharp$. But then $\mathsf{V}_B^\sharp \simeq \mathsf{V}_{\omega_B}^\sharp \simeq \mathsf{V}_{\omega_{B'}}^\sharp \simeq \mathsf{V}_{B'}^\sharp$ because any two nondegenerate alternating forms on V are $\mathrm{GL}(\mathsf{V})$-conjugate.

When $\mathrm{char}(k) = 2$, if $B'(v,v)$ is conjugated to $B(v,v)$ by $\sigma \in \mathrm{GL}(\mathsf{V})$, then the form $B'(\sigma v, \sigma w) - B(v,w)$ is alternating and thus $\mathsf{V}_B^\sharp \simeq \mathsf{V}_{B'}^\sharp$. Note that $(a,v)^2 = (B(v,v), 0)$ for any $a \in k$. Hence, if $\tau\colon \mathsf{V}_B^\sharp \simeq \mathsf{V}_{B'}^\sharp$ is an isomorphism of abstract groups, then the automorphism of V induced by τ takes the quadratic form $B(v,v)$ to the quadratic form $B'(v,v)$. When $k = \mathbb{F}_2$, this induced automorphism of V is automatically k-linear. □

Explicitly, Lemma 2.1.6 gives the following description of Heisenberg $\mathbb{F}_p$-groups.

Example 2.1.7 (Heisenberg $\mathbb{F}_p$-groups, odd p)**.** Suppose $p \neq 2$. By Lemma 2.1.6(a), for every $n \geq 1$ there is a unique (up to isomorphism) Heisenberg $\mathbb{F}_p$-group of order p^{2n+1}, constructed as follows. Given a symplectic $\mathbb{F}_p$-vector space (V, ω) of dimension $2n$, the group $\mathsf{V}_\omega^\sharp$ is the set-theoretic product $\mathbb{F}_p \times \mathsf{V}$ with multiplication

$$(a,v) \cdot (b,w) = (a + b + \tfrac{1}{2}\omega(v,w), v + w).$$

Example 2.1.8 (Heisenberg $\mathbb{F}_2$-groups)**.** Let (V, Q) be a finite-dimensional quadratic space over $\mathbb{F}_2$ of even dimension with Q non-degenerate. Up to isomorphism, there are two possible isomorphism classes of (V, Q) when $\dim(\mathsf{V}) \geq 2$: the split space, isomorphic to k^{2n} with Q given by (A.1), and the nonsplit space, isomorphic to $k^{2n-2} \oplus \ell$, where ℓ/k is a quadratic field extension, with Q given by (A.2). By Lemma 2.1.6(b), there are two isomorphism classes of Heisenberg $\mathbb{F}_2$-groups of every fixed order 2^{2n+1}: one of "positive type" for the split form and one of "negative type" for the nonsplit form. In the simplest nontrivial case, order 8, the positive-type group is the dihedral group D_8 of order 8 and the negative-type group is the quaternion group Q_8. Example 3.4.2 shows that both types of groups are needed in the construction of supercuspidal representations.

In the finite group theory literature, extraspecial 2-groups are described as built up from D_8 and Q_8 as follows. Given Heisenberg $\mathbb{F}_2$-groups P and Q, identify $Z(P)$ and $Z(Q)$ with $\mathbb{F}_2$ and define the central product $P \circ Q$ as the quotient $(P \times Q)/Z$ where Z is the kernel of the multiplication map $Z(P) \times Z(Q) \to \mathbb{F}_2$. Then every Heisenberg $\mathbb{F}_2$-group P of order 2^{2n+1} is isomorphic to a central product

$$P_1 \circ P_2 \circ \cdots \circ P_n$$

where each P_i is either D_8 or Q_8. Two such groups are isomorphic if and only if the number of their quaternionic factors has the same parity, since $D_8 \circ D_8 \simeq Q_8 \circ Q_8$. The positive-type group is $D_8 \circ \cdots \circ D_8$ and the negative-type group is $Q_8 \circ D_8 \circ \cdots \circ D_8$.

2.2 R-representations

In this subsection we describe a certain structure of an "R-representation" carried by every self-dual irreducible representation of a finite group. This structure plays a key role in linearizing projective Weil representations, as we explain in more detail in Section 2.5.

Let A be a finite group. Let (π, V) be an irreducible complex representation of A and let (π^*, V^*) be the dual representation. Let $\mathbb{H}$ denote the ring of quaternions over $\mathbb{R}$.

Suppose that π is irreducible. Following Serre [Ser77, Section 13.2], there are the following three mutually exclusive possible situations, indexed by a ring $R \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ that we call the Frobenius–Schur type of π.[4]

(1) π is complex: $\pi \not\simeq \pi^*$, or equivalently, the character of π is not real-valued.

In the remaining two cases $\pi \simeq \pi^*$, but there are two ways that this can happen, depending on the sign of the form $V \otimes_{\mathbb{C}} V \to \mathbb{C}$ resulting from the isomorphism $\pi \simeq \pi^*$.

(2) π is real: the form is symmetric, or equivalently, there is a representation defined over $\mathbb{R}$ whose extension of scalars to $\mathbb{C}$ is π.

(3) π is quaternionic: the form is alternating, or equivalently, there is a structure of a right $\mathbb{H}$-module on V for which the action of G on V is $\mathbb{H}$-linear.

We index the Frobenius–Schur type of π by a ring R because π can be repackaged as an R-module, as follows. Given $R \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, an R-representation of A is a right R-module W together with an R-linear action of A on W, or in other words, a homomorphism ρ from A to the group $\mathrm{GL}(W, R)$ of R-linear automorphisms of W. Then to each R-representation (ρ, W) of A we associate as follows a complex representation (π, V) of A:

(1) If $R = \mathbb{C}$, then $\pi = \rho$.

(2) If $R = \mathbb{R}$, then $V = \mathbb{C} \otimes_{\mathbb{R}} W$ with π the base change of ρ.

(3) If $R = \mathbb{H}$, then $V = W$ with $\mathbb{C}$-module structure pulled back along an $\mathbb{R}$-algebra embedding $\mathbb{C} \hookrightarrow \mathbb{H}$ and π is the composition of ρ and $\mathrm{GL}(W, R) \to \mathrm{GL}(V)$.

If the complex representation π is irreducible, then it has Frobenius–Schur type R.

[4] This terminology is nonstandard but is inspired by the Frobenius–Schur indicator, which equals 0, +1, or −1 for an irreducible representation of Frobenius–Schur type $\mathbb{C}$, $\mathbb{R}$, or $\mathbb{H}$, respectively.

Lemma 2.2.1. *Let (ρ, W) be an irreducible R-representation of A with associated complex representation (π, V). Suppose (π, V) is irreducible. Then the isomorphism class of ρ as an R-representation is uniquely determined by the isomorphism class of π as a complex representation.*

Proof. If $R = \mathbb{C}$, then there is nothing to prove, so assume $R \in \{\mathbb{R}, \mathbb{H}\}$. Then $\mathrm{GL}(W, R) \subseteq \mathrm{GL}(V, \mathbb{C}) = \mathrm{GL}(V)$ and it suffices to show that if two homomorphisms $\rho_1, \rho_2 \colon A \to \mathrm{GL}(W, R)$ whose associated complex representation is irreducible become conjugate in $\mathrm{GL}(V)$, then they were already conjugate in $\mathrm{GL}(W, R)$.

For this, we use Galois descent. There is a reductive $\mathbb{R}$-group G such that $G(\mathbb{R}) = \mathrm{GL}(W, R)$ and $G(\mathbb{C}) \simeq \mathrm{GL}(V)$: if $R = \mathbb{R}$, then G is isomorphic to $\mathrm{GL}_{n,\mathbb{R}}$ where $n = \dim(V)$, and if $R = \mathbb{H}$, then G is the nonsplit inner form of a general linear group.

Let $\mathrm{Transp}(\rho_1, \rho_2)$ be the elements of $\mathrm{GL}(V)$ that conjugate ρ_1 to ρ_2. To complete the proof, we need to show that this set has a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-fixed point. The centralizer $Z_{\mathrm{GL}(V)}(\rho_1)$ of ρ_1 in $\mathrm{GL}(V)$ acts on $\mathrm{Transp}(\rho_1, \rho_2)$ by right multiplication, and this action turns the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-set $\mathrm{Transp}(\rho_1, \rho_2)$ into a $Z_{\mathrm{GL}(V)}(\rho_1)$-torsor in the sense of [Ser02, Chapter 1, Section 5.2]. Such torsors are classified by the cohomology set $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), Z_{\mathrm{GL}(V)}(\rho_1))$. But $Z_{\mathrm{GL}(V)}(\rho_1) = Z(\mathrm{GL}(V)) \simeq \mathbb{C}^\times$ by Schur's Lemma, since the complex representation associated to ρ_1 is irreducible, and $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on this group in the usual way, by complex conjugation. So the set $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), Z_{\mathrm{GL}(V)}) = H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times)$ is trivial by Hilbert's Theorem 90 and thus the torsor is trivial, implying that it has a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-fixed point. $\square$

Next we turn to projective representations. Let $R \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and let W be a finite-dimensional right R-module. Then

$$Z(\mathrm{GL}(W, R)) = Z(R)^\times \simeq \begin{cases} \mathbb{C}^\times & \text{if } R = \mathbb{C}, \\ \mathbb{R}^\times & \text{if } R \in \{\mathbb{R}, \mathbb{H}\} \end{cases}$$

and we write $\mathrm{PGL}(W, R) := \mathrm{GL}(W, R)/Z(R)^\times$. We define a projective R-representation of a finite group A to be a group homomorphism $\bar{\rho} \colon A \to \mathrm{PGL}(W, R)$, and an R-linearization of $\bar{\rho}$ to be a group homomorphism $\rho \colon A \to \mathrm{GL}(W, R)$ lifting $\bar{\rho}$. If ρ and ρ' are two R-linearizations of $\bar{\rho}$, then there is a character $\chi \colon A \to Z(R)^\times$ such that $\rho' = \chi \otimes \rho$. Since A is finite, the character χ takes values in the maximal compact subgroup

$$Z(R)^\times_{\mathrm{c}} = \begin{cases} \{z \in \mathbb{C} \mid |z| = 1\} & \text{if } R = \mathbb{C}, \\ \{\pm 1\} & \text{if } R \in \{\mathbb{R}, \mathbb{H}\}. \end{cases}$$

Consequently, linearizing real or quaternionic projective representations involves less of a choice than linearizing complex projective representations. In the first case the linearization is unique if and only if A has no characters of order two, but in the second case the linearization is unique if and only if A is perfect, which is a much stronger condition.

Lemma 2.2.2. *Let $R \in \{\mathbb{R}, \mathbb{H}\}$, let A be a finite group, let $P \subseteq A$ be a Sylow 2-subgroup, and let $\bar{\rho}$ be a projective R-representation. Then $\bar{\rho}$ has an R-linearization if and only if $\bar{\rho}|_P$ has an R-linearization.*

This result is similar to [Gér77, Lemma 1.5], but much more general: there is no need to assume that A or π have a particular form.

Proof. The pullback of the short exact sequence

$$1 \longrightarrow \mathbb{R}^\times \longrightarrow \mathrm{GL}(V, R) \longrightarrow \mathrm{PGL}(V, R) \longrightarrow 1.$$

via $\bar{\rho}\colon A \to \mathrm{PGL}(V, R)$ yields an extension of A by $\mathbb{R}^\times$ which is split if and only if $\bar{\rho}$ has an R-linearization. Let $c \in H^2(A, \mathbb{R}^\times)$ be the cocycle class attached to this extension, which is trivial if and only if the extension splits. Note that since A is a finite group, $H^2(A, \mathbb{R}^\times) = H^2(A, \{\pm 1\}) \oplus H^2(A, \mathbb{R}^\times_{>0}) = H^2(A, \{\pm 1\})$. Since the restriction map $H^2(A, \{\pm 1\}) \to H^2(P, \{\pm 1\})$ is injective ([Ser79, Chapter IX, Theorem 4]), we can detect the vanishing of c by restricting to P, and hence if $\bar{\rho}|_P$ has an R-linearization, then so does $\bar{\rho}$. □

2.3 Heisenberg representations

In this subsection we recall the representation theory of Heisenberg $\mathbb{F}_p$-groups, paying special attention to the Frobenius–Schur type—real, complex, or quaternionic—of the Heisenberg representation. Let P be a Heisenberg $\mathbb{F}_p$-group, let $\psi\colon \mathbb{F}_p \to \mathbb{C}^\times$ be a nontrivial character, and let $\mathsf{V}_P := P/Z(P)$, an $\mathbb{F}_p$-vector space.

Lemma 2.3.1 (Stone–von Neumann theorem)**.** *There is (up to equivalence) a unique irreducible representation ω_ψ of P whose restriction to $\mathbb{F}_p$ is ψ-isotypic. Moreover,* $\dim(\omega_\psi) = \sqrt{|\mathsf{V}_P|}$.

Proof. This follows from [Gér77, Lemma 1.2] and Lemma 2.1.5. □

Definition 2.3.2. We call the representation ω_ψ of Lemma 2.3.1 the Heisenberg representation of P corresponding to ψ.

In order to relate the Heisenberg representations of P to the Heisenberg representations of appropriate subgroups of P that are themselves Heisenberg $\mathbb{F}_p$-groups, and to compute their Frobenius–Schur type, we first introduce some additional notation and make a few observations.

By Lemma 2.1.5, a direct calculation, and identifying $Z(P)$ with $\mathbb{F}_p$, the formulas

$$\omega_P(xZ(P), yZ(P)) := [x, z], \qquad Q_P(xZ(P)) := x^2 \quad (p = 2) \tag{2.3.3}$$

define a symplectic form on V_P and, when $p = 2$, a nondegenerate quadratic form on V_P. The nondegeneracy of those forms follows from observing that if $P = \mathsf{V}_B^\sharp$, then under the identification of V_P with V, we have $\omega_P = \omega_B$, and, if $p = 2$, then $Q_P(v) = B(v, v)$ for $v \in \mathsf{V}_P = \mathsf{V}$ and $\omega_B = B_{Q_P}$ using the notation of Appendix A. Following Appendix A, we extend the notions of nondegenerate subspace, isotropic subspace, polarization, and partial polarization to V_P, taking these notions with respect to the nondegenerate alternating form ω_P when $p \neq 2$ and with respect to the nondegenerate quadratic form Q_P when $p = 2$.

Let W be a subspace of V_P. A splitting of W (in P) is a subgroup $0 \times \mathsf{W}$ of P for which the natural projection $P \twoheadrightarrow P/Z(P) = \mathsf{V}_P$ induces an isomorphism $0 \times \mathsf{W} \xrightarrow{\sim} \mathsf{W}$. The subspace W admits a splitting if and only if W is isotropic, and all splittings are conjugate under the inner automorphism group V_P of P. At the opposite extreme, the preimage of W in P is a Heisenberg $\mathbb{F}_p$-group if and only if W is a nondegenerate subspace.

Let $\mathsf{V}_P = \mathsf{V}^+ \oplus \mathsf{V}_0 \oplus \mathsf{V}^-$ be a partial polarization, and let P_0 be the preimage of V_0 in P. Let ω_ψ and $\omega_{0,\psi}$ be the Heisenberg representations of P and P_0, respectively. Choose a splitting $0 \times \mathsf{V}^+$ of V^+ in P and let $\mathsf{V}^+ \times P_0$ be the internal direct product of $0 \times \mathsf{V}^+$ and P_0 in P. Let $\mathrm{triv} \boxtimes \omega_{0,\psi}$ denote the inflation of $\omega_{0,\psi}$ along the resulting projection map $\mathsf{V}^+ \times P_0 \to P_0$.

Lemma 2.3.4. *Let P be a Heisenberg $\mathbb{F}_p$-group. With the notation of the paragraph above,*

(a) $\omega_\psi \simeq \mathrm{Ind}_{\mathsf{V}^+ \times P_0}^{P}(\mathrm{triv} \boxtimes \omega_{0,\psi})$ *(b)* $(\omega_\psi)^{0 \times \mathsf{V}^+} \simeq \omega_{0,\psi}$.

Proof. For the first part, by Lemma 2.3.1, we know that $\dim(\omega_\psi) = \sqrt{|\mathsf{V}_P|}$. Since

$$\dim\big(\mathrm{Ind}_{\mathsf{V}^+ \times P_0}^{P}(\mathrm{triv} \boxtimes \omega_{0,\psi})\big) = \sqrt{|\mathsf{V}_0|} \cdot |\mathsf{V}^-| = \sqrt{|\mathsf{V}_P|} = \dim(\omega_\psi)$$

and this induced representation has central character ψ, it must be the Heisenberg representation. For the second part, we use an identification of P with $\mathsf{V}_B^\sharp$ for some B as in Construction 2.1.3 that sends $0 \times \mathsf{V}^+$ to $\{0\} \times \mathsf{V}^+$. Then $\{0\} \times \mathsf{V}^-$, which we identify with V^- via $(0, v) \mapsto v$, forms a set of coset representatives for $P/(\mathsf{V}^+ \times P_0)$, and by the first part we can describe ω_ψ as the space of functions $f\colon \mathsf{V}^- \to \mathsf{V}_{\omega_{0,\psi}}$ on which $\mathsf{V}_B^\sharp$ acts as follows

$$((a, v^+ + v_0 + v^-)f)(x) = \omega_{0,\psi}(a, v_0)\psi(\omega_B(v^+, x))f(x + v^-)$$

where $x \in \mathsf{V}^-$, $(a, v_0) \in \mathsf{V}_0^\sharp$, $v^+ \in \mathsf{V}^+$, and $v^- \in \mathsf{V}^-$. Such an f is fixed by $0 \times \mathsf{V}^+$ if and only if $f(x) = 0$ for all $x \neq 0$. The assignment $f \mapsto f(0)$ is the desired isomorphism. □

If the partial polarization is a polarization, then Lemma 2.3.4(a) gives a construction of the Heisenberg representation. Indeed, in this case $\mathsf{V}_0 = 0$ and $P_0 = \mathbb{F}_p$ if $p \neq 2$ or P has positive type, and $\dim(\mathsf{V}_0) = 2$ and $P_0 = Q_8$ if P has negative type, meaning we can easily construct $\omega_{0,\psi}$ by hand. We will now use this observation to compute the Frobenius–Schur type of the Heisenberg representation, which will ultimately allow us to reduce the number of choices in the construction of supercuspidal representations (see Remark 2.5.3 and Proposition 3.5.6).

Lemma 2.3.5. *The Heisenberg representation is complex if $p \neq 2$, real if $p = 2$ and P is of positive type, and quaternionic if $p = 2$ and P is of negative type.*

Proof. Let ω_ψ denote the Heisenberg representation. If $p \neq 2$, then ω_ψ is complex because $\omega_\psi^* \simeq \omega_{\psi^{-1}} \not\simeq \omega_\psi$. Now suppose $p = 2$. In general, if a complex representation of a subgroup of a finite group is self-dual, then its induced representation is self-dual of the same Frobenius–Schur type as the original representation. Using Lemma 2.3.4(a) for a polarization of V_P, we may therefore assume that $P = \mathbb{F}_2$ or $P = Q_8$. Now in the positive type case the Heisenberg representation $\mathbb{F}_2 \hookrightarrow \{\pm 1\} \subset \mathbb{C}^\times$ is visibly real, and in the negative type case the Heisenberg representation can be identified with the tautological embedding $Q_8 \hookrightarrow \mathbb{H}^\times = \mathrm{GL}(\mathbb{H})$, which is visibly quaternionic. □

Definition 2.3.6. Let $R \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ be the Frobenius–Schur type of the Heisenberg representation ω_ψ corresponding to ψ. The Heisenberg R-representation corresponding to ψ is the irreducible R-representation of P whose associated complex representation is ω_ψ.

2.4 The pseudosymplectic group

We work in the same setting as Construction 2.1.3: Let k be a field, let V be a finite-dimensional k-vector space, and let $B\colon \mathsf{V} \otimes_k \mathsf{V} \to k$ be a bilinear form. Assume that the associated alternating form ω_B of (2.1.4) is nondegenerate. In this subsection we review Weil's definition [Wei64, Section 31] of the pseudosymplectic group $\mathrm{Ps}(\mathsf{V})$, as well as Blasco's extension [Bla93, Section 1] to the case where $\mathrm{char}(k) = 2$ and the associated quadratic form is not split.

Definition 2.4.1. The pseudosymplectic group $\mathrm{Ps}(\mathsf{V}) = \mathrm{Ps}(\mathsf{V}, B)$ is the set of pairs $(f, \sigma) \in \mathrm{Sym}^2(\mathsf{V}^*) \times \mathrm{GL}(\mathsf{V})$ such that

$$f(v + w) - f(v) - f(w) = B(\sigma v, \sigma w) - B(v, w)$$

with multiplication law

$$(f, \sigma) \cdot (f', \sigma') = (f'', \sigma\sigma'), \qquad f''(v) := f(\sigma' v) + f'(v).$$

Our formula is slightly different from Weil's because he uses the right action of $\mathrm{GL}(\mathsf{V})$ on V while we use the left action.

The pseudosymplectic group $\mathrm{Ps}(\mathsf{V}) = \mathrm{Ps}(\mathsf{V}, B)$ is a subgroup of the group $\mathrm{Aut}_{Z\text{-fix}}(\mathsf{V}_B^\sharp)$ of automorphisms of $V_B^\sharp$ that fix the center of $V^\sharp$, where (f, σ) corresponds to the automorphism $V_B^\sharp \ni (a, v) \mapsto (a + f(v), \sigma v) \in V_B^\sharp$.

When $\mathrm{char}(k) \neq 2$, the projection $(f, \sigma) \mapsto \sigma$ defines an isomorphism $\mathrm{Ps}(\mathsf{V}, B) \simeq \mathrm{Sp}(\mathsf{V}, \omega_B)$, which we may use to identify these two groups. But when $\mathrm{char}(k) = 2$, the map $(f, \sigma) \mapsto \sigma$ fits into a short exact sequence

$$1 \to (\mathsf{V}^*)^{(2)} \to \mathrm{Ps}(\mathsf{V}, B) \to \mathrm{O}(\mathsf{V}, Q_B) \to 1, \qquad Q_B(v) := B(v, v), \tag{2.4.2}$$

where $(\mathsf{V}^*)^{(2)}$ denotes the space of diagonal quadratic forms Q, i.e., those for which $Q(v + w) = Q(v) + Q(w)$. This exact sequence splits if and only if $k = \mathbb{F}_2$ and $\dim(\mathsf{V}) \leq 2$

[Bla93, Section 1.3] (cf. [Gri73, Theorem 1]). Moreover, the algebraic group underlying $\mathrm{Ps}(\mathsf{V})$ is disconnected when $p = 2$. We write $\mathrm{Ps}^\circ(\mathsf{V})$ for (the k-points of the algebraic group underlying) the identity component of $\mathrm{Ps}(\mathsf{V})$. So if $p \neq 2$, then $\mathrm{Ps}^\circ(\mathsf{V}) = \mathrm{Ps}(\mathsf{V}) \simeq \mathrm{Sp}(\mathsf{V})$ and if $p = 2$, then $\mathrm{Ps}^\circ(\mathsf{V})$ is the preimage of $\mathrm{SO}(\mathsf{V})$.

Using $\mathrm{Ps}(\mathsf{V})$, we can describe the automorphism group $\mathrm{Aut}_{Z\text{-fix}}(\mathsf{V}^\sharp_B)$.

Fact 2.4.3 ([Win72, Theorem 1])**.** *Let $k = \mathbb{F}_p$ and recall that $\mathrm{Aut}_{Z\text{-fix}}(\mathsf{V}^\sharp_B)$ denotes the group of automorphisms of $\mathsf{V}^\sharp_B$ that act trivially on the center.*

(a) If $p \neq 2$, then $\mathrm{Aut}_{Z\text{-fix}}(\mathsf{V}^\sharp_B) \simeq \mathsf{V} \rtimes \mathrm{Sp}(\mathsf{V}, \omega_B)$ and $\mathrm{Out}(\mathsf{V}^\sharp_B) \simeq \mathrm{Sp}(\mathsf{V}, \omega_B)$.

(b) If $p = 2$, then $\mathrm{Aut}_{Z\text{-fix}}(\mathsf{V}^\sharp_B) = \mathrm{Aut}(\mathsf{V}^\sharp_B) = \mathrm{Ps}(\mathsf{V})$ and $\mathrm{Out}(\mathsf{V}^\sharp_B) \simeq \mathrm{O}(\mathsf{V}, Q_B)$.

2.5 Weil representations

We remain in the setting of Construction 2.1.3, but assume in addition that $k = \mathbb{F}_p$ is a finite field with p elements, and suppress B from the notation. Let ψ be a nontrivial additive character of k, and let ω_ψ be the corresponding Heisenberg representation of $\mathsf{V}^\sharp$ (see Definition 2.3.2). Given a subgroup A of $\mathrm{Ps}(\mathsf{V})$, since $\mathrm{Ps}(\mathsf{V})$ acts trivially on the center of $\mathsf{V}^\sharp$ and ω_ψ is the unique irreducible representation of $\mathsf{V}^\sharp$ with central character ψ, the action of A preserves ω_ψ up to isomorphism and thereby gives rise to a projective representation of A on the space underlying the Heisenberg representation, which we call the projective Weil representation of A.

Our construction of supercuspidal representations requires us to linearize the projective Weil representation for certain subgroups A. When $p \neq 2$, the whole projective Weil representation $\mathrm{Ps}(\mathsf{V})$ can be linearized ([Gér77, Theorem 2.4(a)]). When $p = 2$ and V is nontrivial, however, such a linearization is not possible (Remark 2.5.1 below). Moreover, without additional constraints, there is ambiguity in the choice of linearization: the character group of A acts transitively, by twisting, on the set of linearizations. Since A is often abelian in our applications, this ambiguity is quite dire.

To deal with both of these problems, linearizing at all and pinning down a specific linearization, we use a special feature of the Heisenberg representation present only when $p = 2$: the structure of an R-representation, for $R \in \{\mathbb{R}, \mathbb{H}\}$. The first problem is resolved by a simple criterion for R-linearizability, Lemma 2.5.5, which builds on the abstract criterion Lemma 2.2.2. The second problem is resolved by the general fact that an R-linearization is unique up to a character of order two, rather than an arbitrary complex character.

Remark 2.5.1. In characteristic 2, the projective Weil representation is not linearizable. If it were, then its restriction to the subgroup $(\mathsf{V}^*)^{(2)}$ of (2.4.2) would be linearizable as well. In other words, since $(\mathsf{V}^*)^{(2)}$ acts on $\mathsf{V}^\sharp$ by the group V of inner automorphisms, there would exist a representation $\pi\colon \mathsf{V} \ltimes \mathsf{V}^\sharp \to \mathrm{GL}(V_{\omega_\psi})$ extending the Heisenberg representation $\mathsf{V}^\sharp \to \mathrm{GL}(V_{\omega_\psi})$. Since ω_ψ is irreducible, for every $v \in \mathsf{V}$ there would then be a scalar $c(v) \in \mathbb{C}^\times$ such that $\pi(v) = c(v)\omega_\psi(0, v)$. The fact that π and ω_ψ are homomorphisms forces the function

c to satisfy a certain identity, which we can use to show that $c(v)$ is contained in the pth roots of unity μ_p and that $c(v)$ gives rise to a splitting of the homomorphism $\mathsf{V}^\sharp \to \mathsf{V}$. This is a contradiction; no splitting exists.

In characteristic $\neq 2$, the same argument shows that the projective Weil representation of $\mathrm{Aut}_{Z\text{-fix}}(\mathsf{V}) \simeq \mathsf{V} \rtimes \mathrm{Sp}(\mathsf{V})$ is not linearizable.

Let A be a subgroup of $\mathrm{Ps}(\mathsf{V})$ and let $R \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ be the Frobenius–Schur type of the Heisenberg representation ω_ψ of $\mathsf{V}^\sharp$. Then for every $a \in A$, by Lemma 2.2.1, the a-twist of the Heisenberg R-representation is isomorphic to the Heisenberg R-representation, and the intertwiner between these two representations is unique up to scaling by $Z(R)^\times$. Hence we obtain a projective R-representation of A, which we call the projective Weil R-representation.

Definition 2.5.2. Let A be a subgroup of $\mathrm{Ps}(\mathsf{V})$ and let R be the Frobenius–Schur type of the Heisenberg representation of $\mathsf{V}^\sharp$. A Weil R-representation of A is defined to be:

(a) When $p \neq 2$, the restriction to A of Gérardin's Weil representation of $\mathrm{Ps}(\mathsf{V}) \simeq \mathrm{Sp}(\mathsf{V})$ [Gér77, Lemma 2.4(a)].

(b) When $p = 2$, some R-linearization (if it exists) of the projective Weil R-representation of A.

A Weil representation of A is the complex representation associated to a Weil R-representation of A.

Remark 2.5.3 (How unique is the Weil representation?)**.** When $p \neq 2$, the projective Weil representation of $\mathrm{Ps}(\mathsf{V})$ has a unique linearization unless $\mathrm{Ps}(\mathsf{V}) = \mathrm{Sp}_2(\mathbb{F}_3)$, in which case there are three linearizations and Gérardin singles out one of them. So a Weil representation of A is unique when $p \neq 2$.

When $p = 2$, the Weil representations all differ from each other by twisting by an order-two character. So if A has no character of order two, then its Weil representation is unique.

To finish this subsection, we give a criterion for a Weil R-representation to exist. We recall from Section 2.1, page 14, that if $\mathsf{V} = \mathsf{V}^+ \oplus \mathsf{V}_0 \oplus \mathsf{V}^-$ is a partial polarization, we may identify the preimage of V_0 in $\mathsf{V}^\sharp$ with $\mathsf{V}_0^\sharp$. If $0 \times \mathsf{V}^+$ is a splitting of V^+ in $\mathsf{V}^\sharp$, then we write $\mathsf{V}^+ \times \mathsf{V}_0^\sharp$ for the internal direct product of $0 \times \mathsf{V}^+$ and $\mathsf{V}_0^\sharp$, which is the preimage of $\mathsf{V}^+ \oplus \mathsf{V}_0$ in $\mathsf{V}^\sharp$.

Definition 2.5.4. Let $\mathsf{V}^+ \subseteq \mathsf{V}$ be an isotropic subspace and $0 \times \mathsf{V}^+$ a splitting of V^+ in $\mathsf{V}^\sharp$. Define $\mathcal{P}(0 \times \mathsf{V}^+)$ to be the subgroup of $\mathrm{Ps}(\mathsf{V})$ consisting of the elements g such that

(a) $g(0 \times \mathsf{V}^+) = 0 \times \mathsf{V}^+$, and

(b) $g(x) \cdot x^{-1} \in 0 \times \mathsf{V}^+$ for all $x \in \mathsf{V}^+ \times \mathsf{V}_0^\sharp$.

Lemma 2.5.5. *Let $\psi\colon \mathbb{F}_p \to \mathbb{C}^\times$ be a nontrivial character, let $R \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ be the Frobenius–Schur type of the Heisenberg representation of $\mathsf{V}^\sharp$ corresponding to ψ, and let H be a subgroup of $\mathrm{Ps}(\mathsf{V})$. Suppose there is a Sylow p-subgroup H_p of H, an isotropic subspace V^+ of V, and a splitting $0 \times \mathsf{V}^+$ of V^+ such that*

$$H_p \subseteq \mathcal{P}(0 \times \mathsf{V}^+).$$

Then the Heisenberg R-representation of $\mathsf{V}^\sharp$ extends to an R-representation of $H \ltimes \mathsf{V}^\sharp$ whose restriction to H is a Weil R-representation.

Proof. By Definition 2.5.2 (and Remark 2.5.3 for $p \neq 2$), a desired R-linear extension to $H \ltimes \mathsf{V}^\sharp$ exists if and only if the restriction of the projective Weil R-representation to H can be lifted to an honest R-representation of H. By [Gér77, Lemma 1.5] when $p \neq 2$ and Lemma 2.2.2 when $p = 2$, it suffices to show that the Heisenberg R-representation ω_ψ^R extends to $H_p \ltimes \mathsf{V}^\sharp$. Write $\mathcal{P} := \mathcal{P}(0 \times \mathsf{V}^+)$. Extend the Heisenberg R-representation $\omega_{0,\psi}^R$ of $\mathsf{V}_0^\sharp$ to the R-representation π of $\mathcal{P} \ltimes (\mathsf{V}^+ \times \mathsf{V}_0^\sharp)$ defined by the formula

$$\pi\colon (p, v, x) \mapsto \omega_{0,\psi}(x), \qquad p \in \mathcal{P},\ v \in 0 \times \mathsf{V}^+,\ x \in \mathsf{V}_0^\sharp.$$

By the definition of $\mathcal{P}$, this formula defines a homomorphism: $p(v, x)p^{-1} = (v + w, x)$ for some $w \in 0 \times \mathsf{V}_+$, and then $\pi(v + w, x) = \pi(v, x)$ because $0 \times \mathsf{V}_+ \subseteq \ker(\pi)$. Using Lemma 2.3.4(a), we deduce that the restriction to $\mathsf{V}^\sharp$ of the induced representation $\mathrm{Ind}_{\mathcal{P}\ltimes(\mathsf{V}^+\times\mathsf{V}_0^\sharp)}^{\mathcal{P}\ltimes\mathsf{V}^\sharp} \pi$ is isomorphic to ω_ψ^R. Restricting $\mathrm{Ind}_{\mathcal{P}\ltimes(\mathsf{V}^+\times\mathsf{V}_0^\sharp)}^{\mathcal{P}\ltimes\mathsf{V}^\sharp}(\pi)$ to $H_p \ltimes \mathsf{V}^\sharp$ yields therefore an extension of ω_ψ^R to an R-representation. □

Definition 2.5.6. We call an R-representation of $H \ltimes \mathsf{V}^\sharp$ as in the conclusion of Lemma 2.5.5, i.e., one whose restriction to H is a Weil R-representation and whose restriction to $\mathsf{V}^\sharp$ is Heisenberg R-representation, a Heisenberg–Weil R-representation. We call the associated complex representation a Heisenberg–Weil representation.

3 Construction of supercuspidal representations

Let F be a non-archimedean local field. Let G be a connected reductive group that splits over a tamely ramified extension of F.

3.1 The input

The input to our construction of supercuspidal representations is analogous to the input that Yu ([Yu01]) uses, but it allows $p = 2$ and removes the genericity assumption (GE2) imposed by Yu ([Yu01, Section 8]). We follow the conventions in [Fin21]; see [Fin21, Remark 2.4] for a comparison of conventions.

Throughout the paper we will use the following weaker notion of generic elements.

Definition 3.1.1 (cf. [Fin, Definition 3.5.2]). Let G' be a connected reductive F-group and let $H \subseteq G'$ be a twisted Levi subgroup that splits over a tamely ramified field extension of F. Let $x \in \mathscr{B}(H, F)$, and let $r \in \mathbb{R}_{>0}$.

(a) An element $X \in \mathrm{Lie}^*(H)^H(F)$ is (G', H)-generic of depth r if it satisfies conditions (GE0) and (GE1) of [Fin, Definition 3.5.2].

(b) A character ϕ of $H(F)$ is (G', H)-generic (relative to x) of depth r if ϕ is trivial on $H(F)_{x,r+}$ and the restriction of ϕ to $H(F)_{x,r}/H(F)_{x,r+}$ is realized by an element of $\mathrm{Lie}^*(H)^H(F)$ that is (G', H)-generic of depth $-r$ (as in [Fin, Definition 3.5.2(b)]).

Notably, these conditions do not require (GE2).

The input to our construction is a tuple (cf. [Fin21, Section 2.1])

$$\Upsilon = ((G_i)_{1 \le i \le n+1}, x, (r_i)_{1 \le i \le n}, \rho, (\phi_i)_{1 \le i \le n})$$

for some non-negative integer n, where

(a) $G = G_1 \supseteq G_2 \supsetneq G_3 \supsetneq \ldots \supsetneq G_{n+1}$ are twisted Levi subgroups of G that split over a tamely ramified extension of F,

(b) $x \in \mathscr{B}(G_{n+1}, F) \subset \mathscr{B}(G, F)$,

(c) $r_1 > r_2 > \ldots > r_n > 0$ are real numbers,

(d) ρ is an irreducible representation of $(G_{n+1})_{[x]}$ that is trivial on $(G_{n+1})_{x,0+}$,

(e) ϕ_i, for $1 \le i \le n$, is a character of $G_{i+1}(F)$ of depth r_i,

satisfying the following conditions

(i) G_{n+1} is elliptic in G, i.e., $Z(G_{n+1})/Z(G)$ is anisotropic,

(ii) the image of the point x in $\mathscr{B}(G_{n+1}^{\mathrm{der}}, F)$ is a vertex,

(iii) $\rho|_{(G_{n+1})_{x,0}}$ is a cuspidal representation of $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$,

(iv) ϕ_i is (G_i, G_{i+1})-generic relative to x of depth r_i for all $1 \le i \le n$.

For brevity, we will refer to such an object Υ as a supercuspidal G-datum, and we will fix such a datum from now on.

3.2 Overview of the construction

We will now construct several objects out of a supercuspidal G-datum Υ, culminating in a supercuspidal representation. The dependence on Υ is implicit, not reflected in the notation.

Define the open, compact-mod-$Z(G)$ subgroups

$$K^+ := G_1(F)_{x,r_1/2} \cdot G_2(F)_{x,r_2/2} \cdots G_n(F)_{x,r_n/2} \cdot N_G(G_1, G_2, \dots, G_n, G_{n+1})(F)_{[x]}\,,$$
$$K := G_1(F)_{x,r_1/2} \cdot G_2(F)_{x,r_2/2} \cdots G_n(F)_{x,r_n/2} \cdot G_{n+1}(F)_{[x]}\,.$$

Then K is a normal, finite-index subgroup of K^+. Note that our K is denoted by K^+ in [Fin21, Section 2.5], but we want to avoid too many tildes and indices.

Lemma 3.2.1. *Let G' be a connected reductive F-group and let $H \subseteq G'$ be a twisted Levi subgroup that splits over a tamely ramified extension of F. Let $x \in \mathscr{B}(H,F)$ and let ϕ be a character of H of depth r. Then there exists a unique character $\hat{\phi}_{(G',x)}$ of $H(F)_{[x]} \cdot G'(F)_{x,r/2+}$ such that $\hat{\phi}_{(G',x)}$ and ϕ agree on $H(F)_{[x]}$ and $(H,G')_{x,r+,r/2+} \subseteq \ker(\hat{\phi}_{(G',x)})$.*

Proof. This follows from the argument at the beginning of [Yu01, Section 4]. □

In particular, for each $1 \le i \le n$ we have a character $\hat{\phi}_i := (\hat{\phi}_i)_{(G,x)}$ of $(G_{i+1})_{[x]} \cdot G_{x,r_i/2+}$ that extends the restriction of ϕ to $(G_{i+1})_{[x]}$.[5]

Our goal is to construct a representation σ of a certain group $\widetilde{K} := N_{K^+}(\rho \otimes \kappa)$ contained between K and K^+. The irreducible supercuspidal representation is then $\text{c-ind}_{\widetilde{K}}^{G(F)}(\sigma)$. If the characters in the input Υ satisfy Yu's additional condition (GE2), then $\widetilde{K} = K$ (see Theorem 3.6.8(b)). The construction takes two steps, which we briefly summarize before describing them in more detail.

First, we define a certain normal subgroup K^- of K, for which the quotient K/K^- is an abelian 2-group if $p = 2$ and is trivial otherwise (see (3.2.2)). Using the Heisenberg–Weil representation, we construct an irreducible representation κ^- of K^- (see Lemma 3.5.8).

Second, we make two choices: an irreducible representation κ of K whose restriction to K^- contains κ^-, and an irreducible representation σ of $\widetilde{K} := N_{K^+}(\rho \otimes \kappa)$ whose restriction to K contains $\rho \otimes \kappa$. These two choices can be studied using Clifford theory, and we reflect on the choices in Section 3.3.

Step 1: Heisenberg–Weil representation. The first step uses the theory of Heisenberg–Weil representations that features in Yu's work ([Yu01]) in the case $p \neq 2$, but which was before this paper not available in the case of $p = 2$.

[5] Our character $\hat{\phi}$ is defined on a slightly larger subgroup than Yu's character $\hat{\phi}$, which was defined on the subgroup $(G_{n+1})_{[x]} \cdot (G_{i+1})_{x,0} \cdot G_{x,r_i/2}$. There is little risk of confusion, however, because our character extends Yu's character.

For $\tilde r, \tilde r' \in \widetilde{\mathbb{R}} \setminus \{\infty\}$ with $\tilde r \geq \tilde r' \geq \tilde r/2 > 0$, let $(G_i)_{x,\tilde r,\tilde r'} = (G_{i+1}, G_i)(F)_{x,\tilde r,\tilde r'}$ as in [Fin21, Section 2.5]. In Lemma 3.4.1 we will show that the group

$$\mathsf{V}_i^\natural := (G_i)_{x,r_i,r_i/2}/((G_i)_{x,r_i,r_i/2+} \cap \ker(\hat\phi_i))$$

is a Heisenberg $\mathbb{F}_p$-group. Let (ω_i, V_{ω_i}) denote the Heisenberg representation of $\mathsf{V}_i^\natural$ with central character $\phi_i|_{(G_i)_{x,r_i,r_i/2+}}$ (Definition 2.3.2).

Define the group

$$K^- := (G_1)_{x,r_1,r_1/2} \cdots (G_n)_{x,r_n,r_n/2} G_{n+1}(F)^-_{[x]} \tag{3.2.2}$$

where if $p \neq 2$, then $(G_{n+1})^-_{[x]} := (G_{n+1})_{[x]}$, and if $p = 2$, then $(G_{n+1})^-_{[x]}$ is defined to be the kernel of the projection map from $(G_{n+1})_{[x]}$ to the finite abelian 2-group

$$((G_{n+1})_{[x]}/(Z(G(F)) \cdot (G_{n+1})_{x,0})) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)}. \tag{3.2.3}$$

The lefthand factor is a finite abelian group by [KP23, Corollary 11.6.3].

The representation κ^- of K^- will have underlying vector space $V_{\kappa^-} := \bigotimes_{i=1}^n V_{\omega_i}$. To define κ^- we give an action of each factor of K^- on each vector space V_{ω_i}, form the tensor product of the actions to produce an action of each factor of K^- on V_{κ^-}, and then check that the actions of the different factors of K^- are compatible, so that they descend to a morphism $\kappa^-\colon K^- \to \mathrm{GL}(V_{\kappa^-})$ (see Lemma 3.5.8). More precisely, let $1 \leq i, j \leq n$. Then the factor $(G_i)_{x,r_i,r_i/2}$ acts on the space V_{ω_j} when $i \neq j$ by the character $\hat\phi_j|_{(G_i)_{x,r_i,r_i/2}}$, and when $i = j$ by the Heisenberg representation ω_i of $\mathsf{V}_i^\natural$ with central character $\hat\phi_i|_{(G_i)_{x,r_i,r_i/2+}}$. As for the factor $(G_{n+1})^-_{[x]}$, we let $(G_{n+1})^-_{[x]}$ act on V_{ω_i} via $\phi_i \otimes \omega_i$, where ω_i denotes the restriction of a (pull back of a) Weil–Heisenberg representation as in Notation 3.5.4 (see also Corollary 3.5.3 and Proposition 3.5.6). While Weil representations in general are only uniquely defined up to twisting by an order-two character (Remark 2.5.3), the resulting representation ω_i is uniquely defined when $q > 2$ because it is inflated from a group with no characters of order two (Proposition 3.5.6).

Step 2: Clifford theory. Recall from Section 1.1 that if A is a group, B is a normal subgroup of A, and π is a representation of B, then we write $\mathrm{Irr}(A, B, \pi)$ for the set of $\sigma \in \mathrm{Irr}(A)$ whose restriction to B contains π.

First, let $\kappa \in \mathrm{Irr}(K, K^-, \kappa^-)$. If $p \neq 2$, then $K^- = K$ and $\kappa = \kappa^-$. By Lemma 3.3.1(a), the character group of K/K^- acts transitively, by twisting, on the set of such κ. We may now inflate the representation ρ from $G_{n+1}(F)_{[x]}$ to K by asking the inflation to be trivial on $G_1(F)_{x,r_1/2} \cdot G_2(F)_{x,r_2/2} \cdots G_n(F)_{x,r_n/2}$. We denote this inflation by ρ as well, and we form the tensor product $\rho \otimes \kappa$ of the inflation with κ.

Second, let $\sigma \in \mathrm{Irr}(\widetilde K, K, \rho \otimes \kappa)$, where we recall that $\widetilde K := N_{K^+}(\rho \otimes \kappa)$. By Lemma 3.3.1(b), these σ are in bijection with the irreducible representations of the intertwining algebra $\mathrm{End}_{\widetilde K}\big(\mathrm{Ind}_K^{\widetilde K}(\rho \otimes \kappa)\big)$.

In Theorem 3.6.9(a) we will show that the representation $\text{c-ind}_{\widetilde K}^{G(F)}(\sigma)$ is irreducible and supercuspidal if $q > 3$. The representations $\text{c-ind}_{\widetilde K}^{G(F)}(\sigma)$ for varying σ can also be recovered as the irreducible subrepresentations of $\text{c-ind}_K^{G(F)}(\rho \otimes \kappa)$, by Theorem 3.6.9(b). If the characters ϕ_i in the input Υ of the construction satisfy Yu's condition (GE2), then $\text{c-ind}_K^{G(F)}(\rho \otimes \kappa)$ itself is irreducible (see Theorem 3.6.9(c)).

3.3 Choices to be made in the construction

Let Υ be a cuspidal G-datum and assume $q > 3$. Ideally, a construction of supercuspidal representations would output a single irreducible representation from each input Υ. For several reasons, however, our construction is not so precise. In this subsection we reflect on the choices in our construction, in addition to Υ, that one needs to make to produce a single supercuspidal representation. We stress that these additional choices are only necessary when either $p = 2$ and G has complicated Bruhat–Tits theory at x, or when p is a torsion prime for the Langlands dual group $\widehat{G}$. Neither of these phenomena occur for the general linear group, where additional choices are not necessary (see Remark 3.3.5).

The only choices the reader has to make in the construction outlined in Section 3.2 are the representations $\kappa \in \operatorname{Irr}(K, K^-, \kappa^-)$ and $\sigma \in \operatorname{Irr}(\widetilde{K}, K, \rho \otimes \kappa)$, which are described by the following lemma.

Lemma 3.3.1.

(a) The character group of K/K^- acts transitively, by twisting, on the set $\operatorname{Irr}(K, K^-, \kappa^-)$.

(b) There is a bijection $\operatorname{Irr}(\widetilde{K}, K, \rho \otimes \kappa) \simeq \operatorname{Irr}\big(\operatorname{End}_{\widetilde{K}}\big(\operatorname{Ind}_K^{\widetilde{K}}(\rho \otimes \kappa)\big)\big)$.

Proof. The first part is a special case of [Kal, Lemma A.4.3] since K/K^- is abelian, and the second is a special case of Lemma B.1(a). □

Remark 3.3.2 (On choosing κ and σ)**.** The choices of κ and σ are of a different nature.

The choice of κ can be accounted for by refactorization, as in the work of Hakim and Murnaghan [HM08, Definition 4.19]. Specifically, suppose κ and κ' are two choices of an element of $\operatorname{Irr}(K, K^-, \kappa^-)$. Then by Lemma 3.3.1(a) there is a character χ of $(G_{n+1})_{[x]}$ trivial on $(G_{n+1})_{[x]}^-$ such that $\kappa' = \chi \otimes \kappa$, where we identify χ with its inflation to K. Let Υ' be the supercuspidal G-datum obtained from Υ by replacing ρ with $\rho' := \chi \otimes \rho$. Since $\rho \otimes \kappa' = \rho' \otimes \kappa$, and the supercuspidal representations we construct depend only on this tensor product rather than its individual factors, we would produce the same set of supercuspidal representations by choosing κ' for Υ or κ for Υ'. All in all, choosing a different κ can be accounted for by instead modifying the depth-zero part of Υ.

The choice of σ is in general of a nonabelian nature and cannot be accounted for by refactorization. In Appendix D, summarized in Remark D.11, we give an example where $\dim(\sigma) > \dim(\rho \otimes \kappa)$, showing that in general σ might not extend $\rho \otimes \kappa$.

Remark 3.3.3 (Why we do not refine the input Υ)**.** Our construction is formulated so that a single G-datum Υ gives rise to a finite set of supercuspidal representations rather than a single one. For many reasons it would be advantageous to reformulate the construction so that it produce an individual representation, starting from a variant datum Σ which would somehow record the choices of σ and κ. However, such a reformulation would come at the price of making the input Σ much more conceptually and notationally complicated than Υ.

In more detail, suppose that κ extends to a representation $\widetilde{\kappa}$ of $\widetilde{K} = N_{K^+}(\rho \otimes \kappa)$ (for any allowed choice of ρ). Then, instead of the representation ρ of $(G_{n+1})_{[x]}$ as the input in Υ, we could take as an input in Σ a depth-zero representation $\widetilde{\rho}$ of a group $K_{\widetilde{\rho}}$, with $K \subseteq K_{\widetilde{\rho}} \subseteq K^+$, such that $N_{K^+}(\widetilde{\rho}|_K \otimes \kappa) = K_{\widetilde{\rho}}$ and $\widetilde{\rho}$ is cuspidal when restricted to $(G_{n+1})_{x,0}$. In terms of our current language, $\sigma = \widetilde{\rho} \otimes \widetilde{\kappa}$. In this new language, however, the input is unpleasant to describe because one needs to already construct K and κ to even define where $\widetilde{\rho}$ lives.

We finish by discussing the case of the general linear group.

Lemma 3.3.4. *Let k be a field, let $G = \mathrm{GL}_n$ over k, and let M be a twisted Levi subgroup of G. Then there are separable field extensions $\ell_1, \dots, \ell_r$ of k and integers $d_1, \dots, d_r$ with $n = \sum_{i=1}^r d_i[\ell_i : k]$ such that $M \simeq \prod_{i=1}^r \mathrm{Res}_{\ell_i/k} \mathrm{GL}_{d_i}$. If, moreover, M is elliptic, then $r = 1$.*

Proof. Let k^{sep} be a separable closure of k, let M_0 be a Levi subgroup of G, let $N_0 = N_G(M_0)$, and let $W_0 = N_G(M_0)/M_0$. There is a section $W_0 \to N_0$ that preserves a fixed pinning of M_0, which we use to write $N_0 \simeq M_0 \rtimes W_0$.

We can identify $(G/N_0)(k)$ with the set of twisted Levi subgroups of G that are $G(k^{\mathrm{sep}})$-conjugate to M_0. If M' is one such Levi subgroup, then the isomorphism class $z' \in H^1(k, \mathrm{Aut}(M_0))$ corresponding to M' is the image of M' under the composite map

$$(G/N_0)(k) \longrightarrow H^1(k, N_0) \longrightarrow H^1(k, \mathrm{Aut}(M_0)),$$

where the first map is the connecting homomorphism and the second is obtained from the conjugation action $N_0 \to \mathrm{Aut}(M_0)$. At the same time, the composition $W_0 \to N_0 \to \mathrm{Aut}(M_0)$ yields a map $H^1(k, W_0) \to H^1(k, \mathrm{Aut}(M_0))$, and for any $w \in H^1(k, W_0)$, the resulting twist $M_{0,w}$ of M_0 is a product of Weil restrictions of general linear groups as in the statement of the lemma. Therefore, it suffices to prove the following claim: The map $f\colon H^1(k, N_0) \to H^1(k, W_0)$ is a bijection.

Since $H^1(k, M_0)$ is trivial by Hilbert's Theorem 90, the fiber of f over the basepoint of $H^1(k, W_0)$ is a singleton. To prove the same for the other fibers, we use a twisting argument, as in [Ser02, Chapter 1, Section 5.6, Corollary 2]. Let $w \in H^1(k, W_0)$ with image $n \in H^1(k, N_0)$ under the section $W_0 \to N_0$. Then there is an exact sequence of sets

$$H^1(k, M_{0,n}) \longrightarrow H^1(k, N_{0,n}) \longrightarrow H^1(k, W_{0,w}).$$

At the same time, since the image of n in $H^1(k, \mathrm{Out}(M_0))$ acts by permuting the irreducible components of the Dynkin diagram of M_0, the twist $M_{0,n}$ of M must be an inner form of a product of Weil restrictions of general linear groups. In other words, $M_{0,n}$ is a product of groups of the form $A^\times$, where A is a central simple algebra over a separable extension of F.

But Hilbert's Theorem 90 holds in this setting as well: $H^1(k, A^\times) = 1$ by [Ser79, Chapter X, Section 1, Exercise 1]. So $H^1(k, M_{0,n})$ is trivial. This completes the proof of the first part.

Lastly, the claim about ellipticity follows since the center of M contains GL_1^r. □

Remark 3.3.5 (No choices for GL_N)**.** When $G = \mathrm{GL}_N$, there is a unique choice for κ and σ.

For σ, the root data of GL_N and its twisted Levi subgroups have no torsion primes. Hence $\widetilde{K} = K$ by Theorem 3.6.8(b), and so $\sigma = \rho \otimes \kappa$.

For κ, we claim that $G_{n+1}(F)^-_{[x]} = G_{n+1}(F)_{[x]}$, from which it follows that $K = K^-$. Indeed, by Lemma 3.3.4, all elliptic tame twisted Levi subgroups of GL_N, including G_{n+1}, are of the form $\mathrm{Res}_{E/F}\,\mathrm{GL}_d$ for E/F a tame extension such that $N = d \cdot [E : F]$. Moreover, using for instance the lattice-chain model of the Bruhat-Tits building of the general linear group (see [KP23, Remark 15.1.33]), we observe that $\mathrm{GL}_d(E)_{[x]} = E^\times \cdot \mathrm{GL}_d(E)_{x,0}$. Now

$$\frac{G_{n+1}(F)_{[x]}}{Z(G(F)) \cdot G_{n+1}(F)_{x,0}} = \frac{E^\times \cdot \mathrm{GL}_d(\mathcal{O}_E)}{F^\times \cdot \mathrm{GL}_d(\mathcal{O}_E)} \simeq \frac{E^\times}{F^\times \cdot \mathcal{O}_E^\times} \simeq \frac{\mathbb{Z}}{e(E/F)\mathbb{Z}}.$$

where $e(E/F)$ is the ramification degree of E/F. Since E/F is tame, if $p = 2$, then $e(E/F)$ is odd. So $K = K^-$.

3.4 Heisenberg $\mathbb{F}_p$-groups arising from p-adic groups

We recall that G is a reductive F-group, and we let H be a twisted Levi subgroup of G that splits over a tamely ramified extension of F. Let $x \in \mathscr{B}(H, F) \subseteq \mathscr{B}(G, F)$, and let $\phi\colon H(F) \to \mathbb{C}^\times$ be a (G, H)-generic character of some positive depth r, as in Definition 3.1.1.

The following lemma is due to Yu ([Yu01, Proposition 11.4]) if $p > 2$ and extends to the case of $p = 2$.

Lemma 3.4.1. *The group* $\dfrac{(H, G)(F)_{x,r,r/2}}{(H, G)(F)_{x,r,r/2+} \cap \ker(\hat{\phi})}$ *is a Heisenberg* $\mathbb{F}_p$*-group.*

Proof. For brevity, denote this quotient group by $\mathsf{V}^\natural$, and write $\mathsf{V} := \dfrac{(H, G)(F)_{x,r,r/2}}{(H, G)(F)_{x,r,r/2+}}$, which is an abelian group of exponent 1 or p.

When $p \neq 2$, the proof of Proposition 11.4 of [Yu01] still works as written also for our more general notion of (G, H)-generic characters so that $\mathsf{V}^\natural$ ($= J/N$ in Yu's notation) is a Heisenberg $\mathbb{F}_p$-group.

When $p = 2$, Lemma 11.1 of [Yu01] still holds with the same proof, i.e., the bi-additive pairing $\mathsf{V} \times \mathsf{V} \to \{\pm 1\} \subset \mathbb{C}^\times$ given by $(aJ_+, bJ_+) \mapsto \hat{\phi}([a, b])$ where $J_+ := (H, G)(F)_{x,r,r/2+}$, is well defined and non-degenerate. Hence the center of $\mathsf{V}^\natural$ is

$$Z(\mathsf{V}^\natural) = \frac{(H, G)(F)_{x,r,r/2+}}{(H, G)(F)_{x,r,r/2+} \cap \ker(\hat{\phi})} \simeq \{\pm 1\},$$

and $\mathsf{V}^\natural/Z(\mathsf{V}^\natural) = \mathsf{V}$ is an abelian 2-group of exponent at most 2. Hence the group $\mathsf{V}^\natural$ is a Heisenberg $\mathbb{F}_2$-group by Definition 2.1.2. □

Example 3.4.2 (Positive- and negative-type Heisenberg $\mathbb{F}_2$-groups in 2-adic groups)**.** In this example we show that both positive- and negative-type Heisenberg $\mathbb{F}_2$-groups can arise in the construction of supercuspidal representations.

Suppose $p = 2$ and F has residue field k_F. Let E/F be a quadratic unramified extension of F and let σ be the nontrivial element of $\mathrm{Gal}(E/F)$. Let $G = \mathrm{GL}(E/F)$ be the group of linear automorphisms of the F vector space E, isomorphic (after choosing an ordered basis of E) to GL_2. Then $T = \mathrm{Res}_{E/F}\,\mathbb{G}_{\mathrm{m}}$ canonically embeds as a maximal torus of G through the multiplication action of $E^\times$ on E. Let x be such that $G(F)_x = \mathrm{GL}(\mathcal{O}_E/\mathcal{O}_F)$. Let $n \geq 1$ be an integer, and let $\phi\colon E^\times \to \mathbb{C}^\times$ be a (G,T)-generic character of depth $2n$. We claim that $(T,G)(F)_{x,2n,n}/((T,G)(F)_{x,2n,n+} \cap \ker(\hat{\phi}))$ is a negative-type Heisenberg $\mathbb{F}_2$-group.

The main problem is to describe the group $(T,G)(E)_{x,2n,n}/(T,G)(E)_{x,2n+,n+}$, and especially the quotients of root groups, together with the action of $\mathrm{Gal}(E/F)$ on this group. The root groups have the following description: There are orthogonal idempotents $e_1 \neq e_2$ in $(E \otimes_F E)^\times$ interchanged by $\mathrm{Gal}(E/F)$ such that one root subgroup of $G(E)$, call it $U_\alpha(E)$, is represented in the ordered basis (e_1, e_2) by matrices of the form $u(a) = \left[\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right]$ with $a \in E$, while the other, call it $U_{-\alpha}(E)$, is represented in this ordered basis by the matrices of the form $v(b) = \left[\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right]$ with $b \in E$. It follows that $\mathrm{Gal}(E/F)$ acts on $U_\alpha(E)_{x,n}/U_\alpha(E)_{x,n+} \oplus U_{-\alpha}(E)_{x,n}/U_{-\alpha}(E)_{x,n+}$ by

$$\sigma(\bar{u}(a) + \bar{v}(b)) = \bar{v}(\sigma a) + \bar{u}(\sigma a), \quad (\mathrm{val}(a), \mathrm{val}(b) \geq n),$$

where $\bar{u}$ and $\bar{v}$ denote the images of u and v in the above quotient spaces. Using the commutator relation for opposite root groups, we see that

$$Q(\bar{u}(a) + \bar{v}(b)) := \big(u(a)v(b)\big)^2 \bmod (T,G)(E)_{x,2n+,n+} \equiv \alpha^\vee(1+ab) \bmod (T,G)(E)_{x,2n+,n+}.$$

In other words, after identifying $U_\alpha(E)_{x,n}/U_\alpha(E)_{x,n+} \oplus U_{-\alpha}(E)_{x,n}/U_{-\alpha}(E)_{x,n+}$ with $k_E \oplus k_E$ and $T(E)_{x,2n}/T(E)_{x,2n+}$ with k_E, the quadratic form Q becomes the split form $Q(a,b) = ab$. On the subspace of $\mathrm{Gal}(E/F)$-invariants in $k_E \oplus k_E$, which we can identify with k_E by matching $a \in k_E$ with $(a, \sigma a)$, the quadratic form restricts to the norm form $Q(a) = a \cdot \sigma a$.

We claim that the nondegenerate quadratic form $Q' := \hat{\phi} \circ Q\colon k_E \to \{\pm 1\}$ is non-split. Note that the vanishing set of Q' has size less than half of k_E as follows by direct computation:

$$|Q'^{-1}(1)| = (\tfrac{1}{2}q - 1)(q+1) + 1 = \tfrac{1}{2}q^2 - \tfrac{1}{2}q < \tfrac{1}{2}|k_E|.$$

Hence Q' cannot be a split quadratic form, and therefore $(T,G)(F)_{x,2n,n}/((T,G)(F)_{x,2n,n+} \cap \ker(\hat{\phi}))$ is of negative type.

At the same time, since the central product $Q_8 \circ Q_8$ is a positive-type Heisenberg $\mathbb{F}_2$-group (see Example 2.1.8), doubling the previous example—that is, replacing G by $G \times G$, T by $T \times T$, ϕ by $\phi \otimes \phi$, and so on—yields an example where $(T,G)(F)_{x,2n,n}/((T,G)(F)_{x,2n,n+} \cap \ker(\hat{\phi}))$ is of positive type. So both possibilities can arise.

3.5 Weil representations of compact-mod-center open subgroups

Let Υ be a supercuspidal G-datum. In this subsection we first construct various auxiliary subgroups and vector spaces from Υ which are needed both for the Heisenberg–Weil extension step and for the proof of supercuspidality, and then we prove the remaining claims used in the construction of smooth representations from Υ outlined in Section 3.2. In Section 3.6 we will then prove that these representations are supercuspidal.

Let $(\mathscr{G}_{n+1})_{x,0}$ be the connected parahoric integral model of G at x and let G_{n+1} be the reductive quotient of its special fiber. Hence G_{n+1} is a reductive k_F-group with $\mathsf{G}_{n+1}(k_F) = (G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$. By Lemma 3.4.1, the group

$$\mathsf{V}_i^\natural := \frac{(G_{i+1}, G_i)(F)_{x,r_i,r_i/2}}{(G_{i+1}, G_i)(F)_{x,r_i,r_i/2+} \cap \ker(\hat{\phi}_i)}$$

is a Heisenberg $\mathbb{F}_p$-group. Moreover, the conjugation action of $G_{n+1}(F)_{[x]}$ induces an action on $\mathsf{V}_i^\natural$. By Fact 2.4.3(b) (for $p = 2$) and [Yu01, Lemma 11.3] (for $p \neq 2$), this yields a homomorphism $G_{n+1}(F)_{[x]} \to \mathrm{Ps}(\mathsf{V}_i)$.

For any tamely ramified finite field extension F'/F, we define the following group and its $\mathbb{F}_p$-vector space quotient

$$\widetilde{\mathsf{V}}_{i,F'}^\natural := \frac{(G_{i+1}, G_i)(F')_{x,r_i,r_i/2}}{(G_{i+1}, G_i)(F')_{x,r_i+,r_i/2+}} \qquad \text{and} \qquad \mathsf{V}_{i,F'} := \frac{(G_{i+1}, G_i)(F')_{x,r_i,r_i/2}}{(G_{i+1}, G_i)(F')_{x,r_i,r_i/2+}}.$$

We may drop the subscript F' if $F' = F$, that is, write $\mathsf{V}_i := \mathsf{V}_{i,F}$ and $\widetilde{\mathsf{V}}_i^\natural := \widetilde{\mathsf{V}}_{i,F}^\natural$.

Note that $\mathsf{V}_i^\natural$ is an intermediate quotient between $\widetilde{\mathsf{V}}_i^\natural$ and V_i. While we are eventually interested in the group $\mathsf{V}_i^\natural$, we will take advantage of the groups $\widetilde{\mathsf{V}}_{i,F'}^\natural$ that allow us to deduce results over F by proving the analogous results after base change.

The remaining objects that we like to introduce depend on two additional choices: T and λ. Let T be a maximally split, tame maximal torus of G_{n+1} with splitting field E/F such that x is in the apartment $\mathscr{A}(T, F)$ of T. Let $\lambda \in X_*(T)^{\mathrm{Gal}(E/F)} \otimes \mathbb{R}$. The following discussion will later be applied to several choices of λ, but we do not record λ in the notation as this should be clear from the context.

Let S be the maximal split subtorus of T, let $\mathscr{S}$ be an integral model of S that is a split maximal torus of $(\mathscr{G}_{n+1})_{x,0}$, and let $\mathsf{S} = \mathscr{S}_{k_F}$, a maximal split torus in G_{n+1}. Then

$$\lambda \in X_*(S) \otimes \mathbb{R} \simeq X_*(\mathsf{S}) \otimes \mathbb{R}.$$

Let P be the parabolic subgroup of G_{n+1} containing S such that

$$\Phi(\mathsf{P}, \mathsf{S}) = \{\alpha \in \Phi(\mathsf{G}_{n+1}, \mathsf{S}) \mid \lambda(\alpha) \geq 0\}$$

and let U be the unipotent radical of P. Define P_E and U_E analogously. Since λ is $\mathrm{Gal}(E/F)$-stable, there is a (parabolic) subgroup P of G_{n+1} containing T such that

$$\Phi(P_E, T_E) = \{\alpha \in \Phi(G_{n+1}, T) \mid \lambda(\alpha) \geq 0\}.$$

Let U be the unipotent radical of P and write

$$U(F)_{x,0} := U(F) \cap G(F)_{x,0}, \qquad P(F)_{x,0} := P(F) \cap G(F)_{x,0}.$$

We will now use λ to also define a partial polarization of each V_i.

For the isotropic subspaces, define the unipotent subgroups $U_{i,E}^{\pm}$ of $G_{i,E}$ by

$$\begin{aligned} U_{i,E}^{+} &:= \big\langle U_{\alpha,E} \mid \alpha \in \Phi(G_i,T) \smallsetminus \Phi(G_{i+1},T),\ \lambda(\alpha) > 0 \big\rangle \\ U_{i,E}^{-} &:= \big\langle U_{\alpha,E} \mid \alpha \in \Phi(G_i,T) \smallsetminus \Phi(G_{i+1},T),\ \lambda(\alpha) < 0 \big\rangle. \end{aligned}$$

Given $\tilde{r} \in \widetilde{\mathbb{R}}$, let $U_{i,E}^{\pm}(E)_{x,\tilde{r}}$ be the compact subgroup generated by the groups $U_\alpha(E)_{x,\tilde{r}}$ with $\alpha \in \Phi(U_{i,E}^{\pm}, T_E)$. Since $\Phi(U_{i,E}^{\pm}, T_E)$ is $\mathrm{Gal}(E/F)$-stable, the group $U_{i,E}^{\pm}$ descends to a unipotent F-group U_i contained in G_i. We define $U_i^{\pm}(F)_{x,\tilde{r}} := G(F) \cap U_{i,E}^{\pm}(E)_{x,\tilde{r}}$ for $\tilde{r} \in \widetilde{\mathbb{R}}$. Let

$$\mathsf{V}_{i,E}^{\pm} := \frac{U_{i,E}^{\pm}(E)_{x,r_i/2}}{U_{i,E}^{\pm}(E)_{x,r_i/2+}}, \qquad \mathsf{V}_i^{\pm} := \frac{U_i^{\pm}(F)_{x,r_i/2}}{U_i^{\pm}(F)_{x,r_i/2+}} = (\mathsf{V}_{i,E}^{\pm})^{\mathrm{Gal}(E/F)},$$

where the last equality follows from the same arguments used to prove [Yu01, Corollary 2.3]. Via the inclusions of $U_{i,E}^{\pm}(E)_{x,r_i/2}$ and $U_{i,E}^{\pm}(F)_{x,r_i/2}$ into the appropriate subgroups of $G(E)$ and $G(F)$, we can identify $\mathsf{V}_{i,E}^{\pm}$ with a subgroup of $\widetilde{\mathsf{V}}_{i,E}^{\natural}$, and $\mathsf{V}_i^{\pm}$ with subgroups of $\widetilde{\mathsf{V}}_i^{\natural}$, of $\mathsf{V}_i^{\natural}$, and of V_i.

For the nondegenerate part of the partial polarization, write $H_i = Z_{G_i}(\lambda)$ and $H_{i+1} = Z_{G_{i+1}}(\lambda)$ and define the following subgroups of $\widetilde{\mathsf{V}}_i^{\natural}$ and $\widetilde{\mathsf{V}}_{i,E}^{\natural}$:

$$\widetilde{\mathsf{V}}_{i,0}^{\natural} := \frac{(H_{i+1},H_i)(F)_{x,r_i,r_i/2}}{(H_{i+1},H_i)(F)_{x,r_i+,r_i/2+}} \subseteq \widetilde{\mathsf{V}}_i^{\natural} \qquad \text{and} \qquad \widetilde{\mathsf{V}}_{i,0,E}^{\natural} := \frac{(H_{i+1},H_i)(E)_{x,r_i,r_i/2}}{(H_{i+1},H_i)(E)_{x,r_i+,r_i/2+}} \subseteq \widetilde{\mathsf{V}}_{i,E}^{\natural}.$$

By [Yu01, Proposition 2.2],

$$\widetilde{\mathsf{V}}_i^{\natural} = (\widetilde{\mathsf{V}}_{i,E}^{\natural})^{\mathrm{Gal}(E/F)}, \qquad \widetilde{\mathsf{V}}_{i,0}^{\natural} = (\widetilde{\mathsf{V}}_{i,0,E}^{\natural})^{\mathrm{Gal}(E/F)}.$$

We also define the following Heisenberg $\mathbb{F}_p$-group with its quotient $\mathbb{F}_p$-vector space

$$\mathsf{V}_{i,0}^{\natural} := \frac{(H_{i+1},H_i)(F)_{x,r_i,r_i/2}}{(H_{i+1},H_i)(F)_{x,r_i,r_i/2+} \cap \ker(\hat{\phi}_i)} \twoheadleftarrow \widetilde{\mathsf{V}}_{i,0}^{\natural} \qquad \text{and} \qquad \mathsf{V}_{i,0} := \frac{(H_{i+1},H_i)(F)_{x,r_i,r_i/2}}{(H_{i+1},H_i)(F)_{x,r_i,r_i/2+}}.$$

Note that using the notation from Section 2.3, we have $\mathsf{V}_i = \mathsf{V}_i^{\natural}/Z(\mathsf{V}_i^{\natural}) = \mathsf{V}_{\mathsf{V}_i^{\natural}}$.

Lemma 3.5.1. $\mathsf{V}_i = \mathsf{V}_i^{+} \oplus \mathsf{V}_{i,0} \oplus \mathsf{V}_i^{-}$ *is a partial polarization in the sense of Section 2.3.*

Proof. It suffices to show that $\mathsf{V}_{i,0}$ is a nondegenerate subspace and that the subspaces V_i^{+} and V_i^{-} are isotropic and orthogonal to $\mathsf{V}_{i,0}$. The subspace $\mathsf{V}_{i,0}$ is nondegenerate by

Lemma 3.4.1 applied to $H_{i+1} \subseteq H_i$ and the character $\phi_i|_{H_{i+1}}$, which is (H_i, H_{i+1})-generic of depth r_i because

$$\Phi(H_i, T) \setminus \Phi(H_{i+1}, T) \subseteq \Phi(G_i, T) \setminus \Phi(G_{i+1}, T),$$

and because ϕ_i can be represented by an element in $\mathfrak{g}_i(F)^*$ of depth $-r_i$ that is trivial on the sum $\mathfrak{t}^\perp$ of the root subspaces of $\mathfrak{g}_i(F)$ with respect to T hence its restriction to $\mathfrak{h}_i$ has also depth $-r_i$. The subspaces V_i^+ and V_i^- are isotropic because they embed as abelian subgroups of $\mathsf{V}_i^\natural$. To see that V_i^+ and V_i^- are orthogonal to $\mathsf{V}_{i,0}$, it is enough to show that they are normalized by $\mathsf{V}_{i,0}^\natural$, or equivalently, by $\widetilde{\mathsf{V}}_{i,0}^\natural$, and this can be checked over E by Galois descent. Using the commutator relations for root groups, see, e.g., [Yu01, Section 6], we see that for $\alpha \in \Phi(H_i, T) \smallsetminus \Phi(H_{i+1}, T), \beta \in \Phi(H_{i+1}, T)$, the images of the root groups $U_\alpha(E)_{x,r_i/2}$ and $U_\beta(E)_{x,r_i}$ and of $T(E)_{x,r_i}$ normalize the groups $\mathsf{V}_{i,E}^\pm$. Hence $\widetilde{\mathsf{V}}_{i,0,E}^\natural$ normalizes $\mathsf{V}_{i,E}^\pm$. □

Recall that we also view V_i^+ as a subgroup of $\widetilde{\mathsf{V}}_i^\natural$.

Lemma 3.5.2. *The action of $G_{n+1}(F)_{[x]}$ on $\widetilde{\mathsf{V}}_i^\natural$ induced by conjugation satisfies the following properties.*

(a) We have $g(\mathsf{V}_i^+) = \mathsf{V}_i^+$ and $g(\mathsf{V}_i^+ \times \widetilde{\mathsf{V}}_{i,0}^\natural) = \mathsf{V}_i^+ \times \widetilde{\mathsf{V}}_{i,0}^\natural$ for all $g \in \mathsf{P}(k_F)$

(b) We have $g(x) \cdot x^{-1} \in \mathsf{V}_i^+$ for all $g \in \mathsf{U}(k_F)$ and all $x \in \mathsf{V}_i^+ \times \widetilde{\mathsf{V}}_{i,0}^\natural$.

Proof. We first analyze the situation over E, then pass to F. Since T is split over E, these three claims reduce to a commutator calculation with root groups, and follow from the following observations. Let $\alpha \in \Phi(G_{n+1}, T)$ with $\lambda(\alpha) \geq 0$ (a root of P_E), let $\beta \in \Phi(G_i, T) \smallsetminus \Phi(G_{i+1}, T)$ with $\lambda(\beta) \geq 0$ (a potential "root" of $\mathsf{V}_i^+ \times \widetilde{\mathsf{V}}_{i,0}^\natural$), and suppose $i\alpha + j\beta \in \Phi(G_i, T)$ with $i, j > 0$ (a root whose root group might appear in the commutator of the previous two root groups). The following three claims are proved by the subsequent observations about roots:

- $\mathsf{P}_E(k_E)$ preserves $\mathsf{V}_{i,E}^+$: If $\lambda(\beta) > 0$, then $\lambda(i\alpha + j\beta) > 0$.
- $\mathsf{P}_E(k_E)$ preserves $\mathsf{V}_{i,E}^+ \times \widetilde{\mathsf{V}}_{i,0,E}^\natural$: If $\lambda(\beta) \geq 0$, then $\lambda(i\alpha + j\beta) \geq 0$.
- $g(x) \cdot x^{-1} \in \mathsf{V}_{i,E}^+$ for all $g \in \mathsf{U}_E(k_E)$ and all $x \in \mathsf{V}_{i,E}^+ \times \widetilde{\mathsf{V}}_{i,0,E}^\natural$: If $\lambda(\alpha) > 0$ and $\lambda(\beta) \geq 0$, then $\lambda(i\alpha + j\beta) > 0$.

Now the result over F follows from Galois descent, using $\mathsf{V}_i^+ = (\mathsf{V}_{i,E}^+)^{\mathrm{Gal}(E/F)}$, $\widetilde{\mathsf{V}}_{i,0}^\natural = (\widetilde{\mathsf{V}}_{i,0,E}^\natural)^{\mathrm{Gal}(E/F)}$, $\mathsf{U}(k_F) \subseteq \mathsf{U}_E(k_E)^{\mathrm{Gal}(E/F)}$, and $\mathsf{P}(k_F) \subseteq \mathsf{P}_E(k_E)^{\mathrm{Gal}(E/F)}$. □

Recall the normal subgroup $(G_{n+1})_{[x]}^- \subseteq (G_{n+1})_{[x]}$ from Section 3.2.

Corollary 3.5.3. *Let H be the image of $(G_{n+1})_{[x]}^-$ under the map $(G_{n+1})_{[x]}^- \to \mathrm{Ps}(\mathsf{V}_i)$ induced by conjugation. Then the Heisenberg representation ω_i of $\mathsf{V}_i^\natural$ extends to a Heisenberg–Weil representation of $H \ltimes \mathsf{V}_i^\natural$.*

Proof. If $p \neq 2$, the result follows from [Gér77, Lemma 2.4(a)], which we already used in the definition of Weil representations (see Definition 2.5.2), so we assume $p = 2$ to avoid further case distinctions. By the definition of $(G_{n+1})^-_{[x]}$ and the fact that the image of $Z(G_{n+1})$ in $\mathrm{Ps}(\mathsf{V}_i)$ is trivial, there is a Borel subgroup B of G_{n+1} with unipotent radical N such that the image H_p of $\mathsf{N}(k_F)$ in $\mathrm{Ps}(\mathsf{V}_i)$ is a Sylow p-subgroup of H. We choose $\lambda \in X_*(T)^{\mathrm{Gal}(E/F)} \otimes \mathbb{R}$ such that $\mathsf{P} = \mathsf{B}$ and $\mathsf{U} = \mathsf{N}$. It follows from Lemma 3.5.2(a) that H_p is contained in $\mathcal{P}(\mathsf{V}_i^+)$, where the anisotropic subspace $\mathsf{V}_i^+ \subseteq \mathsf{V}_i$ is viewed as a subgroup of $\mathsf{V}_i^\natural$ via the above described splitting and $\mathcal{P}$ is as defined in Definition 2.5.4. Hence the existence of the Heisenberg–Weil representation follows from Definition 2.5.6 and Lemma 2.5.5. □

Notation 3.5.4. We denote the composition of $(G_{n+1})^-_{[x]} \twoheadrightarrow H$ with the Heisenberg–Weil representation of Corollary 3.5.3 also by ω_i.

There is a potential ambiguity in the construction of ω_i when $p = 2$, because a priori the Heisenberg–Weil representation of a finite group is only well-defined up to a character of this finite group that has order one or two. However, the next result will imply that this finite group has no characters of order two if $q > 2$, implying that the extension ω_i is uniquely defined (see Proposition 3.5.6).

Lemma 3.5.5. *If $p = 2$ and $q > 2$, then $(G_{n+1})^-_{[x]}/(Z(G) \cdot (G_{n+1})_{x,0+})$ has no characters of order two.*

Proof. The group $(G_{n+1})^-_{[x]}/(Z(G) \cdot (G_{n+1})_{x,0+})$ fits into the short exact sequence

$$1 \longrightarrow \frac{(G_{n+1})_{x,0}}{Z(G)_0 \cdot (G_{n+1})_{x,0+}} \longrightarrow \frac{(G_{n+1})^-_{[x]}}{Z(G(F)) \cdot (G_{n+1})_{x,0+}} \longrightarrow \frac{(G_{n+1})^-_{[x]}}{Z(G(F)) \cdot (G_{n+1})_{x,0}} \longrightarrow 1.$$

By definition, the righthand quotient has no characters of order two. So it suffices to show that the lefthand kernel has no characters of order two. But already the quotient $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$, the $\mathbb{F}_q$-points of a reductive $\mathbb{F}_q$-group, has no characters of order two by Lemma C.5. □

Proposition 3.5.6. *Suppose $q > 2$. Let H be the image of $(G_{n+1})^-_{[x]}$ under the map $(G_{n+1})^-_{[x]} \to \mathrm{Ps}(\mathsf{V}_i)$ induced by conjugation. Then the Heisenberg representation ω_i of $\mathsf{V}_i^\natural$ extends uniquely to a Heisenberg–Weil representation of $H \ltimes \mathsf{V}_i^\natural$.*

Proof. By Corollary 3.5.3 it remains to prove uniqueness of the extension. This follows from Lemma 3.5.5: the group H here is a quotient of the group appearing there because $(G_{n+1})_{x,0+}$ and $Z(G)$ act trivially on $\mathsf{V}_i^\natural$. □

To finish this subsection, we combine the representations ω_i to make a representation of K^-. This step is almost exactly as in [Yu01], but we spell it out in detail for clarity.

Construction 3.5.7 (Homomorphisms from iterated semidirect products)**.** Suppose we are given groups $A_1, \dots, A_n$ together with an action $(b, a) \mapsto {}^b a$ of A_j on A_i for every $1 \le i < j \le n$. If the "cocycle condition" ${}^{c}b({}^{c}a) = {}^{cb}a$ is satisfied for all $i < j < k$ and $a \in A_i$, $b \in A_j$, $c \in A_k$, then we can form the iterated semidirect product $A^{\rtimes} := A_1 \rtimes \cdots \rtimes A_n$. It is straightforward to check that under these circumstances the semidirect product is associative, like the direct product, so that there is no need to worry about the order of inserting parentheses in this iterated semidirect product.

Suppose we are given another group A' and homomorphisms $f_i \colon A_i \to A'$. Then the induced map $f^{\rtimes} \colon A^{\rtimes} \to A'$ defined by $(a_1, \dots, a_n) \mapsto f_1(a_1) \cdots f_n(a_n)$ is a homomorphism if and only if $f_i({}^b a) = f_j(b) f_i(a) f_j(b)^{-1}$ for every $1 \le i < j \le n$ and $a \in A_i$ and $b \in A_j$.

Such iterated semidirect products naturally arise from the following situation. Suppose that B is an ambient group containing the A_i as subgroups and that A_j normalizes A_i for $1 \le i < j \le n$. Using the conjugation action, we see that the cocycle condition is satisfied (because $cbc^{-1} \cdot c = cb$) and thus we may form the iterated semidirect product $A^{\rtimes}$. Multiplication induces a homomorphism $A^{\rtimes} \to B$, and we write $A_1 \cdots A_n$ for its image. In the situation of the second paragraph of this construction, the homomorphism $f^{\rtimes} \colon A^{\rtimes} \to A'$ descends to a homomorphism $f \colon A_1 \cdots A_n \to A'$ if and only if $f_1(a_1) \cdots f_n(a_n) = 1$ whenever $a_1 \cdots a_n = 1$ and $a_i \in A_i$.

Lemma 3.5.8. *There exists a unique representation κ^- of K^- with underlying vector space $V_{\kappa^-} := \bigotimes_{i=1}^n V_{\omega_i}$ such that*

(a) the restriction of κ^- to $(G_{n+1})_{[x]}^-$ is $\otimes_{i=1}^n (\phi_i|_{(G_{n+1})_{[x]}^-} \otimes \omega_i)$, and

(b) the restriction of κ^- to $(G_j)_{x,r_j,r_j/2}$ for $1 \le j \le n$ is

$$\bigotimes_{i=1}^{j-1} \hat{\phi}_i|_{(G_j)_{x,r_j,r_j/2}} \otimes \omega_j \otimes \bigotimes_{i=j+1}^{n} \hat{\phi}_i|_{(G_j)_{x,r_j,r_j/2}},$$

where ω_j denotes the composition of $(G_j)_{x,r_j,r_j/2} \twoheadrightarrow \mathsf{V}_j^\natural$ with the Heisenberg representation ω_j.

Proof. We will prove the existence of κ^-. Uniqueness will then follow immediately. To make the notation more uniform, write

$$K_j := \begin{cases} (G_j)_{x,r_j,r_j/2} & \text{if } 1 \le j < n+1, \\ (G_{n+1})_{[x]}^- & \text{if } j = n+1. \end{cases}$$

Note for future use in the proof that if $j \ne i$, then K_j is contained in the domain of $\hat{\phi}_i$.

We apply the observations from Construction 3.5.7. The group K^- is a quotient of the iterated semidirect product $K_1 \rtimes \cdots \rtimes K_{n+1}$. Fix i with $1 \le i \le n$. For each $1 \le j \le n+1$, define the homomorphism $\kappa_{ij}^-\colon K_j \to \mathrm{GL}(V_{\omega_i})$ by

$$\kappa_{ij}^- = \begin{cases} \hat{\phi}_i|_{K_j} & \text{if } j \neq i \text{ and } j \neq n+1 \\ \omega_i & \text{if } j = i \neq n+1 \\ \phi_i|_{K_j} \otimes \omega_i & \text{if } j = n+1. \end{cases}$$

Note that in the second case ω_i denotes a Heisenberg representation, and in the third case it denotes a Weil representation, both restrictions of a Heisenberg–Weil representation. We claim that for each fixed i, the representations $(\kappa_{ij}^-)_{1\le j\le n+1}$ induce a representation κ_i^- of K^-. This claim suffices because one can then define $\kappa^- := \otimes_{i=1}^n \kappa_i^-$.

To prove the claim, we have to first show that the map on the iterated semidirect product induced by the representations $(\kappa_{ij}^-)_{1\le j\le n+1}$ is a homomorphism. For this we use the criterion of Construction 3.5.7, which requires us to check that

$$\kappa_{ij}^-(aba^{-1}) = \kappa_{ik}^-(a)\kappa_{ij}^-(b)\kappa_{ik}^-(a)^{-1} \tag{3.5.9}$$

for all $j < k$, $a \in K_k$, $b \in K_j$. We distinguish four cases. First, if $j = i$ and $k = n+1$, then (3.5.9) holds by the definition of the Heisenberg–Weil representation. In the remaining cases one of κ_{ij}^- or κ_{ik}^- is a character and so (3.5.9) amounts to showing that $[K_k, K_j] \subseteq \ker(\kappa_{ij}^-)$. Second, if $j = i$ and $k \neq n+1$ then (3.5.9) holds because

$$[K_k, K_i] \subseteq [(G_i)_{x,0+}, K_i] \subseteq (G_i)_{x,r_i+,r_i/2+} \subseteq \ker(\omega_i),$$

by a root-group commutator calculation and Galois descent. In the remaining cases $\kappa_{ij}^- = \hat{\phi}_i|_{K_j}$ because $j \neq i$. Third, if $j < i$, then (3.5.9) holds because $[K_k, K_j] \subseteq K_j \subseteq \ker(\hat{\phi}_i)$. Fourth, if $i < j$, then (3.5.9) holds because $[K_k, K_j] \subseteq [G_{i+1}(F), G_{i+1}(F)] \subseteq \ker(\phi_i)$, as ϕ_i is a character of $G_{i+1}(F)$, and $\ker(\phi_i) \cap [K_k, K_j] \subseteq \ker(\hat{\phi}_i)$ since $\hat{\phi}_i$ extends $\phi_i|_{[K_k,K_j]}$.

It remains to show that $(\kappa_i^-)^{\rtimes}$ descends to a representation κ_i^- of K^-, for which we need to prove that $\kappa_{i1}^-(a_1)\cdots\kappa_{i,n+1}^-(a_{n+1}) = \mathrm{Id}$, the identity linear transformation, whenever $a_1 \cdots a_{n+1} = 1$ with $a_j \in K_j$ for $1 \le j \le n+1$. So suppose $a_1 \cdots a_{n+1} = 1$ with $a_j \in K_j$ for $1 \le j \le n+1$. Then $\kappa_{ii}^-(a_i) = \hat{\phi}_i(a_i)\,\mathrm{Id}$ because $a_i \in K_i \cap \prod_{1\le j\le n+1, j\neq i} K_j \subseteq (G_i)_{x,r_i,r_{i-1}} \subseteq (G_i)_{x,r_i,r_i/2+}$, and $\kappa_{i,n+1}^-(a_{n+1}) = \hat{\phi}_i(a_{n+1})\,\mathrm{Id}$ because $a_{n+1} \in K_n \cap \prod_{j=1}^n K_j \subseteq (G_{n+1})_{x,0+}$, a group on which ω_i is trivial. Hence $\kappa_{i1}^-(a_1)\cdots\kappa_{i,n+1}^-(a_{n+1}) = \hat{\phi}_i(a_1)\cdots\hat{\phi}_i(a_{n+1})\,\mathrm{Id} = \hat{\phi}_i(a_1\cdots a_{n+1})\,\mathrm{Id} = \mathrm{Id}$.

□

For later use, let us record the following intertwining property.

Lemma 3.5.10. *Suppose $q > 2$. For every $k \in K$, we have ${}^k\kappa^- \simeq \kappa^-$. In particular, if κ is an irreducible representation of K that contains κ^- when restricted to K^-, then κ is κ^--isotypic.*

Proof. Let $k \in K$. Since K^- normalizes κ^-, and $K = K^- \cdot (G_{n+1})_{[x]}$, we may assume without loss of generality that $k \in (G_{n+1})_{[x]}$. Then k is contained in the group $K' = (G_1)_{x,r_1/2+} \cdots (G_n)_{x,r_n/2+}(G_{n+1})_{[x]}$, and hence normalizes the character $\hat{\phi} := \otimes_{i=1}^n \hat{\phi}_i|_{K'}$. Since the restriction of κ^- to the normal subgroup

$$K_+ := (G_1)_{x,r_1/2+} \cdots (G_n)_{x,r_n/2+}(G_{n+1})_{x,0+}$$

is $\hat{\phi}|_{K_+}$-isotypic, and since the restriction of κ^- to

$$K_{0+} := (G_1)_{x,r_1/2} \cdots (G_n)_{x,r_1/n}(G_{n+1})_{x,0+}$$

is by the theory of Heisenberg representations the unique (up to isomorphism) irreducible representation that is $\hat{\phi}|_{K_+}$-isotypic when restricted to K_+, we obtain ${}^k\kappa^-|_{K_{0+}} \simeq \kappa^-|_{K_{0+}}$. Thus it remains to show that under this isomorphism ${}^k\kappa^-(g)$ and $\kappa^-(g)$ agree for $g \in (G_{n+1})_{[x]}^-$. Since $\otimes_{i=1}^n \phi_i|_{(G_{n+1})_{[x]}^-}$ is the restriction of the character $\otimes_{i=1}^n \phi_i|_{(G_{n+1})_{[x]}}$, by Lemma 3.5.8, it suffices to show that Weil representations ${}^k\omega_i^-(g)$ and $\omega_i^-(g)$ agree for $1 \leq i \leq n$ under the isomorphism that matches the underlying Heisenberg representations. This follows from the uniqueness of the extension of the Heisenberg representation (see Proposition 3.5.6). □

3.6 Supercuspidal representations

We keep the notation from the previous subsections. In particular, Lemma 3.5.8 provides us with a representation κ^- of K^-, and we denote by κ an irreducible representation of K that contains κ^- when restricted to K^-, and by σ an irreducible representation of $N_{K^+}(\rho \otimes \kappa)$ that contains $(\rho \otimes \kappa)$ when restricted to K. Our objective is to prove that $\text{c-ind}_{N_{K^+}(\rho\otimes\kappa)}^{G(F)}(\sigma)$ is irreducible supercuspidal if $q > 3$ (see Theorem 3.6.9(a)).

Since our proof is similar to the proof of [Fin21, Theorem 3.1], we will mostly focus on the modifications necessary to accommodate the new Heisenberg–Weil representation theory approach for $p = 2$ and deal with the more complicated intertwining set when (GE2) fails.

For the first half of the proof, which follows [Yu01], in particular Sections 8 and 9, we need to generalize some of Yu's results, which is done in Lemmas 3.6.2 and 3.6.3. Due to the more complicated structure of the intertwining set when (GE2) fails, we also work with the image of the simply connected cover at times, on which the desired characters we work with vanish (see Corollary 3.6.5). We also introduce two general lemmas, Lemmas 3.6.6 and 3.6.7, that are used to avoid the need of [Gér77, Theorem 2.4] in the second part of the proof of supercuspidality as Gérardin's result does not apply to our Heisenberg–Weil representations construction for $p = 2$.

We begin with a lemma that is not strictly necessary for the proof of supercuspidality, but that allows a better understanding of the structure of the group from which we induce when (GE2) fails. We write $H_\alpha := d\alpha^\vee(1)$.

Lemma 3.6.1. *Let G' be a split reductive group over F with split maximal torus T. Let $X \in \mathrm{Lie}^*(T)_r$ for some integer r, and let $\overline{X}$ be the image of X in $\mathrm{Lie}^*(T)_r/\mathrm{Lie}^*(T)_{r+}$. Let*

$W = (N_{G'}(T)/T)(F)$, and let W' be the subgroup of W generated by the reflections s_α with $\mathrm{val}(X(H_\alpha)) > r$, for $\alpha \in \Phi(G', T)$. Then the group $Z_W(\overline{X})/W'$ is a p-group.

Proof. Using $\mathrm{Lie}^*(T)_{-r}/\mathrm{Lie}^*(T)_{(-r)+} \simeq X^*(T) \otimes k_F$, the group W' is generated by the reflections s_α with $\overline{X}(H_\alpha) = 0$, and we may apply [Ste75, Theorem 4.5] to (in the notation of [Ste75]) $H = \langle \overline{X} \rangle$ with $Z_w(H)^0 = W'$ and X/X^0 being an $\mathbb{F}_p$-vector space. □

The following is a generalization of [Yu01, Lemma 8.3] and the second part of the proof is a variant of Yu's arguments.

Lemma 3.6.2 (cf. [Yu01, Lemma 8.3])**.** *Let H be a connected reductive F-group and let $H' \subseteq H$ be a twisted Levi subgroup that splits over a tame extension of F. Let $\widetilde{H}$ be a possibly disconnected reductive F-group and let $f \colon \widetilde{H} \to \mathrm{Aut}(H)$ be an algebraic action such that the induced map $f^\circ : \widetilde{H}^\circ \to H^{\mathrm{ad}}$ is surjective with central kernel.*

Let $X \in \mathrm{Lie}^*(H')^{H'}(F)$ be (H, H')-generic of depth $-r$. Then there is a subgroup $\widetilde{H}'$ of $N_{\widetilde{H}}(H')$ containing the identity component of $N_{\widetilde{H}}(H')$ such that:

(a) If $h \in \widetilde{H}(F)$, and $Y_1, Y_2 \in \mathrm{Lie}^*(H')_{x,-r}$ are regular semisimple that satisfy

$$Y_1 \equiv Y_2 \equiv X \pmod{\mathrm{Lie}^*(H')_{x,(-r)+}} \quad \text{and} \quad \mathrm{Ad}(h)Y_1 = Y_2,$$

then $h \in \widetilde{H}'(F)$.

(b) There is a short exact sequence of groups $1 \to A' \to \pi_0(\widetilde{H}')(F^{\mathrm{sep}}) \to A \to 1$ where $A \subseteq \pi_0(\widetilde{H})(F^{\mathrm{sep}})$ and A' is a p-group which is trivial if ϕ satisfies (GE2).

Proof. To start with, we give the construction of $\widetilde{H}'$. Let T be a maximal torus of H' and let E be a finite Galois field extension of F over which T is split. Note that we do not assume E/F to be tamely ramified. We may identify $\mathrm{Lie}^*(T)$ with the subspace $\mathrm{Lie}^*(H')^T \subseteq \mathrm{Lie}^*(H')$ on which the adjoint action of T is trivial, and $X \in \mathrm{Lie}^*(T)$ under this identification because $\mathrm{Lie}^*(T) \supseteq \mathrm{Lie}^*(H')^{H'}$. Let $\widetilde{N}_T := N_{\widetilde{H}}(H', T)$, let $N_T := N_{\widetilde{H}^\circ}(H', T)$, and let N'_T be the normalizer of T in $f^{\circ -1}(H'/Z(H))$. So $N'_T \subseteq N_T \subseteq \widetilde{N}_T$, each group of finite index in the next one. We define $\widetilde{H}'$ to be the F-subgroup of $N_{\widetilde{H}}(H')$ for which

$$\widetilde{H}'_E = Z_{\widetilde{N}_T}(\overline{X})_E \cdot f^{\circ -1}(H'/Z(H))_E,$$

where $\overline{X}$ is the image of X modulo $\mathrm{Lie}^*(T)_{x,(-r)+}$. Note that this construction is independent of the choice of E as replacing E by a larger field extension yields the same group $\widetilde{H}'$. Moreover, this construction is also independent of the choice of T: If T' is any other maximal torus of H', then we may take E to split both T and T', and the construction of $\widetilde{H}'_E$ produces the same group for both tori because T_E and T'_E are $H'(E)$-conjugate.

The exact sequence of (b) is constructed by observing that N'_T is a normal subgroup of $Z_{N_T}(\overline{X})$ and $\pi_0(\widetilde{H}')$ fits into the short exact sequence

$$1 \longrightarrow \frac{Z_{N_T}(\overline{X})}{N'_T} \longrightarrow \pi_0(\widetilde{H}') \longrightarrow \frac{Z_{\widetilde{N}_T}(\overline{X})}{Z_{N_T}(\overline{X})} \longrightarrow 1.$$

By Lemma 3.6.1, the first group in this sequence is a p-group, and it follows immediately from the definition of (GE2), [Yu01, page 596], that this group is trivial if X also satisfies (GE2).

It remains to prove (a), so let h, Y_1 and Y_2 be as in (a). Set $T_j := Z_{H'}(Y_j)^\circ$ for $j = 1, 2$, which is a maximal torus of H'. Let E/F be a finite Galois extension splitting T_1 and T_2. Take $h' \in f^{\circ-1}(H'/Z(H))(E)$ such that $T_2 = f(h')(T_1)$. Then $Y_0 := f(h')(Y_1) \in \mathrm{Lie}^*(T_2)_{-r}$ by [Yu01, Lemma 8.2], where we identify $\mathrm{Lie}^*(T_2)$ with the subspace $\mathrm{Lie}^*(H')^{T_2} \subseteq \mathrm{Lie}^*(H')$ on which the adjoint action of T_2 is trivial. We have $\mathrm{Lie}^*(T_2) \supseteq \mathrm{Lie}^*(H')^{H'}$, so that $X \in \mathrm{Lie}^*(T_2)$ under this identification. Using [Yu01, Lemma 8.2] we obtain

$$Y_0 \equiv f(h')(X) = X \equiv Y_2 \pmod{\mathrm{Lie}^*(T_2)_{(-r)+}}.$$

The element $n := hh'^{-1} \in \widetilde{H}(E)$ normalizes T_2. Moreover,

$$f(n)(X) \equiv f(n)(Y_0) = Y_2 \equiv X \pmod{\mathrm{Lie}^*(T_2)_{(-r)+}}.$$

Since X is (H, H')-generic of depth $-r$, if $\alpha \in \Phi(H, T_2)$, then

$$\mathrm{val}(X(H_\alpha)) = \begin{cases} \infty & \text{if } \alpha \in \Phi(H', T_2) \\ -r & \text{if not.} \end{cases}$$

As $n \in \widetilde{H}(E)$ preserves the set $\Phi(H, T_2)$, it follows that the element n also preserves the subset $\Phi(H', T_2)$ and hence normalizes H'. Hence $n \in Z_{\widetilde{N}_{T_2}}(\overline{X})(E)$, and therefore

$$h = nh' \in \widetilde{H}(F) \cap (Z_{\widetilde{N}_{T_2}}(\overline{X})(E) \cdot f^{\circ-1}(H'/Z(H))(E)) = \widetilde{H}'(F). \qquad \square$$

Replacing [Yu01, Lemma 8.3] by Lemma 3.6.2 in Yu's work we obtain the analogue of (the first half of) [Yu01, Theorem 9.4] in our more general setting. However, due to the more complicated intertwining set when (GE2) fails, we will have to work with a more general statement that only considers the restriction to the image of the simply connected covers of appropriate groups. Recall that we denote the image of $G^{\mathrm{sc}}(F)$ in $G(F)$ by $G(F)^\natural$.

Lemma 3.6.3 (cf. [Yu01, Theorem 9.4])**.** *Let $\widetilde{H}$ be a possibly disconnected reductive F-group with identity component H, let $H' \subseteq H$ be a twisted Levi subgroup that splits over a tame extension of F, and let $\phi\colon H'(F) \to \mathbb{C}^\times$ be a character that is (H, H')-generic relative to x of depth r. Then there exists a subgroup $\widetilde{H}'$ of $N_{\widetilde{H}}(H')$ with identity component H' satisfying the following properties:*

(a) If $h \in \widetilde{H}(F)$ intertwines the restriction of $\hat{\phi}_{(H,x)}$ to $(H', H)_{x,r,r/2+} \cap H(F)^\natural$, then $h \in H(F)_{x,r/2} \cdot \widetilde{H}'(F) \cdot H(F)_{x,r/2}$.

(b) There is a short exact sequence of groups $1 \to A' \to \pi_0(\widetilde{H}')(F^{\mathrm{sep}}) \to A \to 1$ where $A \subseteq \pi_0(\widetilde{H})(F^{\mathrm{sep}})$ and A' is a p-group which is trivial if ϕ satisfies (GE2).

In particular, if $\widetilde{H}$ is connected and p is not a torsion prime for $\widehat{G}$, then $\widetilde{H}' = H'$.

Proof. Let $c\colon H^{\mathrm{sc}} \to H$ be the simply-connected cover of $H^{\mathrm{der}} \subseteq H$. Since $\mathrm{Out}(H^{\mathrm{der}}) \subseteq \mathrm{Out}(H^{\mathrm{sc}})$, where $\mathrm{Out}(-)$ denotes the algebraic group of outer automorphisms, the conjugation action of $H(F)$ on H^{der} lifts uniquely to an action of $H(F)$ on H^{sc}. Passage to F-points gives an action of $H(F)$ on $H^{\mathrm{sc}}(F)$ lifting the conjugation action on $H^{\mathrm{der}}(F)$.

Let $H^{\mathrm{sc}\prime} := H' \times_H H^{\mathrm{sc}}$ and denote by ψ the composition of $c|_{(H^{\mathrm{sc}\prime}, H^{\mathrm{sc}})_{x,r,r/2+}}$ with $\hat{\phi}_{(H,x)}$. As $c((H^{\mathrm{sc}\prime}, H^{\mathrm{sc}})_{x,r,r/2+}) \subseteq (H', H)_{x,r,r/2+} \cap H(F)^\natural$, if an element $h \in \widetilde{H}(F)$ intertwines the restriction of $\hat{\phi}_{(H,x)}$ to $(H', H)_{x,r,r/2+} \cap H(F)^\natural$, then h also intertwines[6] ψ.

Since ϕ is (H, H')-generic relative to x of depth r, there exists an element $X \in \mathrm{Lie}^*(H')^{H'}(F)$ that is (H, H')-generic of depth $-r$ such that $\phi|_{H'(F)_{x,r}}$ is realized by X. By the construction of $\hat{\phi}_{(H,x)}$ its restriction to $(H', H)_{x,r,r/2+}$ is therefore also realized by $X \in (\mathfrak{h}')^*_{x,-r} \subseteq \mathrm{Lie}^*(H')(F) \subseteq \mathrm{Lie}^*(H)(F)$ (where the last inclusion is obtained by identifying $\mathrm{Lie}^*(H')(F)$ with $\mathrm{Lie}^*(H)^{Z(H')}(F)$), i.e., is given by composing

$$(H', H)_{x,r,r/2+}/H_{x,r+} \simeq (\mathfrak{h}', \mathfrak{h})_{x,r,r/2+}/\mathfrak{h}_{x,r+}$$

with $\Lambda \circ X$ for a fixed additive character $\Lambda : F \to \mathbb{C}^\times$ that is nontrivial on the ring of integers $\mathcal{O}$ of F, but trivial on the maximal ideal of $\mathcal{O}$.By precomposition with $\mathrm{Lie}(H^{\mathrm{sc}}) \to \mathrm{Lie}(H)$, we can view X also as an element in $\mathrm{Lie}^*(H^{\mathrm{sc}})(F)$. Note that then $X \in \mathrm{Lie}^*(H^{\mathrm{sc}\prime})_{x,-r} \cap \mathrm{Lie}^*(H^{\mathrm{sc}\prime})^{H^{\mathrm{sc}\prime}}(F)$, and since the derivative of c maps H_α in $\mathrm{Lie}(H^{\mathrm{sc}}(F)$ to H_α in $\mathrm{Lie}(H)(F)$, the element X is $(H^{\mathrm{sc}}, H^{\mathrm{sc}\prime})$-generic of depth $-r$. Since we have a diagram

$$\begin{array}{ccc} (H^{\mathrm{sc}\prime}, H^{\mathrm{sc}})_{x,r,r/2+}/H^{\mathrm{sc}}_{x,r+} & \longrightarrow & (H', H)_{x,r,r/2+}/H_{x,r+} \\ \downarrow & & \downarrow \\ (\mathfrak{h}^{\mathrm{sc}\prime}, \mathfrak{h}^{\mathrm{sc}})_{x,r,r/2+}/\mathfrak{h}^{\mathrm{sc}}_{x,r+} & \longrightarrow & (\mathfrak{h}', \mathfrak{h})_{x,r,r/2+}/\mathfrak{h}_{x,r+}, \end{array}$$

which commutes by the construction of the Moy–Prasad isomorphism given in the proof of [KP23, Theorem 13.5.1] together with the functoriality of the Moy–Prasad isomorphism for tori [KP23, Proposition B.6.9], the character ψ is represented by X.

Let $\widetilde{H}'$ be the group obtained from Lemma 3.6.2 applied to the group $\widetilde{H}$ acting on H^{sc}, the twisted Levi subgroup $H^{\mathrm{sc}\prime} \subseteq H^{\mathrm{sc}}$, and the Lie algebra element $X \in \mathrm{Lie}^*(H^{\mathrm{sc}\prime})^{H^{\mathrm{sc}\prime}}(F)$.

[6]Here we use the above action of $H(F)$ on $H^{\mathrm{sc}}(F)$ and the following slight generalization of the usual notion of intertwining: given a group A, a subgroup B, and a representation λ of B, an automorphism σ of A intertwines λ if there is a nonzero $B \cap \sigma(B)$-equivariant homomorphism from $\lambda \circ \sigma^{-1}$ to λ.

Now we can generalize the arguments from the proof of [Yu01, Theorem 9.4] as follows to apply to our more general setting of $h \in \widetilde{H}(F)$ intertwining ψ using Lemma 3.6.2 in place of [Yu01, Lemma 8.3].

More precisely, suppose $h \in \widetilde{H}(F)$ intertwines the restriction of $\hat{\phi}_{(H,x)}$ to $(H', H)_{x,r,r/2+} \cap H(F)^\natural$. Then h also intertwines ψ. Now note that Proposition 1.6.7 of [Adl98] (which Yu recorded in his setting as [Yu01, Theorem 5.1]) holds if, in the notation there, g is an algebraic F-automorphism of G rather than simply (conjugation by) an element of G. Thus there are regular semisimple elements Y_1, Y_2 in $X + (\mathfrak{h}^{\mathrm{sc}\prime}, \mathfrak{h}^{\mathrm{sc}})^*_{x,(-r)+,-r/2}$ such that $\mathrm{Ad}(h)Y_1 = Y_2$. Using [Yu01, Lemma 8.6], we can find $k_1, k_2 \in H^{\mathrm{sc}}(F)_{x,r/2}$ such that $Z_i := \mathrm{Ad}(k_i)Y_i \in X + \mathfrak{h}^{\mathrm{sc}\prime *}_{x,(-r)+}$ for $i = 1, 2$. But $Z_i = \mathrm{Ad}(c(k_i))Y_i$ as well, and $c(k_i) \in H(F)_{x,r/2}$. The element $h' := c(k_2) \cdot h \cdot c(k_1)^{-1}$ satisfies $\mathrm{Ad}(h')Z_1 = Z_2$. Using Lemma 3.6.2 we obtain that $h' \in \widetilde{H}'(F)$, as desired.

The last claim follows from generic characters (in our sense) automatically satisfying (GE2) if p is not a torsion prime for $\widehat{G}$ by [Yu01, Lemma 8.1]. □

Lemma 3.6.4. *Let H be a reductive group over F and let $y \in \mathscr{B}(H, F)$. Let φ be a character of $H(F)$. Then the restriction of φ to the intersection $H(F)^\natural_{y,0+}$ of the image of the simply connected cover $H(F)^\natural$ and $H(F)_{y,0+}$ is trivial.*

Proof. We start by reviewing a few facts about semisimple anisotropic groups. Recall that a reductive group is isotropic if it contains a unipotent element, or equivalently, if its derived subgroup contains a nontrivial split torus. First, if H is a simply-connected isotropic F-group, then $H(F) = [H(F), H(F)]$ by [PR84, 6.15]. Hence $H(F)$ has no nontrivial characters in this case. Second, if H is semisimple and anisotropic then H splits over an unramified extension, H is of type A, and the quasi-split inner form of H is split (see [KP23, Remark 10.3.2]). Hence $H^{\mathrm{sc}}(F)$ is compact and a product of groups of the form $\mathrm{SL}_1(D)$ where D is a division algebra over a finite separable extension of F. Third, the derived subgroup of $\mathrm{SL}_1(D)$ is the pro-unipotent radical $\mathrm{SL}_1(D)_{0+}$ (see [Rie70, page 504 and Corollary on page 521]).

Let $c\colon H^{\mathrm{sc}} \to H$ be the simply-connected covering map. Let H_i, $1 \le i \le n$, be the almost-simple subgroups of H corresponding to the irreducible factors of the relative Dynkin diagram of H. Using $H^{\mathrm{sc}} = \prod_{i=1}^n H_i^{\mathrm{sc}}$, we can factor any element $h \in H(F)^\natural_{y,0+}$ as a product $h_1 h_2 \cdots h_n$ with h_i in the image of $H_i^{\mathrm{sc}}(F)$. At the same time, φ is trivial on the image of $H_i^{\mathrm{sc}}(F)$ whenever H_i is isotropic. Replacing H by the subgroup generated by the anisotropic H_i, we are reduced to the case where H is anisotropic and semisimple, which we assume for the rest of the proof.

To finish the proof, it suffices to show that

$$c^{-1}(H(F)_{0+}) = \ker(c) \cdot H^{\mathrm{sc}}(F)_{0+}, \tag{3.6.4a}$$

since the restriction of φ to $H(F)^\natural_{0+}$ inflates to a character of $c^{-1}(H(F)_{0+})$ restricted from a character of $H^{\mathrm{sc}}(F)$. To prove (3.6.4a), note that if $H \to H'$ is an isogeny and (3.6.4a) holds with H replaced by H', then (3.6.4a) holds for H. Hence we may assume that H is an adjoint group. But then H is a product of almost-simple groups, and each factor is therefore of the form $\mathrm{Res}_{E/F}(H')$, where E/F is a finite separable extension by [BT65, 6.21(ii)]. The E-group H' is anisotropic, so that $H'(F) = \mathrm{PGL}_1(D)$ for some division algebra over E. Since $H'(F)_{0+} = \mathrm{PGL}_1(D)_{0+}$ (see [Fin22, Proposition A.12]), after replacing E by F, we are reduced to the case where $H^{\mathrm{sc}} = \mathrm{SL}_1(D)$ and $H = \mathrm{PGL}_1(D)$ for D a division algebra over F.

Let $\dim_F(D) = n^2$. Now $\mathrm{PGL}_1(D)_{0+} \simeq D^\times_{0+}/F^\times_{0+}$ by [Kal19, Lemma 3.3.2(1)], so (3.6.4a) amounts to the claim that if $z \in \mathrm{SL}_1(D)$ satisfies $z \in F^\times \cdot D^\times_{0+}$, then $z \in \mu_n(F) \cdot \mathrm{SL}_1(D)_{0+}$, where $\mu_n(F)$ denotes nth roots of 1 in F. This follows from the fact that if $a \in F^\times$ satisfies $a^n \in F^\times_{0+}$, then $a \in \mu_n(F) \cdot F^\times_{0+}$. □

Corollary 3.6.5. *Let $1 \le i \le n$, let $y \in \mathscr{B}(G_{n+1}, F)$ and $g \in N_G(G_{i+1}, G_{n+1})(F)$. Then the restriction of ${}^g\phi_i$ to $(G_{i+1})^\natural_{y,0+}$ is trivial. In particular, the restriction of ${}^g\phi_i$ to $U^\pm_j(F)_{y,r_j/2}$ is trivial for all $i < j \le n$.*

Proof. Apply Lemma 3.6.4 to the group $H = G_{i+1}$ and the character $\varphi = {}^g\phi_i$. □

In order to generalize the second half of the proof of supercuspidality in [Fin21, Theorem 3.1] we need two more lemmas that allow us to avoid [Gér77, Theorem 2.4], which is not available for the Heisenberg–Weil representations in characteristic 2.

Lemma 3.6.6. *Let P and H be finite groups and $U \trianglelefteq P$ a normal subgroup. Let P act on H by automorphisms, and let (π, V) be a representation of $P \ltimes H$ such that $\pi|_H$ is irreducible. Suppose that U acts trivially on H and $U \subseteq [P, U]$. Then $\pi|_U$ is trivial.*

Proof. Since U and H commute, the elements of $\pi(U)$ act H-equivariantly on V, hence are scalars by Schur's Lemma. So $\pi|_U$ is isotypic for a character ϕ of U. Moreover, since P normalizes U, it normalizes $\pi|_U$, hence ϕ. In other words, $\phi([p, u]) = 1$ for all $p \in P$ and $u \in U$. Hence ϕ is trivial. □

Lemma 3.6.7. *Let k be a field, let H be a quasi-split reductive k-group, let P be a parabolic subgroup of H, and let U be the unipotent radical of P. If either $|k| > 3$, or if $|k| = 3$ and H_{ad} has no factor isomorphic to PGL_2 or SO_5, then $U(k) \subseteq [P(k), U(k)]$.*

Our hypothesis on k is not entirely optimal when $|k| = 3$, but some assumptions are needed because the conclusion is false for $\mathrm{Spin}_5(\mathbb{F}_3) = \mathrm{Sp}_4(\mathbb{F}_3)$.

Proof. Let S be a maximal split torus of H contained in P and fix $\alpha \in \Phi(G, S)$. We will show that $U_\alpha(k) \subseteq [P(k), U(k)]$.

We claim that there is $s \in S(k)$ such that $\alpha(s) \ne 1$. Indeed, if $|k| > 3$, then there is $t \in k^\times$ such that $t^2 \ne 1$, and $\alpha(\alpha^\vee(t)) = t^2 \ne 1$. If $|k| = 3$, then our additional assumption gives $\beta \in \Phi(G, S)$ such that $\langle \alpha, \beta^\vee \rangle = -1$, and then $\alpha(\beta^\vee(t)) \ne 1$ for any $t \in k^\times \smallsetminus \{1\}$.

It suffices to show that $U_\alpha(k) \subseteq [S(k), U_\alpha(k)]$. Take $s \in S(k)$ such that $\alpha(s) \ne 1$. If $2\alpha \notin \Phi(G, S)$, then $U_\alpha(k)$ is abelian and $S(k)$-equivariantly isomorphic to its Lie algebra $\mathfrak{g}_\alpha(k)$, and the claim follows from the fact that the endomorphism $\mathrm{Ad}(s) - 1$ of $\mathfrak{g}_\alpha$ is multiplication by $\alpha(s)-1$ and hence invertible. If $2\alpha \in \Phi(G,S)$, then the quotient $U_\alpha(k)/U_{2\alpha}(k)$ is $S(k)$-equivariantly isomorphic to the root space $\mathfrak{g}_\alpha(k)$, and the same argument as above shows that for every $u \in U_\alpha(k)$ there is $u' \in U_\alpha(k)$ and $s \in S(k)$ such that $[s,u'] = u \pmod{U_{2\alpha}(k)}$. Now we are done because by the previous case, $U_{2\alpha}(k) \subseteq [S(k), U_{2\alpha}(k)]$. □

Now we are in a position to prove the key intertwining result, Theorem 3.6.8, following the proof of [Fin21, Theorem 3.1]. This result will immediately imply the main theorem, Theorem 3.6.9.

Theorem 3.6.8. *Let $\widetilde{K} = N_{K^+}(\rho\otimes\kappa)$ and let $\sigma \in \mathrm{Irr}(\widetilde{K}, K, \rho\otimes\kappa)$. Suppose $q > 3$.*

(a) If $g \in G(F)$ intertwines σ, then $g \in \widetilde{K}$.

(b) The group $\widetilde{K}/K$ is a finite p-group which is trivial if all ϕ_i satisfy (GE2), for example, if p is not a torsion prime for $\widehat{G}$.

Proof. Suppose $g \in G(F)$ intertwines σ. Since σ restricted to the normal subgroup $K \trianglelefteq \widetilde{K} = N_{K^+}(\rho\otimes\kappa)$ is $(\rho\otimes\kappa)$-isotypic, the element g also intertwines $(K, \rho\otimes\kappa)$. Moreover, by Lemma 3.5.10, σ further restricted to $K^- \trianglelefteq \widetilde{K}$ is a direct sum of copies of $(\rho\otimes\kappa^-)$, and hence g also intertwines $(K^-, \rho\otimes\kappa^-)$.

We first claim that $g \in K\widetilde{G}_{n+1}(F)K$ for some subgroup $\widetilde{G}_{n+1} \subseteq N_G(G_1, \ldots, G_n, G_{n+1})$ whose identity component is G_{n+1} and whose component group is a finite p-group which is trivial if all ϕ_i satisfy (GE2). We show this by induction following the first part of the proof of [Fin21, Theorem 3.1], focusing on the differences arising from our more general setup.

Let $1 \le i \le n$ and suppose the induction hypothesis that $g \in K\widetilde{G}_i(F)K$ where $\widetilde{G}_i$ is a subgroup of $N_G(G_1, \ldots, G_{i-1}, G_i)$ whose identity component is G_i and whose component group is a finite p-group which is trivial if all ϕ_j with $j < i$ satisfy (GE2). Let $\widetilde{G}_{i+1}$ be the group $\widetilde{H}'$ from Lemma 3.6.3 applied to $\widetilde{H} = \widetilde{G}_i$, $H' = G_{i+1}$, and $\phi = \phi_i$. Then by induction, $\widetilde{G}_{i+1} \subseteq N_{\widetilde{G}_i}(G_{i+1}) \subseteq N_G(G_1, \ldots, G_i, G_{i+1})$ and $\pi_0(\widetilde{G}_{i+1})(F^{\mathrm{sep}})$ is a finite p-group which is trivial if all ϕ_j with $j \le i$ satisfy (GE2). We will show that $g \in K\widetilde{G}_{i+1}(F)K$. Since K intertwines $\rho\otimes\kappa^-$ by Lemma 3.5.10, we may assume that $g \in \widetilde{G}_i(F)$. As in [Fin21, Theorem 3.1], the restriction of $\rho\otimes\kappa^-$ to $(G_i)_{x,r_i,(r_i/2)+}$ is the restriction of $\prod_{j=1}^{i} \hat{\phi}_j$. Hence g intertwines $(\prod_{j=1}^{i} \hat{\phi}_j)|_{(G_i)_{x,r_i,(r_i/2)+}}$. Moreover, for $1 \le j \le i-1$, the restriction of $\hat{\phi}_j$ to the intersection $(G_i)^{\natural}_{x,r_i,(r_i/2)+}$ of $(G_i)_{x,r_i,(r_i/2)+}$ with the image of $G_i^{\mathrm{sc}}(F)$ agrees with the restriction of ϕ_j and is trivial by Corollary 3.6.5. Hence g intertwines $\hat{\phi}_i|_{(G_i)^{\natural}_{x,r_i,(r_i/2)+}}$. Thus Lemma 3.6.3 implies that $g \in G_i(F)_{x,r_i/2}\widetilde{G}_{i+1}(F)G_i(F)_{x,r_i/2} \subseteq K\widetilde{G}_{i+1}(F)K$, which finishes the induction step.

Therefore, by induction $g \in K\widetilde{G}_{n+1}(F)K$, and to finish the proof of Part (a) we may assume that $g \in \widetilde{G}_{n+1}(F)$. It suffices to show that $g \in \widetilde{G}_{n+1}(F)_{[x]}$, because then g normalizes K and hence, since g intertwines $\rho\otimes\kappa$, we have $g \in N_{K^+}(\rho\otimes\kappa) = \widetilde{K}$, as desired.

Suppose to the contrary that $g[x] \neq [x]$. Note that $gx \in \mathscr{B}(G_{n+1}, F)$, so we can choose a tame maximal, maximally split torus T of G_{n+1} whose associated apartment contains x and gx. We let $\lambda \in X_*(T)^{\mathrm{Gal}(E/F)} \otimes \mathbb{R}$ be such that $gx = x + \lambda$, where E denotes a splitting field of T. We are now in the setting of Section 3.5 and use the notation defined there. Note in particular that $\mathsf{U}(k_F)$ is non-trivial, because x is a minimal facet.

Let f be a nonzero element of $\mathrm{Hom}_{K^- \cap {}^g K^-}({}^g(\rho \otimes \kappa^-), (\rho \otimes \kappa^-))$ and let V_f be the image of f. We adjust the arguments of the bottom of page 2739 and the top of page 2740 of [Fin21] to our setting, using the image of the simply connected cover at various places instead of the groups considered in [Fin21], and using Corollary 3.6.5, to show that

$$V_f \subseteq V_\rho \otimes_{\mathbb{C}} \bigotimes_{i=1}^{n} V_{\omega_i}^{U_i^+(F)_{x,r_i/2}},$$

and that the action of

$$U := ((G_{n+1})_{x,0} \cap (G_{n+1})^\natural_{gx,0+})(G_{n+1})^\natural_{x,0+}$$

on V_f via $\rho \otimes \kappa^-$ is trivial. Noting that the image of U in $\mathsf{G}_{n+1}(k_F)$ is $\mathsf{U}(k_F)$, it will then suffice to show that U acts also trivially on $V_{\omega_i}^{U_i^+(F)_{x,r_i/2}}$ for $1 \leq i \leq n$, because this will contradict that ρ is cuspidal.

For the convenience of the reader, we spell out a few more details. The restriction of $\rho \otimes \kappa^-$ to $(G_{n+1})^\natural_{x,0+}$ is the restriction of the character $\prod_{i=1}^n \phi_i|_{G_{n+1}(F)}$ (times the identity), and hence is the identity by Corollary 3.6.5. Recall that $gx \in \mathscr{B}(G_{n+1}, F)$ and ${}^g G_{n+1} = G_{n+1}$. Hence the group $(G_{n+1})_{x,0} \cap (G_{n+1})^\natural_{gx,0+}$ acts on V_f via the restriction of the character $\prod_{i=1}^n {}^g\phi_i|_{G_{n+1}(F)}$, whose restriction to $(G_{n+1})^\natural_{gx,0+}$ is also trivial by Corollary 3.6.5. Thus the action of U on V_f via $\rho \otimes \kappa^-$ is trivial, as desired. Moreover, by definition of λ and $U_i^+(F)_{x,r_i/2}$, we have $U_i^+(F)_{x,r_i/2} \subseteq (G_i)_{gx,r_i,r_i/2+}$, and hence $U_i^+(F)_{x,r_i/2}$ acts on V_f via the restriction of the character $\prod_{j=1}^{i-1} {}^g\phi_j|_{G_{n+1}(F)}$ to $U_i^+(F)_{x,r_i/2}$, hence by Corollary 3.6.5, the action of $U_i^+(F)_{x,r_i/2}$ on V_f is trivial. On the other hand, also using Corollary 3.6.5, $U_i^+(F)_{x,r_i/2}$ acts also trivially on V_{ω_j} for $1 \leq j \leq n$ with $i \neq j$. Hence we conclude that $V_f \subseteq V_\rho \otimes_{\mathbb{C}} \bigotimes_{i=1}^n V_{\omega_i}^{U_i^+(F)_{x,r_i/2}}$, as claimed.

To finish the proof of Part (a), it suffices to show that the action of U on $V_{\omega_i}^{U_i^+(F)_{x,r_i/2}}$ is trivial. Since $U \subseteq (G_{n+1})^\natural_{gx,0+} \cap (G_{n+1})^\natural_{x,0+}$, the restriction of ϕ_i to U is trivial by Corollary 3.6.5, so it suffices to show that the restriction of the Heisenberg–Weil representation to $\mathsf{U}(k_F)$ acting on $V_{\omega_i}^{U_i^+(F)_{x,r_i/2}} = V_{\omega_i}^{\mathsf{V}_i^+}$ is trivial. Since the image V_i^+ of $U_i^+(F)_{x,r_i/2}$ in the Heisenberg $\mathbb{F}_p$-group $\mathsf{V}_i^\natural$ is a splitting of an isotropic subspace, we can identify $V_{\omega_i}^{\mathsf{V}_i^+}$ as a representation of $\mathsf{V}_{i,0}^\natural$ with the irreducible Heisenberg representation for $\mathsf{V}_{i,0}^\natural$ (with same central character) by Lemma 2.3.4(b). At the same time, by Lemma 3.5.2(a), the group $\mathsf{P}(k_F)$ acts by conjugation on V_i^+ and on the quotient

$$\mathsf{V}_{i,0}^\natural \simeq \mathsf{V}_{i,0}^\natural \mathsf{V}_i^+ / \mathsf{V}_i^+,$$

and the action of the subgroup $\mathsf{U}(k_F)$ on the quotient is trivial. Hence the action of $\mathsf{P}(k_F)$ on V_{ω_i} preserves $V_{\omega_i}^{\mathsf{V}_i^+}$. Since $q > 3$, we may apply Lemma 3.6.7 to show that $\mathsf{U}(k_F) \subseteq [\mathsf{P}(k_F), \mathsf{U}(k_F)]$, and then apply Lemma 3.6.6 to conclude that $\mathsf{U}(k_F)$ acts trivially on $V_{\omega_i}^{\mathsf{V}_i^+} = V_{\omega_i}^{U_i^+(F)_{x,r_i/2}}$. This completes the proof of (a).

For the second part, we observed earlier in the proof, during the inductive argument, that $\pi_0(\widetilde{G}_{n+1})(F^{\mathrm{sep}})$ is a finite p-group which is trivial if all ϕ_i satisfy (GE2). Moreover, we have seen that if g intertwines σ, then $g \in K\widetilde{G}_{n+1}(F)_{[x]}K = K \cdot \widetilde{G}_{n+1}(F)_{[x]}$, and since all elements of $\widetilde{K}$ intertwine σ, we have $K \trianglelefteq \widetilde{K} \trianglelefteq K \cdot \widetilde{G}_{n+1}(F)_{[x]}$. Hence there is a chain of inclusions

$$\frac{\widetilde{K}}{K} \subseteq \frac{\widetilde{G}_{n+1}(F)_{[x]}}{G_{n+1}(F)_{[x]}} \subseteq \frac{\widetilde{G}_{n+1}(F)}{G_{n+1}(F)} \subseteq \pi_0(\widetilde{G}_{n+1})(F),$$

and so $\widetilde{K}/K$ is also a finite p-group which is trivial if all ϕ_i satisfy (GE2). □

Theorem 3.6.9. *Let $\widetilde{K} = N_{K^+}(\rho \otimes \kappa)$ and let $\sigma \in \mathrm{Irr}(\widetilde{K}, K, \rho \otimes \kappa)$. Suppose $q > 3$.*

(a) Let $\sigma \in \mathrm{Irr}(\widetilde{K}, K, \rho \otimes \sigma)$. Then $\text{c-ind}_{\widetilde{K}}^{G(F)}(\sigma)$ is irreducible supercuspidal.

(b) $\text{c-ind}_{K}^{G(F)}(\rho \otimes \kappa) \simeq \bigoplus_{\sigma \in \mathrm{Irr}(\widetilde{K}, K, \rho\otimes\kappa)} \left(\text{c-ind}_{\widetilde{K}}^{G(F)}(\sigma)\right)^{\oplus m_\sigma}$ with $m_\sigma \in \mathbb{N}_{\geq 1}$.

(c) If all ϕ_i satisfy (GE2), for example, if p is a torsion prime for $\widehat{G}$, then $\widetilde{K} = K$ and the representation $\text{c-ind}_K^G(\rho \otimes \kappa)$ is irreducible supercuspidal.

Proof. By Theorem 3.6.8(a), if $g \in G(F)$ intertwines σ, then $g \in \widetilde{K} = N_{K^+}(\rho \otimes \kappa)$. Moreover, K is a normal, finite-index subgroup of K^+ that contains $Z(G(F))$ and $K/Z(G(F))$ is compact. Now the result follows by applying Lemma B.1, using the well-known fact that for such a compactly-induced representation, irreducibility implies supercuspidality (see [Fin, Lemma 3.2.1]). □

A Alternating, symmetric, and quadratic forms

In this section we review the notions of symmetric forms, alternating forms, and quadratic forms, paying close attention to the features of these objects in characteristic 2 (see [KMRT98, pp. xvii–xxi]). Fix a base field k and a finite-dimensional k-vector space V.

Let B be a bilinear form on V. Recall that B is

- alternating if $B(v, v) = 0$ for all $v \in V$,
- symmetric if $B(v, w) = B(w, v)$ for all $v, w \in V$, and
- skew-symmetric if $B(v, w) = -B(w, v)$ for all $v, w \in V$.

If $\mathrm{char}(k) \neq 2$, then alternating is equivalent to skew-symmetric. If $\mathrm{char}(k) = 2$, then skew-symmetric is equivalent to symmetric. Moreover, if $\mathrm{char}(k) = 2$, then alternating implies skew-symmetric but not conversely, as we see by considering the simplest nontrivial bilinear pairing, $(a, b) \mapsto a \cdot b$ on the one-dimensional vector space k.

A bilinear form B is called nondegenerate if for every nonzero $v \in V$ there exists $w \in W$ such that $B(v, w) \neq 0$, and a nondegenerate alternating bilinear form is called a symplectic form.

A quadratic form Q on V is an element of $\mathrm{Sym}^2(V^*)$, that is, a homogeneous polynomial function on V of degree 2. Any quadratic form Q defines a symmetric bilinear form $B_Q \in \mathrm{Sym}^2(V)^*$ by the formula

$$B_Q\colon (v, w) \mapsto Q(v + w) - Q(v) - Q(w).$$

A quadratic form Q is defined to be nondegenerate if B_Q is nondegenerate.

The assignment $Q \mapsto B_Q$ defines a map

$$\mathrm{Sym}^2(V^*) \to \mathrm{Sym}^2(V)^*$$

whose behavior depends on $\mathrm{char}(k)$. If $\mathrm{char}(k) \neq 2$, then the map is an isomorphism with inverse $B \mapsto (v \mapsto \frac{1}{2}B(v, v))$, giving a bijection between quadratic forms and symmetric bilinear forms. If $\mathrm{char}(k) = 2$, then the map $Q \mapsto B_Q$ is not an isomorphism. Instead, its kernel is the space $(V^*)^{(2)}$ of diagonal quadratic forms and therefore, by a dimension count, its image is the space $\mathrm{Alt}^2(V)^*$ of alternating bilinear forms.

Assume for simplicity in the remainder of this section that $\dim(V) = 2n$ is even.

Let ω be a nondegenerate alternating form. A subspace W of V is isotropic if $\omega(w, w') = 0$ for all $w, w' \in W$. A partial polarization of V is a decomposition $V = V^+ \oplus V_0 \oplus V^-$ in which V^+ and V^- are isotropic, V_0 is orthogonal to $V^+ \oplus V^-$, and the restriction of ω to V_0 is nondegenerate. A polarization is a partial polarization in which $V_0 = 0$.

Similarly, let Q be a quadratic form. A subspace W of V is isotropic if every $w \in W$ is isotropic, meaning that $Q(w) = 0$.[7] A subspace W of V is called anisotropic if $Q(w) \neq 0$ for every $w \in W - \{0\}$. A partial polarization of V is a decomposition $V = V^+ \oplus V_0 \oplus V^-$ in which V^+ and V^- are isotropic, V_0 is orthogonal to $V^+ \oplus V^-$ (with respect to B_Q), and the restriction of Q to V_0 is nondegenerate. A polarization is a partial polarization in which V_0 is anisotropic.

The Witt index of Q is the dimension of one (equivalently, by a theorem of Witt, any [Lam05, Section I.4]) maximal isotropic subspace of V. We say Q is split if Q has Witt index n, in which case (V, Q) is isomorphic to the k-vector space k^{2n} equipped with the quadratic form

$$Q\colon \sum_{i=1}^{n}(x_i e_i + x_{-i}e_{-i}) \longmapsto \sum_{i=1}^{n} x_i \cdot x_{-i}, \tag{A.1}$$

[7] Some authors call a subspace "isotropic" if it contains some isotropic vector and "totally isotropic" if every vector is isotropic.

where $\{e_i : i \in \{\pm 1, \ldots, \pm n\}\}$ is a basis of V. If the Witt index of Q is $n-1$, then there is a separable quadratic extension ℓ/k such that (V, Q) is isomorphic to the k-vector space $k^{2n-2} \oplus \ell$ equipped with the quadratic form

$$Q\colon \sum_{i=1}^{n-1}(x_i e_i + x_{-i}e_{-i}) + ye_0 \longmapsto \sum_{i=1}^{n-1} x_i \cdot x_{-i} + \mathrm{Nm}_{\ell/k}(y), \qquad x_i \in k,\ y \in \ell, \tag{A.2}$$

where $\{e_i : i \in \{\pm 1, \ldots, \pm(n-1)\}\}$ is a basis of k^{2n-2} and e_0 is a non-zero element of ℓ.

Given Q nondegenerate, we can form the orthogonal group $\mathrm{O}(V) = \mathrm{O}(Q)$ of $g \in \mathrm{GL}(V)$ that preserve Q, meaning that $Q(gv) = Q(v)$ for all $v \in V$. Let $\mathrm{SO}(V)$ be the index-two subgroup of $\mathrm{O}(V)$ defined as the kernel of a map $\mathrm{O}(V) \to \mathbb{Z}/2\mathbb{Z}$ which is the determinant if $\mathrm{char}(k) \neq 2$ and the Dickson invariant if $\mathrm{char}(k) = 2$. See [Con14, Appendix C.2] for more discussion of the definition of the special orthogonal group in characteristic 2. The group $\mathrm{SO}(V)$ is reductive and of type D_n over the algebraic closure of k. Moreover, $\mathrm{SO}(V)$ is split if and only if Q is split. As [Tit66, Table II] explains, the group $\mathrm{SO}(V)$ is quasi-split but not split if and only if Q has Witt index $n-1$.

B Basic Clifford theory and the intertwining criterion

Let B be a group, let C be a finite-index normal subgroup of B, and let ρ be an irreducible representation of C. Clifford theory concerns two closely related problems: decomposing the induced representation $\mathrm{Ind}_C^B(\rho)$ and describing the set $\mathrm{Irr}(B, C, \rho)$ of irreducible representations of B whose restriction to C contains ρ. In this appendix we collect some results from basic Clifford theory and combine them with the classical intertwining criterion for irreducibility of a compactly-induced representation.

Lemma B.1. *Let $C \trianglelefteq B \leq A$ be groups with C normal and finite-index in B, and let ρ be a finite-dimensional irreducible representation of C.*

(a) Sending σ to the σ-isotypic component of $\mathrm{Ind}_C^B(\rho)$ defines a bijection

$$\mathrm{Irr}(B, C, \rho) \longleftrightarrow \mathrm{Irr}\big(\mathrm{End}_B(\mathrm{Ind}_C^B(\rho))\big).$$

Suppose in addition that A is locally profinite and ρ is smooth. Then

(b) $\text{c-ind}_C^A(\rho) \simeq \bigoplus_\sigma \text{c-ind}_{N_B(\rho)}^A(\sigma) \otimes V_\sigma$, where the sum ranges over $\sigma \in \mathrm{Irr}(N_B(\rho), C, \rho)$ and each V_σ is a finite-dimensional vector space with trivial A-action.

Finally, suppose in addition that C is open and has compact image in $A/Z(A)$. If every element of A intertwining ρ lies in B, then the following holds.

(c) For every $\sigma \in \mathrm{Irr}(N_B(\rho), C, \rho)$, the representation $\text{c-ind}_{N_B(\rho)}^A(\sigma)$ is irreducible.

Proof. For the first part, since $\mathrm{Ind}_C^B(\rho)$ is semisimple (see, e.g., [Kal, Fact A.3.2]), it decomposes as a finite direct sum

$$\mathrm{Ind}_C^B(\rho) \simeq \bigoplus_\sigma \sigma \otimes V_\sigma$$

where $\sigma \in \mathrm{Irr}(B)$ and V_σ is a vector space with trivial B-action recording the (finite) multiplicity of σ in $\mathrm{Ind}_C^B(\rho)$. By Frobenius reciprocity, σ contributes to this direct sum if and only if $\sigma|_C$ contains ρ. Using Schur's Lemma (see, e.g., [Ren10, Section B.II]), we obtain that

$$\mathrm{End}_B(\mathrm{Ind}_C^B(\rho)) \simeq \bigoplus_\sigma \mathrm{End}_B(V_\sigma).$$

The claim now follows from the fact that a finite-dimensional matrix algebra has, up to isomorphism, a unique irreducible representation.

The second part follows from transitivity of compact induction together with the proof of the first part where B is replaced by $N_B(\rho)$, which shows that

$$\text{c-ind}_C^{N_B(\rho)}(\rho) \simeq \bigoplus_\sigma \sigma \otimes V_\sigma.$$

For the third part, we first claim that if $a \in A$ intertwines σ, then $a \in N_B(\rho)$. Suppose $a \in A$ intertwines σ, then a intertwines $\sigma|_C$ and thus ρ because $\sigma|_C$ is ρ-isotypic. Hence $a \in B$ by assumption, and therefore $a \in N_B(\rho)$ because B normalizes C. The proof is now completed using the standard intertwining criterion for irreducibility of a compactly-induced representations. This criterion is stated when $A = G(F)$ in [Fin, Lemma 3.2.3], and the proof adapts to our setting using the Mackey decomposition for locally profinite groups (see [Kut77, Yam22]), after we note that $Z(A) \subseteq B$ because $Z(A)$ intertwines every representation of a subgroup of A. □

C Commutators in simply-connected quasi-split groups

Let k be a field and let H be a simply-connected quasi-split reductive k-group. In this appendix we review for convenience a classical result of Tits, Corollary C.4, showing that $H(k)$ usually equals its own commutator subgroup, except for some degenerate cases where $k = \mathbb{F}_2$ or $\mathbb{F}_3$ which we explicitly list. Although this result has been well-known for many years, we were unable to find a source in the literature that states it.

Let $H(k)^+$ be the subgroup of $H(k)$ generated by the subgroups $U(k)$ where U is the unipotent radical of some parabolic subgroup of H.

Theorem C.1. *Suppose that either*

$$\begin{aligned} &|k| \geq 4, \text{ or} \\ &k \simeq \mathbb{F}_3 \text{ and } H \text{ has no factor isomorphic to } \mathrm{SL}_2, \text{ or} \\ &k \simeq \mathbb{F}_2 \text{ and } H \text{ has no factor isomorphic to } \mathrm{SL}_2,\ \mathrm{Sp}_4,\ G_2, \text{ or } \mathrm{SU}_3. \end{aligned} \tag{C.2}$$

Then $[H(k)^+, H(k)^+] = H(k)^+$.

Proof. This is explained in [Tit64, Section 3.4]. □

Lemma C.3. *If H satisfies (C.2), then $H(k) = H(k)^+$.*

Proof. This is claimed without proof in [Tit78, Section 1.1.2], using the standard notation for the Whitehead group $W(H, k) := H(k)/H(k)^+$; see [MO486102] for a proof. □

Corollary C.4. *H satisfies (C.2) if and only if $[H(k), H(k)] = H(k)$.*

Proof. The forward implication follows from Theorem C.1 and Lemma C.3. The reverse implication is proved by direct computation: Clearly $\mathrm{SL}_2(\mathbb{F}_2) \simeq S_3$, and the remaining groups are worked out, for example, in [Wil09], specifically Section 3.3.1 ($\mathrm{SL}_2(\mathbb{F}_3)$), Section 3.5.2 ($\mathrm{Sp}_4(\mathbb{F}_2) \simeq S_6$), Section 4.4.4 ($G_2(\mathbb{F}_2)$), and Exercise 3.24 ($\mathrm{SU}_3(\mathbb{F}_2)$). □

Lemma C.5. *Let H be a reductive $\mathbb{F}_q$-group. If $H^{\mathrm{sc}}(\mathbb{F}_q)$ has trivial abelianization, for instance, if $q > 3$, then the abelianization of $H(\mathbb{F}_q)$ has order prime to q.*

Proof. Let $\widetilde{H} \to H$ be a z-extension (see [KP23, Section 11.4] for a discussion of this notion): a surjective map of reductive $\mathbb{F}_q$-groups whose kernel is an induced torus and for which $\widetilde{H}^{\mathrm{der}} = \widetilde{H}^{\mathrm{sc}}$. Then the map $\widetilde{H}(\mathbb{F}_q) \to H(\mathbb{F}_q)$ is surjective, and thus induces a surjection on abelianizations. So without loss of generality, after replacing H by $\widetilde{H}$, we may assume that $H^{\mathrm{der}} = H^{\mathrm{sc}}$, and hence that $H^{\mathrm{der}}(\mathbb{F}_q)$ has trivial abelianization. We have a short exact sequence

$$1 \longrightarrow H^{\mathrm{der}}(\mathbb{F}_q) \longrightarrow H(\mathbb{F}_q) \longrightarrow (H/H^{\mathrm{der}})(\mathbb{F}_q) \longrightarrow 1$$

in which H/H^{der} is a torus. Since $H^{\mathrm{der}}(\mathbb{F}_q)$ has trivial abelianization, the abelianizations of $H(\mathbb{F}_q)$ and $(H/H^{\mathrm{der}})(\mathbb{F}_q)$ are isomorphic. So we are reduced to the case where $H = T$ is a torus, where the claim follows from the fact that every element of $T(\mathbb{F}_q)$ is semisimple, and thus has order prime to q. That $H^{\mathrm{sc}}(\mathbb{F}_q)$ has trivial abelianization when $q > 3$ follows from Corollary C.4. □

See Corollary C.4 and (C.2) for a list of when $H^{\mathrm{sc}}(\mathbb{F}_q)$ fails to be perfect if $q = 2$ or 3.

D An example in the spin group

In this section we give an example of the failure of (GE2) which illustrates the need for Clifford theory in our construction of supercuspidal representations. Our example shows that the dimension of σ can be strictly larger than the dimension of $\rho \otimes \kappa$, as Remark D.11 spells out. The example is an extension of [Ste75, 2.20 Example].

We start by isolating a certain class of tori that is well adapted for making examples.

Definition D.1. Let k be a field. A -1-torus is a k-torus T splitting field over a quadratic extension ℓ/k such that $\mathrm{Gal}(\ell/k)$ acts on $X^*(T)$ by negation.

Concretely, a -1-torus is isomorphic to $(U^1_{\ell/k})^n$ for some n.

Lemma D.2. *Let T be a -1-torus with splitting field ℓ.*

(a) If a torus T' admits an isogeny to or from T then T' is a -1-torus.

Let G be a reductive group containing T as a maximal torus.

(b) $W(G,T)(k) = W(G,T)(\bar{k})$.

(c) If $k \simeq \mathbb{F}_q$ then any other maximal torus T' which is a -1-torus is $G(k)$-conjugate to T.

(d) If k is a nonarchimedean local field and ℓ/k is unramified then the $G(k)$-conjugacy class of T is uniquely determined by the point $\mathscr{B}(T,F)$ in $\mathscr{B}(G,F)$.

Proof. The first two parts are easy to check. The third part follows from [DeB06, Lemma 4.2.1], and the last part follows from the third and DeBacker's description of unramified maximal tori in reductive p-adic groups [DeB06, Theorem 3.4.1]. □

Lemma D.3. *Let ℓ/k be a separable quadratic extension, let T be a maximal torus of $G := \mathrm{SL}_2$ isomorphic to $U^1_{\ell/k}$, and let $N := N_G(T)$. Then*

$$N(k)/T(k) = \begin{cases} 1 & \text{if } -1 \notin \mathrm{Nm}_{\ell/k}(\ell^\times) \\ \mathbb{Z}/2\mathbb{Z} & \text{if } -1 \in \mathrm{Nm}_{\ell/k}(\ell^\times). \end{cases}$$

Proof. Let $\widetilde{G} := \mathrm{GL}_2$, let $\widetilde{T} := Z_{\widetilde{G}}(T)$, and let $\widetilde{N} := N_{\widetilde{G}}(\widetilde{T})$. We claim that

$$\widetilde{N}(k) \simeq \widetilde{T}(k) \rtimes \mathrm{Gal}(\ell/k) = \ell^\times \rtimes \mathrm{Gal}(\ell/k).$$

To see this, choose a basis for the 2-dimensional k-vector space ℓ, yielding an isomorphism of the matrix group GL_2 with the group $\mathrm{GL}(\ell/k)$ of k-linear automorphisms of ℓ. Then $\ell^\times \subseteq \mathrm{GL}(\ell/k)$ through its multiplication action, and $\mathrm{Gal}(\ell/k) \subseteq \mathrm{GL}(\ell/k)$ through its natural action on ℓ, proving the claim. Moreover, writing σ for the nontrivial element of $\mathrm{Gal}(\ell/k)$, the normal basis theorem shows that $\det(\sigma) = -1$. Returning to SL_2, we see that $N = \widetilde{N} \cap G$. We are finished after observing that $\det|_{\widetilde{T}} = \mathrm{Nm}_{\ell/k}$. □

Corollary D.4. *Let G be a reductive k-group such that $G_{\mathrm{ad}} \simeq (\mathrm{PGL}_2)^n$, let ℓ/k be a separable quadratic extension, and let T be a -1-torus of G splitting over ℓ that is maximal in G. If $-1 \in \mathrm{Nm}_{\ell/k}(\ell^\times)$ then*

$$N_G(T)(k)/T(k) \simeq W(G,T)(\bar{k}) \simeq (\mathbb{Z}/2\mathbb{Z})^n.$$

This concludes our general discussion of -1-tori.

In the rest of the section, let F be a nonarchimedean local field of residue characteristic $p = 2$ and let G be the split group over F of type Spin_8. We use Bourbaki's model for the root system $\Phi(D_4)$ and its Weyl group $W(D_4)$ [Bou02, Plate IV]:

$$\Phi(D_4) = \{\pm e_i \pm e_j : 1 \le i, j \le 4,\ i \ne j\}$$

with basis $\Delta = \{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_3 + e_4\}$. So $W(D_4) \simeq (\mathbb{Z}/2\mathbb{Z})^3 \rtimes S_4$, of order $2^6 \cdot 3$. Let x be a special vertex of $\mathscr{B}(G, F)$. Recall that $\mathrm{SO}_4 \simeq (\mathrm{SL}_2)^2/\mu_2$, where μ_2 is embedded diagonally. The group $(\mathrm{SO}_4)^2$ is a subgroup of SO_8, and its preimage H in Spin_8 is isomorphic to $(\mathrm{SL}_2)^4/\mu_2$, where μ_2 is again embedded diagonally.

Lemma D.5. *There exists an unramified -1-torus T maximal in H, hence G, such that the $\mathscr{B}(T, F) = \{x\}$.*

Proof. Since maximal tori of the special fiber of a parahoric group lift to unramified tori [DeB06, Lemma 2.3.1], it suffices to show that $(\mathrm{SL}_{2,k_F})^4/\mu_2$ contains a -1-torus. But $(\mathrm{SL}_{2,k_F})^4$ contains a -1-torus because each SL_2 factor does, and we are done by Lemma D.2(a). □

In the remainder of this section, let T be a maximal torus of G as in Lemma D.5.

Lemma D.6. $N_G(T)(F)/T(F) \simeq W(D_4)$.

Proof. Let $W := W(G, T)(F)$, so that $W = W(G, T)(\overline{F}) \simeq W(D_4)$ by Lemma D.2(b), and let $W' := N_G(T)(F)/T(F)$, which we view as a subgroup of W. Let $A := N_H(T)(F)/T(F)$. Then $A \subseteq W'$ and $A \simeq (\mathbb{Z}/2\mathbb{Z})^4$ by Corollary D.4.

Next, we show that W' contains a subgroup of order 64, a Sylow 2-subgroup. We will exhibit this subgroup using the four copies $\mathrm{SL}_2^{(1)}, \dots \mathrm{SL}_2^{(4)}$ of SL_2 in H with root systems

$$\Phi(T \cdot \mathrm{SL}_2^{(1)}, T) = \{\pm(\bar{e}_1 - \bar{e}_2)\}, \quad \Phi(T \cdot \mathrm{SL}_2^{(2)}, T) = \{\pm(\bar{e}_1 + \bar{e}_2)\},$$
$$\Phi(T \cdot \mathrm{SL}_2^{(3)}, T) = \{\pm(\bar{e}_3 - \bar{e}_4)\}, \quad \Phi(T \cdot \mathrm{SL}_2^{(4)}, T) = \{\pm(\bar{e}_3 + \bar{e}_4)\}.$$

To avoid confusion, we write $\bar{e}_i$ for elements of $X^*(T)$ and e_i for elements of $X^*(S)$ with S a maximal split torus, which will appear momentarily. Write $\Phi_i := \Phi(T \cdot \mathrm{SL}_2^{(i)}, T)$, a two-element subsystem of $\Phi := \Phi(G, T)$.

Let S be a split maximal torus of H whose apartment contains $\mathscr{B}(T, F)$. Choose a pinning $\mathcal{P} = (S, B, \{X_i\}_{1 \le i \le 4})$ of H where $X_i \in \mathrm{Lie}(\mathrm{SL}_2^{(i)})$ and $\Phi(B, S) = \{e_1 \pm e_2, e_3 \pm e_4\}$ and where $x_{\mathcal{P}} \in \mathscr{B}(G, F)$ is equal to x. Let $\sigma_0 \in W(G, S)$ act by the involution $e_1 \leftrightarrow e_3, e_2 \leftrightarrow e_4$. Then for any lift $n_0 \in N_G(S)(F)$ of σ_0, conjugation by n_0 permutes the SL_2 subgroups in the same way:

$$\mathrm{SL}_2^{(1)} \longleftrightarrow \mathrm{SL}_2^{(3)}, \qquad \mathrm{SL}_2^{(2)} \longleftrightarrow \mathrm{SL}_2^{(4)}. \tag{D.7}$$

Take n_0 to be the Tits lift of σ_0 (the element denoted by $\phi(\sigma_0)$ in [Spr09, Section 9.3.3]) with respect to $\mathcal{P}$. Then $\mathrm{Ad}(n_0)$ takes $\{X_i\}_{1\le i\le 4}$ to $\{\varepsilon_i X_i\}_{1\le i\le 4}$ for some $\varepsilon_i \in \{\pm 1\}$. Since $x = x_{\mathcal{P}} = x_{\mathrm{Ad}(n_0)\mathcal{P}}$, the action of n_0 fixes x and thus preserves the $H(F)$-conjugacy class of T by Lemma D.2(d). Hence there is an element $h \in H(F)$ such that $n := hn_0$ normalizes T, and n permutes the SL_2-factors according to (D.7). Making a similar argument with $\mathrm{SL}_2^{(1)}$ and $\mathrm{SL}_2^{(2)}$ exchanged, we find in summary that W' contains elements that act on the four-element set $\{\Phi_i : 1 \le i \le 4\}$ by

$$(\Phi_1 \longleftrightarrow \Phi_3, \quad \Phi_2 \longleftrightarrow \Phi_4), \qquad (\Phi_1 \longleftrightarrow \Phi_4, \quad \Phi_2 \longleftrightarrow \Phi_3).$$

Thus, the group of order 4 generated by those elements together with the 8 elements of A, which act trivially on this set, generate a subgroup of order 64 in W'.

Since $|W| = 64 \cdot 3$, to finish the proof it suffices to show that W' contains another of the three Sylow 2-subgroups of order 64. Let H' be the reductive subgroup of G that contains T and for which

$$\Phi(H', T) = \{\pm e_1 \pm e_3, \pm e_2 \pm e_4\}.$$

Then H' is isomorphic to H over $\overline{F}$, and the proof is finished if we can show that H is isomorphic to H' over F; if so, then we can rerun the previous arguments in the proof for this other copy of $(\mathrm{SL}_2)^4/\mu_2$. For this, write G_x and T_x for the special fibers of the parahorics at x of G and T, respectively. Since SL_2 has no nontrivial inner forms over k_F, we see that T_x is contained in a copy of $(\mathrm{SL}_2)^4/\mu_2$ in G_x whose root system is identified with that of H'. This shows that H' must be split, since the maximal reductive quotient of the special fiber of the nonsplit form of SL_2 is a torus, rather than SL_2. Hence W' contains another Sylow 2-subgroup; so $W' = W$. □

To define our exceptional character of $T(F)$, first observe that $T(F) \simeq U^1_{E/F}(F) \otimes_{\mathbb{Z}} X_*(T)$. Since Spin_8 is simply-connected, $X^*(T)$ is the weight lattice of type D_4. Number the fundamental weights ϖ_i, $1 \le i \le 4$, as in [Bou02, Plate IV, p. 272], so that ϖ_2 is dual to the central vertex of the Dynkin diagram.

Lemma D.8. *Let $q \ge 4$ and let r be an odd integer with $1 \le r < \mathrm{val}(2)$.*

(a) There exist two order-two characters $\phi_i\colon U^1_{E/F}(F) \to \mathbb{C}^\times$, $i = 1, 2$ such that

$$\mathrm{depth}(\phi_1) = \mathrm{depth}(\phi_2) = \mathrm{depth}(\phi_1\phi_2) = r.$$

(b) For all such ϕ_i, the following character $\phi\colon T(F) \to \mathbb{C}^\times$ is (G, T)-generic of depth r:

$$\phi(t) = \phi_1(t^{\varpi_2}) \cdot \phi_2(t^{\varpi_1+\varpi_3+\varpi_4}).$$

Here we use the exponential notation $t^\alpha := \alpha(t)$.

Proof. For the first part, consider the maximal 2-torsion quotient

$$U^1_{E/F}(F)/U^1_{E/F}(F)^{\times 2},$$

where $(-)^{\times 2}$ forms the subgroup of squares. If $a \in E_0^\times$ and $\mathrm{val}(a-1) < \mathrm{val}(2)$ then

$$\mathrm{val}(a^2-1) = \mathrm{val}((a-1)^2 - 2(a-1)) = 2\,\mathrm{val}(a-1);$$

moreover, if $\mathrm{val}(a-1) = 0$ then $\mathrm{val}((a-1)^2) = \mathrm{val}(a-1)$ Since r is odd, the map

$$U^1_{E/F}(F)_{r:r+} \to U^1_{E/F}(F)/\big(U^1_{E/F}(F)^{\times 2} \cdot U^1_{E/F}(F)_{r+}\big)$$

is injective, and hence we can freely extend any character of the group $U^1_{E/F}(F)_{r:r+}$ to an order-two character of $U^1_{E/F}(F)$. Since the group $U^1_{E/F}(F)_{r:r+} \simeq k_F$ has cardinality q and $q \geq 4$, this group has two linearly independent characters, completing the proof.

For the second part, coordinatize $X^*(T)$ as in Bourbaki: $X^*(T) = \mathbb{Z}^4 + \mathbb{Z}\varpi_4$, where $\varpi_4 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$. Since $\varpi_1 + \varpi_3 + \varpi_4 = 2e_1 + e_2 + e_3$, we can rewrite ϕ as

$$\phi(t) = \phi_1(t_1 t_2) \cdot \phi_2(t_2 t_3).$$

After passing to the dual Lie algebra at depth $-r$, this description becomes

$$X = a_1 \otimes (e_1 + e_2) + a_2 \otimes (e_2 + e_3)$$

for some $a_i \neq 0$. It is now straightforward to check that X is (G,T)-generic of depth $-r$. □

Recall that $W(D_4) \simeq (\mathbb{Z}/2\mathbb{Z})^3 \rtimes S_4$. The group A_4 contains a unique Sylow 2-subgroup, the Klein four-group, which is normal. Let P be the subgroup of $W(D_4)$ generated by the inversions $(\mathbb{Z}/2\mathbb{Z})^3$ and this Klein four-group. Then $P \simeq (\mathbb{Z}/2\mathbb{Z})^3 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$ is a nonabelian group of order 32, normal in $W(D_4)$.

Lemma D.9. *The centralizer in $N_G(T)(F)/T(F)$ of the character ϕ from Lemma D.8 is P.*

Proof. The inversions $(\mathbb{Z}/2\mathbb{Z})^3$ centralize ϕ because the ϕ_i have order two. To see that the Klein four-group centralizes ϕ, use that $\phi_1(t_1t_2) = \phi_1(t_3t_4)$ and $\phi_2(t_2t_3) = \phi_2(t_1t_4)$ because $e_1 + e_2 + e_3 + e_4 \in 2X^*(T)$. □

Corollary D.10. *Let $\phi\colon T(F) \to \mathbb{C}^\times$ be as in Lemma D.8 and let $N_G(T)(F)_P$ be the preimage of P under the projection to $W(G,T)$. Then*

$$Z_{G(F)_{[x]}}(\hat{\phi}) = N_G(T)(F)_P \cdot G(F)_{x,r/2}.$$

Remark D.11. Consider the supercuspidal G-datum $((G,T), x, r, 1, \phi)$ where ϕ is as in Lemma D.8, so that $\rho \otimes \kappa = \hat{\phi}$ and $N_{K^+}(\hat{\phi}) = N_G(T)(F)_P \cdot G(F)_{x,r/2}$ by Corollary D.10. Let σ be a representation of $N_{K^+}(\hat{\phi})$ that is $\hat{\phi}$-isotypic on K^+. Then σ is determined by its restriction σ_0 to $N_G(T)(F)_P$, and conversely, any representation σ_0 of $N_T(G)(F)_P$ that is

ϕ-isotypic on $T(F)$ determines such a σ by the formula $\sigma(kn) = \hat{\phi}(k)\sigma_0(n)$ for $k \in G(F)_{x,r}$ and $n \in N_G(T)(F)_P$. So we might as well work with σ_0.

Since σ_0 and ϕ are trivial on $\ker(\phi)$, we can interpret them as representations of the finite group $\widetilde{P} := N_G(T)(F)_P/\ker(\phi)$. And since $T(F)/\ker(\phi) \simeq \mathbb{Z}/2\mathbb{Z}$, the group $\widetilde{P}$ fits into a short exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{P} \to P \to 1.$$

Choosing σ_0 amounts to choosing a representation of $\widetilde{P}$ whose restriction to $\mathbb{Z}/2\mathbb{Z}$ is nontrivial. Hence there certainly exists a σ_0 for which $\dim(\sigma_0) > \dim(\phi) = 1$, since P is nonabelian. Moreover, if we choose two σ_0 and σ_0' with $\dim(\sigma_0) \neq \dim(\sigma_0')$, then the resulting supercuspidal representations π and π' are not isomorphic because their formal degrees are different:

$$\operatorname{fdeg}(\pi) = \frac{\dim(\sigma_0)}{\operatorname{vol}(N_{K^+}(\hat{\phi}))} \neq \frac{\dim(\sigma_0')}{\operatorname{vol}(N_{K^+}(\hat{\phi}))} = \operatorname{fdeg}(\pi').$$

So our use of Clifford theory is unavoidable.

Selected notation

Selected terminology

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