REDUCTIVE GROUPS II: BOREL-WEIL-BOTT, LINKAGE, TRANSLATION

DAVID SCHWEIN

ABSTRACT. In this talk for the Oberwolfach Arbeitsgemeinschaft "Geometric representation theory", we discuss the Borel-Weil-Bott theorem, the linkage principle, and the translation functors, following Chapters 5, 6, and 7 of Jantzen's book on representations of algebraic groups.

NOTATION

k	algebraically closed field	R	root system of (G,T)
G	reductive algebraic k-group	R^+	set of positive roots
T	maximal torus of G	X	character lattice of T
W	Weyl group of (G,T)	X^+	dominant weights

1. THE DOT ACTION AND THE AFFINE WEYL GROUP

The affine Weyl group of G is the semidirect product

$$W_{\mathrm{aff}} \overset{\mathsf{def}}{=} W \ltimes \mathbb{Z}R$$

where W acts in the usual way on the root lattice $\mathbb{Z}R$. The two factors of the affine Weyl group act on X: the ordinary Weyl group in the usual way and $\mathbb{Z}R$ by translation. These two actions are compatible and give rise to an action of W_{aff} on X. For our application, however, a slight modification of this action is needed.

Let

$$\rho \overset{\mathsf{def}}{=} \frac{1}{2}\sum_{\alpha\in R^+}\alpha$$

be the half-sum of the positive weights, or equivalently, the sum of the fundamental coweights. Let ℓ be a positive integer. The dot action of W_{aff} on X with parameter ℓ is obtained from the usual action of W_{aff} by shifting the origin to $-\rho$ and scaling the action of $\mathbb{Z}R$ by ℓ:

$$(wt_\lambda)\bullet_\ell \mu \overset{\mathsf{def}}{=} w(\mu+\ell\lambda+\rho)-\rho,$$

where t_λ denotes translation by $\lambda\in\mathbb{Z}R$. The dot action of the ordinary Weyl group is independent of ℓ and we will therefore suppress it from the notation in that case.

One fundamental domain for the (induced) dot action on $X\otimes\mathbb{R}$ is the alcove $\overline{C}=\overline{C}_\ell$ consisting of the $\lambda\in X\otimes\mathbb{R}$ such that for all $\alpha\in R^+$,

$$0\le\langle\lambda+\rho,\alpha^\vee\rangle\le\ell.$$

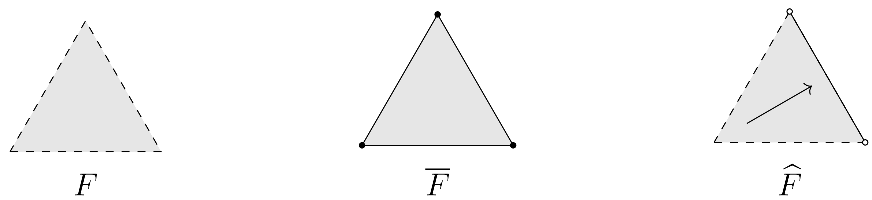
Let $\overline{C}_{\ell,\mathbb{Z}}=\overline{C}_{\mathbb{Z}}\overset{\mathsf{def}}{=}\overline{C}\cap X$.

Date: 4 April 2022.

More generally, the facets of $X \otimes \mathbb{R}$ with respect to the dot action are the subsets F consisting of the λ such that

$$\begin{cases} n_\alpha \ell < \langle \lambda + \rho, \alpha^\vee \rangle < (n_\alpha + 1)\ell & \text{if } \alpha \in R_0^+ \\ n_\alpha \ell = \langle \lambda + \rho, \alpha^\vee \rangle & \text{if } \alpha \in R_1^+ \end{cases}$$

where $R^+ = R_0^+ \sqcup R_1^+$ is a partition of the positive roots and the n_α's are integers. The (topological) closure $\overline{F}$ of F is the set of points satisfying the above constraints but with all $<$'s replaced by $\leq$'s. We similarly define the upper closure $\widehat{F}$ by replacing only the righthand $<$'s with $\leq$'s. Unlike the topological closure, the upper closure depends on the choice of positive roots; in the picture below, the arrow indicates the direction of the positive Weyl chamber.

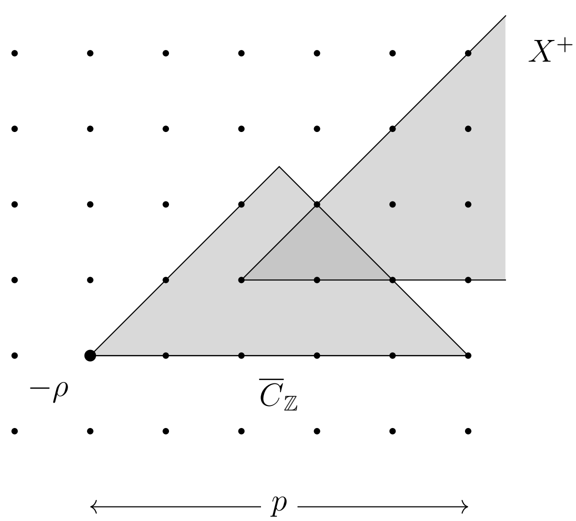


Example 1. For $G = \mathrm{PGL}_2$ we have $X = \mathbb{Z}$, $R = \{\pm 1\}$, $X^\vee = 2\mathbb{Z}$, $R^\vee = \{\pm 2\}$, $\rho = \frac{1}{2}$, and

$$C = (-1/2, (p-1)/2), \quad \overline{C} = [-1/2, (p-1)/2], \quad \widehat{C} = (-1/2, (p-1)/2],$$

where we form the upper closure with respect to the Weyl chamber $[0, \infty)$, that is, the positive root $+1$.

For $G = \mathrm{Sp}_4$ we have $X = \mathbb{Z}^2$, the simple roots are $e_1 - e_2$ and $2e_2$, the other positive roots are $e_1 + e_2$ and $2e_1$, $\rho = 2e_1 + e_2$, and the positive Weyl chamber is a cone with apex the origin and walls the x-axis and the line $x = y$. The chamber $\overline{C}$ is an isosceles right-angled triangle, pointing up with horizontal hypotenuse of length p and leftmost vertex at $(-2, -1)$. In the picture below, $p = 5$.



These examples show that not every facet of $(W_{\mathrm{aff}}, \bullet_p)$ need intersect X. In particular, if p is very small then a facet of maximal dimension need not intersect X; this happens in the previous example if $p = 2$ or 3.

2. Borel-Weil-Bott

We have seen that a weight $\lambda \in X$ is dominant if and only if $\mathrm{H}^0(\lambda) \neq 0$, and that in this case, $\mathrm{H}^{>0}(\lambda) = 0$. What can be said about the cohomologies of the line bundles associated to a general weight, not necessarily dominant? The Borel-Weil-Bott theorem states roughly that the dot action of a simple reflection increments cohomology by one degree.

In what follows, if $\operatorname{char} k = p > 0$ then let $\overline{C}_{\mathbb{Z}} = \overline{C}_{p,\mathbb{Z}}$ and if $\operatorname{char} k = 0$ then let

$$\overline{C}_{\mathbb{Z}} \stackrel{\mathsf{def}}{=} \left\{\lambda \in X \mid \langle \lambda + \rho, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in R^+\right\} = -\rho + (X \otimes \mathbb{R})^+.$$

Theorem 2 (Borel-Weil-Bott)**.** *Let $\lambda \in \overline{C}_{\mathbb{Z}}$.*

(1) If $\lambda \notin X^+$ then $\mathrm{H}^i(w \bullet \lambda) = 0$ for all $i \geq 0$ and $w \in W$.
(2) If $\lambda \in X^+$ then $\mathrm{H}^i(w \bullet \lambda) = \mathrm{H}^0(\lambda)$ for $i = \ell(w)$ and 0 otherwise.

If $\operatorname{char} k = 0$ then every element of X is of the form $w \bullet \lambda$ for some $w \in W$ and $\lambda \in X^+$ and the Borel-Weil-Bott theorem therefore computes completely the cohomologies in this case.

Classically, in characteristic zero, it was an important part of the statement that $\mathrm{H}^0(\lambda)$ is simple for λ dominant. We can deduce this in all characteristics using Serre duality.

Corollary 3. *If $\lambda \in \overline{C}_{\mathbb{Z}}$ then $\mathrm{H}^0(\lambda)$ is simple.*

Proof. The proof of the corollary rests on Serre duality, which we quickly review. The dualizing sheaf on G/B is $\mathcal{L}(-2\rho)$, and $\mathcal{L}(\lambda)^* \simeq \mathcal{L}(-\lambda)$. So in this setting, Serre duality gives an isomorphism

$$\mathrm{H}^i(\lambda)^* \simeq \mathrm{H}^{|R^+|-i}(-\lambda - 2\rho).$$

In particular, letting w_0 denote the longest element of the Weyl group, since $-w_0$ preserves the positive Weyl chamber, $w_0\rho = -\rho$, and it follows that

$$w_0 \bullet (-w_0\lambda) = -\lambda - 2\rho.$$

These facts imply that

$$\mathrm{H}^0(\lambda)^* \simeq \mathrm{H}^{|R^+|}(w_0 \bullet (-w_0\lambda)).$$

Since $-w_0\overline{C}_{\mathbb{Z}} = \overline{C}_{\mathbb{Z}}$, we can apply Borel-Weil-Bott to show that

$$\mathrm{H}^0(\lambda)^* \simeq \mathrm{H}^0(-w_0\lambda).$$

Combining this isomorphism with the Weyl-module description of simple modules yields

$$\operatorname{soc}_G \mathrm{H}^0(\lambda) \stackrel{\mathsf{def}}{=} L(\lambda) \simeq V(\lambda)/\operatorname{rad}_G V(\lambda) \simeq \mathrm{H}^0(\lambda)/\operatorname{rad}_G \mathrm{H}^0(\lambda).$$

If $\operatorname{rad}_G \mathrm{H}^0(\lambda)$ were nonzero then it would have to contain some $L(\mu)$, and the only possibility is $L(\lambda)$, the maximal semisimple submodule of $\mathrm{H}^0(\lambda)$. But since $L(\lambda)$ has multiplicity one in $\mathrm{H}^0(\lambda)$ this is impossible and $\operatorname{rad}_G \mathrm{H}^0(\lambda) = 0$. □

Remark 4. The ρ-shift arises in many places in the representation theory of reductive groups. In a slightly different setting, p-adic reductive groups, one defines the normalized parabolic induction (say, from a Borel subgroup with split maximal torus) of a representation as the twist of the parabolic induction by the square root of the modulus character of $B(\mathbb{Q}_p)$. This correction term is precisely the inflation of the unramified character of $T(\mathbb{Q}_p)$ corresponding to ρ.

Remark 5. We cannot improve on the Borel-Weil-Bott theorem in characteristic p by relaxing the hypothesis that $\lambda \in \overline{C}_{\mathbb{Z}}$. It is known [Jan03, II.5.18] that for any simple root α,

$$\mathrm{H}^1(-p^n\alpha) \neq 0.$$

On the other hand, as soon as the Dynkin diagram of G has a connected component with two or more vertices, $-p^n\alpha$ does not lie in $s \bullet X^+$ for any simple reflection s. Indeed, the only simple reflection that could move $-p^n\alpha$ to be dominant is s_α, and

$$s_\alpha \bullet (-p^n\alpha) = (p^n - 1)\alpha$$

since $s_\alpha\rho = \rho - \alpha$. But this element is not dominant: we can find simple roots α and β so that $\langle \alpha, \beta^\vee \rangle < 0$ by our hypothesis on G.

3. Linkage principle

Let $\mathrm{Rep}(G)$ be the category of finite-dimensional[1] algebraic representations of G. When $\mathrm{char}\, k = 0$, this category decomposes as a direct sum over X^+:

$$\mathrm{Rep}(G) = \sum_{\lambda \in X^+} \mathrm{Rep}_\lambda(G),$$

where $\mathrm{Rep}_\lambda(G)$ is the category of $L(\lambda)$-isotypic modules. In this section we'll give a related decomposition of G-Mod when $p \overset{\mathsf{def}}{=} \mathrm{char}\, k > 0$, which we assume from now on. The decomposition is a consequence of the following theorem, of which we will prove a special case in Section 5.

Theorem 6 (Linkage principle)**.** *Let $\lambda, \mu \in X^+$. If $\mathrm{Ext}^1(L(\lambda), L(\mu)) \neq 0$ then $\lambda \in W_{\mathrm{aff}} \bullet_p \mu$.*

Consequently,

$$\mathrm{Rep}(G) = \bigoplus_{\gamma \in X/(W_{\mathrm{aff}}, \bullet_p)} \mathrm{Rep}_\gamma(G) \tag{1}$$

where $\mathrm{Rep}_\gamma(G)$ is the Serre subcategory generated by the simple modules $L(\mu)$ with $\mu \in \gamma \cap X^+$. In other words, $V \in \mathrm{Rep}_\gamma(G)$ if and only if every composition factor of V has highest weight in $\mu \in \gamma \cap X^+$.

Remark 7. It is tempting to call each subcategory $\mathrm{Rep}_\gamma(G)$ a block of $\mathrm{Rep}(G)$. However, this terminology is not strictly correct because it can happen that $\mathrm{Rep}_\gamma(G)$ decomposes further. Here is the general result, due to Donkin [Don80]. As a block is uniquely determined, and in fact generated by, the simple modules it contains, we can identify blocks with subsets of X^+. The subsets of X^+ corresponding to blocks are of the following form [Jan03, II.7.2]. Given $\lambda \in X^+$, let

$$r \overset{\mathsf{def}}{=} \min_{\alpha \in R} \mathrm{ord}_p \langle \lambda + \rho, \alpha^\vee \rangle$$

and make the subset $W_{\mathrm{aff}} \bullet_{p^r} \lambda \cap X^+$. (Since $\langle \lambda + \rho, \alpha^\vee \rangle > 0$, the constant r is finite.)

[1] The assumption of finite-dimensionality is not essential here.

4. Translation functors

As before, assume that $p \overset{\text{def}}{=} \operatorname{char} k > 0$. The decomposition (1) reduces the study of $\operatorname{Rep}(G)$ to the study of the finitely-many categories $\operatorname{Rep}_\gamma(G)$, which, however, may be quite complicated. In this section we will see that these categories are related to each other by so-called translation functors.

The decomposition (1) gives rise to functors

$$\operatorname{pr}_\lambda\colon \operatorname{Rep}(G) \to \operatorname{Rep}_\lambda(G) \overset{\text{def}}{=} \operatorname{Rep}_\gamma(G),$$

where $\lambda \in X^+$ and $\gamma \overset{\text{def}}{=} W_{\text{aff}} \bullet_p \lambda$. Namely, we define $\operatorname{pr}_\lambda V$ to be the sum of the submodules of V all of whose composition factors have highest weight in γ.

Definition 8. Let $\lambda, \mu \in \overline{C}_{\mathbb{Z}}$ and let $X^+ \cap W(\mu - \lambda) = \{\nu\}$. Define the translation functor T_λ^μ from λ to μ as

$$T_\lambda^\mu V \overset{\text{def}}{=} \operatorname{pr}_\mu(L(\nu) \otimes \operatorname{pr}_\lambda V).$$

In many cases, translation functors are equivalences of categories.

Theorem 9. *If $\lambda, \mu \in \overline{C}_{\mathbb{Z}}$ belong to the same facet then $T_\lambda^\mu\colon \operatorname{Rep}_\lambda(G) \to \operatorname{Rep}_\mu(G)$ is an equivalence of categories.*

Although weights in different facets need not yield isomorphic categories, we can compare their categories if one facet is in the closure of another: translation functors propagate information from facets of larger dimension.

Proposition 10. *Let $\lambda, \mu \in \overline{C}_{\mathbb{Z}}$ and let F be the facet of $(W_{\text{aff}}, \bullet_p)$ containing λ. Say $\mu \in \overline{F}$.*

(1) For all $w \in W_{\text{aff}}$ and $i \in \mathbb{N}$,

$$T_\lambda^\mu\big(\mathrm{H}^i(w \bullet_p \lambda)\big) \simeq \mathrm{H}^i(w \bullet_p \mu).$$

(2) For all $w \in W_{\text{aff}}$ such that $w \bullet_p \lambda \in X^+$,

$$T_\lambda^\mu L(w \bullet_p \lambda) = \begin{cases} L(w \bullet_p \mu) & \text{if } w \bullet \mu \in \widehat{F'} \text{ (where } F' \overset{\text{def}}{=} w \bullet F) \\ 0 & \text{if not.} \end{cases}$$

Finally, we finish with a discussion of characters. Recall that the Euler characteristics

$$\chi(\lambda) \overset{\text{def}}{=} \sum_i (-1)^i \operatorname{ch} \mathrm{H}^i(\lambda)$$

with $\lambda \in X^+$ form a basis for the space $\mathbb{Z}[X]^W$ in which formal characters live. In particular, every formal character is a linear combination of such $\chi(\lambda)$.

Proposition 11. *Let $\lambda, \mu \in \overline{C}_{\mathbb{Z}}$ and let $w \in W_{\text{aff}}$ such that $w \bullet_p \lambda \in X^+$ and $w \bullet_p \mu$ is in the upper closure of the facet containing $w \bullet_p \lambda$. If*

$$\operatorname{ch} L(w \bullet_p \lambda) = \sum_{w' \in W_{\text{aff}}} a_{w,w'} \chi(w' \bullet_p \lambda)$$

then

$$\operatorname{ch} L(w \bullet_p \mu) = \sum_{w' \in W_{\text{aff}}} a_{w,w'} \chi(w' \bullet_p \mu).$$

Remark 12. We call any subcategory $\mathrm{Rep}_\lambda(G)$ with λ in a facet F of maximal dimension a principal block. The results above reduce some problems to the principal block. This strategy is not entirely successful, however, both because the upper closure of F is smaller than its topological closure, and because if p is very small then $F \cap X$ can sometimes be empty, as we saw in Example 1.

5. Proof of linkage principle

In this section we'll prove the linkage principle in the special case where the derived subgroup of G is simply connected[2] and $X/\mathbb{Z}R$ has no p-torsion, following Riche [Ric, §2.4]. The proof consists of analyzing separately two kinds of central characters, infinitesimal and global.

The rough idea is quite simple. If $L(\lambda)$ and $L(\mu)$ had different central characters then any extension of one by the other would split. But since $\mathrm{Ext}^1(L(\lambda), L(\mu)) \neq 0$, the central characters must agree. This agreement forces λ and μ to be linked.

Start with the infinitesimal character. Let $U(-)$ denote the universal enveloping algebra. The decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{t} \oplus \mathfrak{b}^+$ together with the Poincaré-Birkhoff-Witt theorem gives a linear projection $\phi\colon U(\mathfrak{g}) \to U(\mathfrak{t})$ with kernel $\mathfrak{b}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{b}^+$. The restriction of ϕ to $U(\mathfrak{g})^G$ is an isomorphism

$$U(\mathfrak{g})^G \to U(\mathfrak{t})^{(W,\bullet)}$$

called the Harish-Chandra isomorphism.[3] Here the superscript $(W, \bullet)$ denotes the dot-action invariants of the Weyl group. Every G-module V inherits, by differentiation, the structure of a $U(\mathfrak{g})$-module, and the actions are compatible in the sense that

$$\pi(g)\, \mathrm{d}\pi(X)(v) = \pi\big(\mathrm{ad}(g)(X)\big)(v)$$

for all $g \in G$, $X \in \mathfrak{g}$, and $v \in V$, where $\pi\colon G \to \mathrm{GL}(V)$ denotes the G-action and $\mathrm{d}\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ its differential. It follows that the restriction of $\mathrm{d}\pi$ to $U(\mathfrak{g})^G$ maps to the G-equivariant endomorphisms of V. In particular, if V is simple then this restriction is a character of $U(\mathfrak{g})^G$, which by the Harish-Chandra isomorphism can be identified with a character of $U(\mathfrak{t})^{(W,\bullet)}$. We call this restriction the infinitesimal central character of V. A character of $U(\mathfrak{t})^{(W,\bullet)}$ is just a point of the quotient $\mathfrak{t}^*/(W, \bullet)$, which we can identify with $(X \otimes k)/(W, \bullet)$ via the differential map. When $V = L(\lambda)$, it should not come as a surprise that the infinitesimal central character is the class of (the differential of) λ. It follows that λ and μ have the same image in $(X \otimes k)/(W, \bullet)$. In other words, there is $w \in W$ such that

$$\lambda - w \bullet \mu \in pX.$$

The global central character is simpler: restrict $L(\lambda)$ to $Z(G)$. The resulting character is an element of the dual group of $Z(G)$, namely $X/\mathbb{Z}R$. Since λ and μ agree in this group,

$$\lambda - \mu \in \mathbb{Z}R.$$

We can now complete the proof. Since $\mu - w \bullet \mu \in \mathbb{Z}R$ for any $\mu \in X$ and $w \in W$,

$$\lambda - w \bullet \mu \in \mathbb{Z}R \cap pX$$

[2] The case of general G can probably be reduced to the simply-connected case, so the second hypothesis is the essential one.

[3] It seems that this map is an isomorphism only when the derived subgroup of G is simply connected; I don't understand why this assumption is needed, however. [?]

for some $w \in W$. But since $X/\mathbb{Z}R$ has no p-torsion, $\mathbb{Z}R \cap pX = p\mathbb{Z}R$. Hence $\lambda = w \bullet_p \mu$ for some $w \in W_{\mathrm{aff}}$. $\square$

Remark 13. In the classical statement of the Harish-Chandra isomorphism, the algebra $U(\mathfrak{g})^G$ is replaced by the center $Z(U(\mathfrak{g}))$. In positive characteristic, however, the center is too large. Here $\mathfrak{g}$ has an additional structure of a restricted Lie algebra: an operation $x \mapsto x^{[p]}$ satisfying certain axioms, but which can be defined as the usual pth power in a fixed linear representation of $\mathfrak{g}$. It turns out that for all $x \in \mathfrak{g}$,

$$\xi(x) \overset{\mathsf{def}}{=} x^p - x^{[p]} \in Z(U(\mathfrak{g})),$$

and that furthermore, the image under $\xi\colon \mathfrak{g} \to Z(U(\mathfrak{g}))$ of a linearly independent set is algebraically independent [Jan98, 2.3]. Hence $Z(U(\mathfrak{g}))$ contains at least $\dim(G)$ algebraically independent elements, so it is much larger than $U(\mathfrak{t})^{(W,\bullet)}$.

References

[Don80] Stephen Donkin, *The blocks of a semisimple algebraic group*, J. Algebra **67** (1980), no. 1, 36–53. MR 595017

[Jan98] Jens Carsten Jantzen, *Representations of Lie algebras in prime characteristic*, Representation theories and algebraic geometry (Montreal, PQ, 1997), NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., vol. 514, Kluwer Acad. Publ., Dordrecht, 1998, Notes by Iain Gordon, pp. 185–235. MR 1649627

[Jan03] ———, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003. MR 2015057

[Ric] Simon Riche, *Lectures on modular representation theory of reductive algebraic groups.*