On Derived Equivalences of Triangles, Rectangles and Lines

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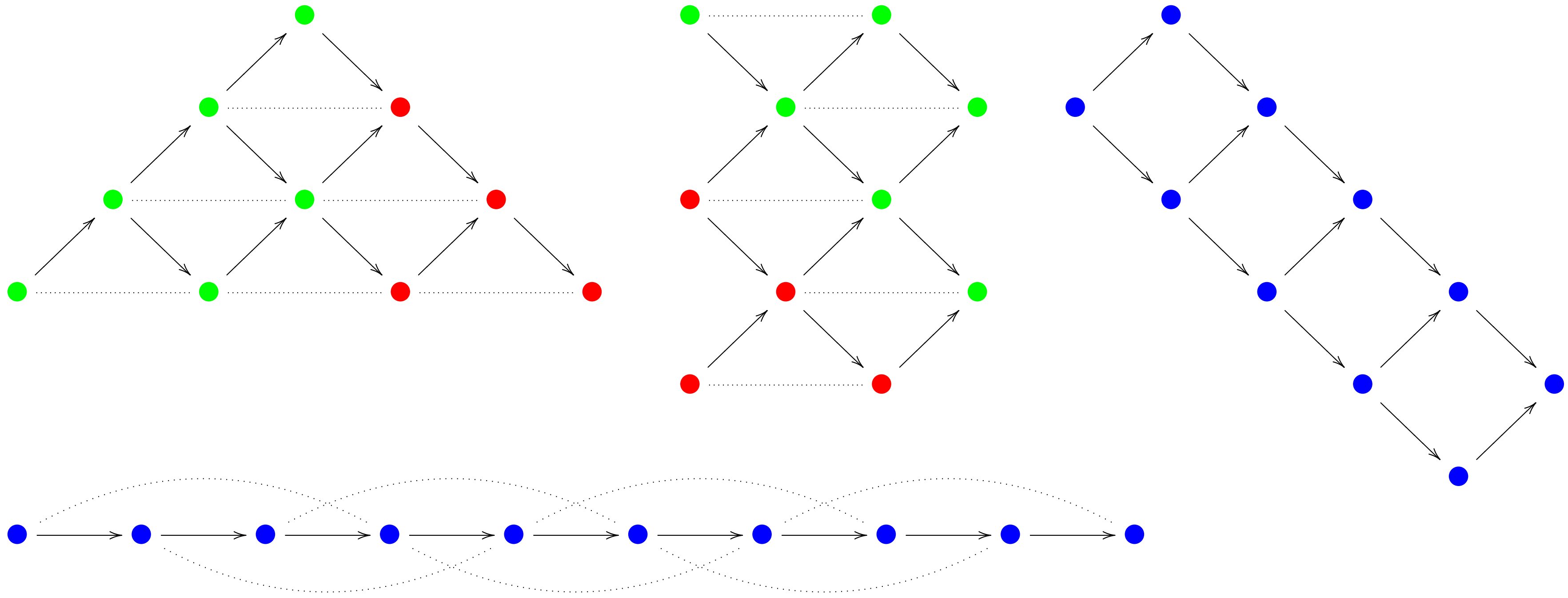
What is the connection between . . .



$A_1, A_2, A_3, D_4, D_5, E_6, E_7, E_8$

Context

- *Test algebras* [Lenzing - de la Peña 2008]
- Structured equivalence of *Euler forms* as an indicator of derived equivalence.
- Categories of singularities; *weighted projective lines* [Lenzing et al.]
- *Auslander algebras* and *initial modules* [Geiss-Leclerc-Schröer]
- *Cluster algebra* structures on . . .
  - Upper-triangular *unipotent matrices* [Geiss-Leclerc-Schröer]
  - *Grassmannians* [Scott 2006]

Lines

k – field, $\overrightarrow{A_n}$ – the quiver

$$\bullet_1 \xrightarrow{x} \bullet_2 \xrightarrow{x} \bullet_3 \xrightarrow{x} \cdots \xrightarrow{x} \bullet_n$$

The *path algebra* $k\overrightarrow{A_n}$ is the *incidence algebra* of the linear order on $\{1, 2, \dots, n\}$.

For $r \geq 2$, consider $A(n,r) = k\overrightarrow{A_n}/(x^r)$ –
the path algebra modulo the ideal generated by all the relations x^r.

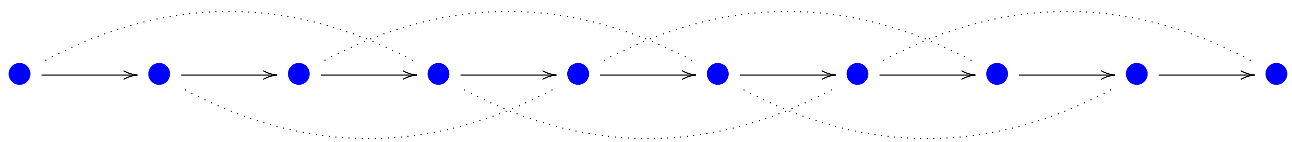
- $A(n,r)$ is of finite representation type,
- We are interested in its *derived equivalence* class, following [Lenzing - de la Peña, 2008].

The algebras $A(n,r)$

- $A(n,2) \sim k\overrightarrow{A_n}$.
- The derived equivalence class of $A(n,3)$ for $1 \leq n \leq 11$:

$$A_1, A_2, A_3, D_4, D_5, E_6, E_7, E_8, C(2,3,5), C(2,3,6), C(2,3,7)$$

where $C(2,p,q)$ is the *canonical algebra* of weight type $(2,p,q)$ [Lenzing - de la Peña 2008].

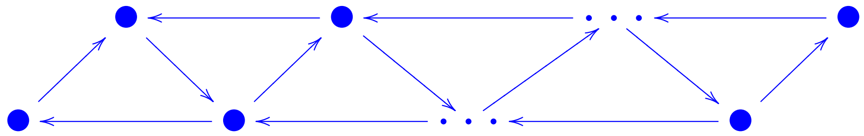


$\sim \quad C(2,3,6)$

- Characterization of the pairs (n,r) for which $A(n,r)$ is *piecewise hereditary* [Happel - U. Seidel].

The ADE Chain: $A_1, A_2, A_3, D_4, D_5, E_6, E_7, E_8$

- The *cluster type* of . . .
  - the quiver

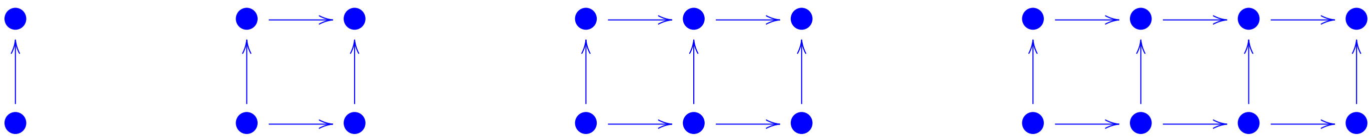


with n vertices [Barot-Geiss-Zelevinsky 2006].

  - the coordinate rings of the *Grassmannians* [Scott 2006]

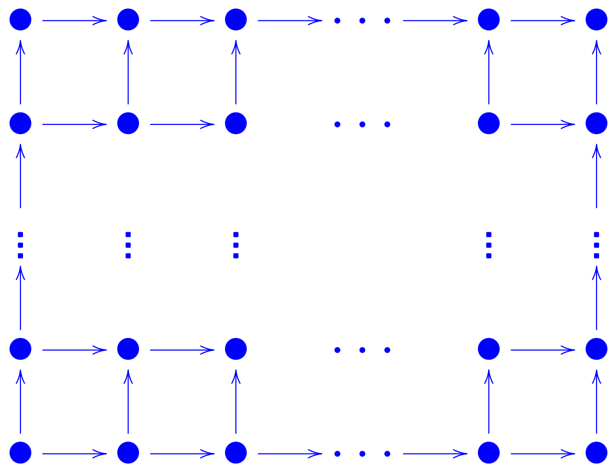
$$\mathrm{Gr}_{3,5}, \mathrm{Gr}_{3,6}, \mathrm{Gr}_{3,7}, \mathrm{Gr}_{3,8}$$

- The derived equivalence class of



Rectangles

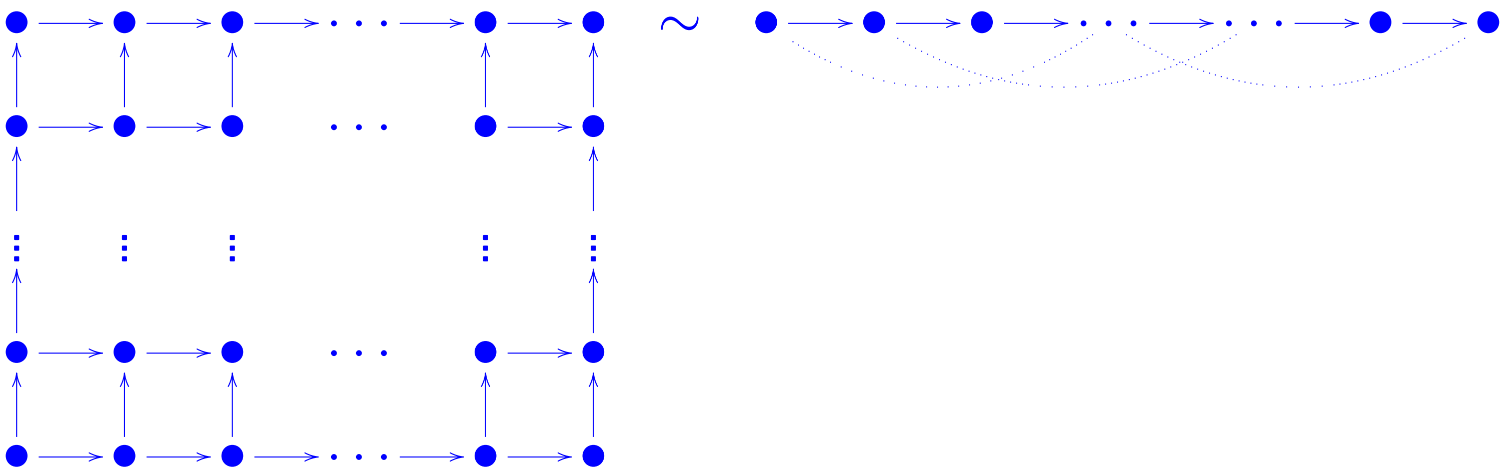
$n, m \geq 1$. Consider the incidence algebra of $\overrightarrow{A_n} \times \overrightarrow{A_m}$,



- Fully commutative quiver.
- Global dimension 2 (when $m, n \geq 2$).
- Periodic *Coxeter transformation*.

Derived equivalence of rectangles and lines

Theorem 1. $k(\overrightarrow{A_n} \times \overrightarrow{A_m}) \sim A(m \cdot n, m+1)$.



Generalizes $A(n,2) \sim k\overrightarrow{A_n}$ and $A(2n,3) \sim k(\overrightarrow{A_n} \times \overrightarrow{A_2})$, hence can be viewed as *higher ADE chains*.

Invariants of derived equivalence

Derived equivalent algebras (with finite global dimension)

$\Downarrow$

Equivalent Euler forms

with respect to bases of indecomposable projectives: *Cartan matrices*

$\Downarrow$

Similar Coxeter transformations

$\Downarrow$

Same Coxeter polynomial

Examining the Cartan matrices

$$\overrightarrow{A_3} \times \overrightarrow{A_4} \qquad\qquad A(12,4)$$

$$\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&0&1&1&0&1&1&0&1&1\\
0&0&1&0&0&1&0&0&1&0&0&1\\
0&0&0&1&1&1&1&1&1&1&1&1\\
0&0&0&0&1&1&0&1&1&0&1&1\\
0&0&0&0&0&1&0&0&1&0&0&1\\
0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&0&1&1&0&1&1\\
0&0&0&0&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\qquad
\begin{pmatrix}
1&1&1&1&0&0&0&0&0&0&0&0\\
0&1&1&1&1&0&0&0&0&0&0&0\\
0&0&1&1&1&1&0&0&0&0&0&0\\
0&0&0&1&1&1&1&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0\\
0&0&0&0&0&1&1&1&1&0&0&0\\
0&0&0&0&0&0&1&1&1&1&0&0\\
0&0&0&0&0&0&0&1&1&1&1&0\\
0&0&0&0&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}$$

$$A = \begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix} \qquad A^T = \begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}$$

A statement on matrices . . .

Proposition. Let A be a square invertible matrix over a commutative ring K. Then the bilinear forms represented by the matrices

$$C = \begin{pmatrix} A & A & \dots & A & A \\ 0 & A & A & \ddots & A \\ \vdots & 0 & A & \ddots & \vdots \\ \vdots & & \ddots & \ddots & A \\ 0 & \dots & \dots & 0 & A \end{pmatrix} \quad and \quad \begin{pmatrix} A & A^T & 0 & \dots & 0 \\ 0 & A & A^T & \ddots & \vdots \\ \vdots & 0 & A & \ddots & 0 \\ \vdots & & \ddots & \ddots & A^T \\ 0 & \dots & \dots & 0 & A \end{pmatrix} = C'$$

are *equivalent* over K.

Proof. Find P such that $P^T C P = C'$.

There exists such P whose blocks are 0 or powers of $S = -A^{-1}A^T$.

. . . interpreted as derived equivalence

Λ – finite-dimensional algebra over k with $\mathrm{gl.dim}\,\Lambda < \infty$.

$D\Lambda = \mathrm{Hom}_k(\Lambda, k)$, with multiplication maps

$$\Lambda \otimes_\Lambda D\Lambda \to D\Lambda, \qquad D\Lambda \otimes_\Lambda \Lambda \to D\Lambda, \qquad D\Lambda \otimes D\Lambda \to 0.$$

Theorem 2.

$$\Lambda \otimes_k k\overrightarrow{A_n} = \begin{pmatrix} \Lambda & \Lambda & \dots & \Lambda & \Lambda \\ 0 & \Lambda & \Lambda & \ddots & \Lambda \\ \vdots & 0 & \Lambda & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \Lambda \\ 0 & \dots & \dots & 0 & \Lambda \end{pmatrix} \sim \begin{pmatrix} \Lambda & D\Lambda & 0 & \dots & 0 \\ 0 & \Lambda & D\Lambda & \ddots & \vdots \\ \vdots & 0 & \Lambda & \ddots & 0 \\ \vdots & & \ddots & \ddots & D\Lambda \\ 0 & \dots & \dots & 0 & \Lambda \end{pmatrix} = \Gamma$$

Corollary. Taking $\Lambda = k\overrightarrow{A_m}$ we get Theorem 1.

A tilting complex

$\Lambda \otimes_k k\overrightarrow{A_n}$ module: $M_1 \to M_2 \to \cdots \to M_n$ $M_i \in \operatorname{mod} \Lambda$

$T = T_0 \oplus T_1 \oplus \cdots \oplus T_{n-1}$ is a *tilting complex* with $\operatorname{End}_\Lambda T \simeq \Gamma$, where

$$T_0 : \quad \Lambda \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0$$

$$T_1 : \quad 0 \longrightarrow F\Lambda \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0$$

$$\vdots$$

$$T_{n-1} : \quad 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow F^{n-1}\Lambda$$

for $F = \nu[1]$, $\nu = - \overset{\mathbf{L}}{\otimes}_\Lambda D\Lambda$.

There are generalized versions for certain other auto-equivalences F.

Relevance

- Stable category of vector bundles on *weighted projective lines* [Kussin-Lenzing-Meltzer-de la Peña]

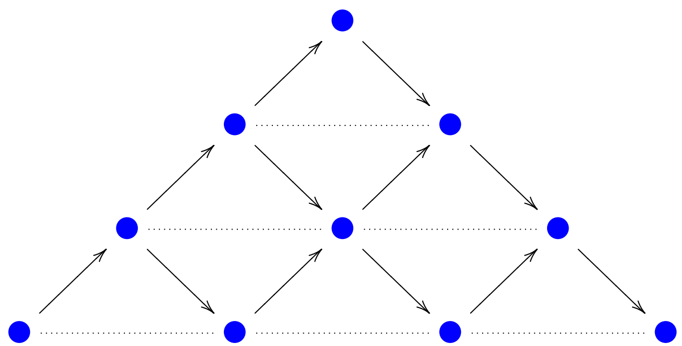
$$\underline{\mathrm{vect}}\,\mathbb{X}_{2,3,p} \simeq \mathcal{D}^b\,(A(2(p-1),3))$$

- Categories of (graded) singularities [loc. cit.]

$$x^2 + y^3 + z^p$$

- The cluster algebra structure on the coordinate ring of the *Grassmannian* $\mathrm{Gr}_{m+1,n+m+2}$ is related to $A_n \times A_m$ [Scott 2006].

Triangles



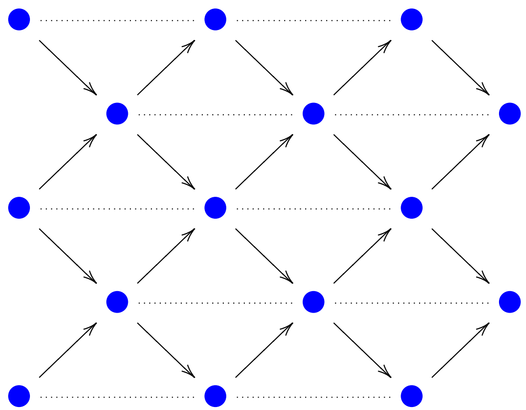
The *Auslander algebra* of $\overrightarrow{A_n}$ with linear orientation,

$$\text{Auslander}(k\overrightarrow{A_n}) = \text{End}_{k\overrightarrow{A_n}}\Big(\bigoplus_M M\Big)$$

where M runs over all indecomposable $k\overrightarrow{A_n}$-modules.

Variants

The Auslander algebra of $\overleftrightarrow{A_{2n+1}}$ with *bipartite* orientation:



More generally, consider the *initial modules* [Geiss-Leclerc-Schröer]

$$kQ \oplus \tau^{-1}kQ \oplus \cdots \oplus \tau^{-r}kQ$$

for an acyclic quiver Q, where τ is the *Auslander-Reiten* translation.

Auslander algebras and rectangles

Theorem 3. Let Q be an acyclic quiver such that

$$\tau^{-1}kQ, \tau^{-2}kQ, \ldots, \tau^{-r}kQ$$

are kQ-modules. Then

$$\mathrm{End}_{kQ}\Big(\bigoplus_{i=0}^{r} \tau^{-i}kQ\Big) \sim kQ \otimes_k k\overrightarrow{A_{r+1}}$$

Corollary. Auslander$(k\overleftrightarrow{A_{2n+1}}) \sim k(A_{2n+1} \times A_{n+1})$.

Strategy of proof

- Examine the *Euler forms* (this time, with respect to the basis of simples)

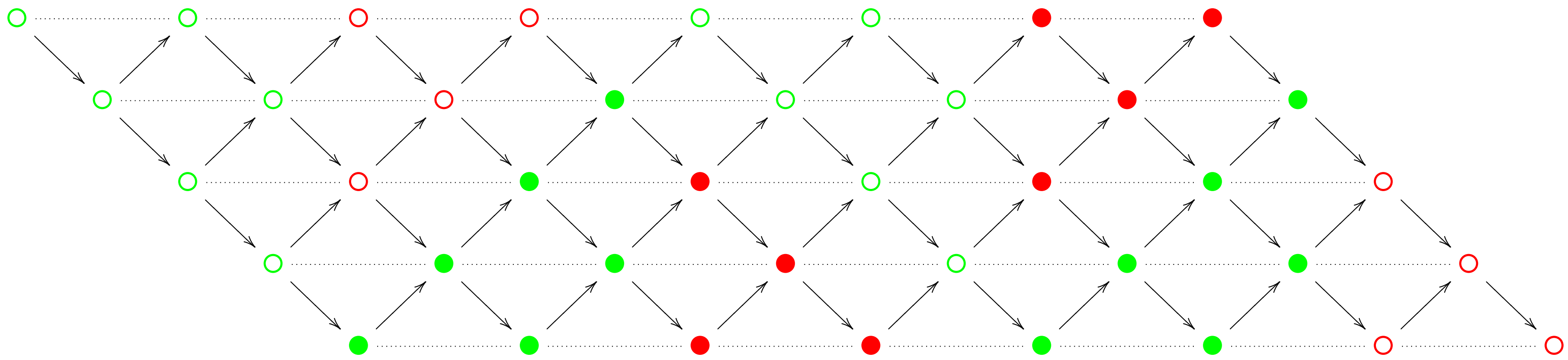
$$C = \begin{pmatrix} A & -A & 0 & \dots & 0 \\ 0 & A & -A & \ddots & \vdots \\ \vdots & 0 & A & \ddots & 0 \\ \vdots & & \ddots & \ddots & -A \\ 0 & \dots & \dots & 0 & A \end{pmatrix} \qquad \begin{pmatrix} A & A^T & 0 & \dots & 0 \\ 0 & A & A^T & \ddots & \vdots \\ \vdots & 0 & A & \ddots & 0 \\ \vdots & & \ddots & \ddots & A^T \\ 0 & \dots & \dots & 0 & A \end{pmatrix} = C'$$

- Observe structured equivalence $C' = P^T C P$ with

$$P = \mathrm{diag}(I, S, S^2, \dots, S^r), \qquad S = -A^{-1}A^T$$

- Construct appropriate tilting complex.
- Generalized version.

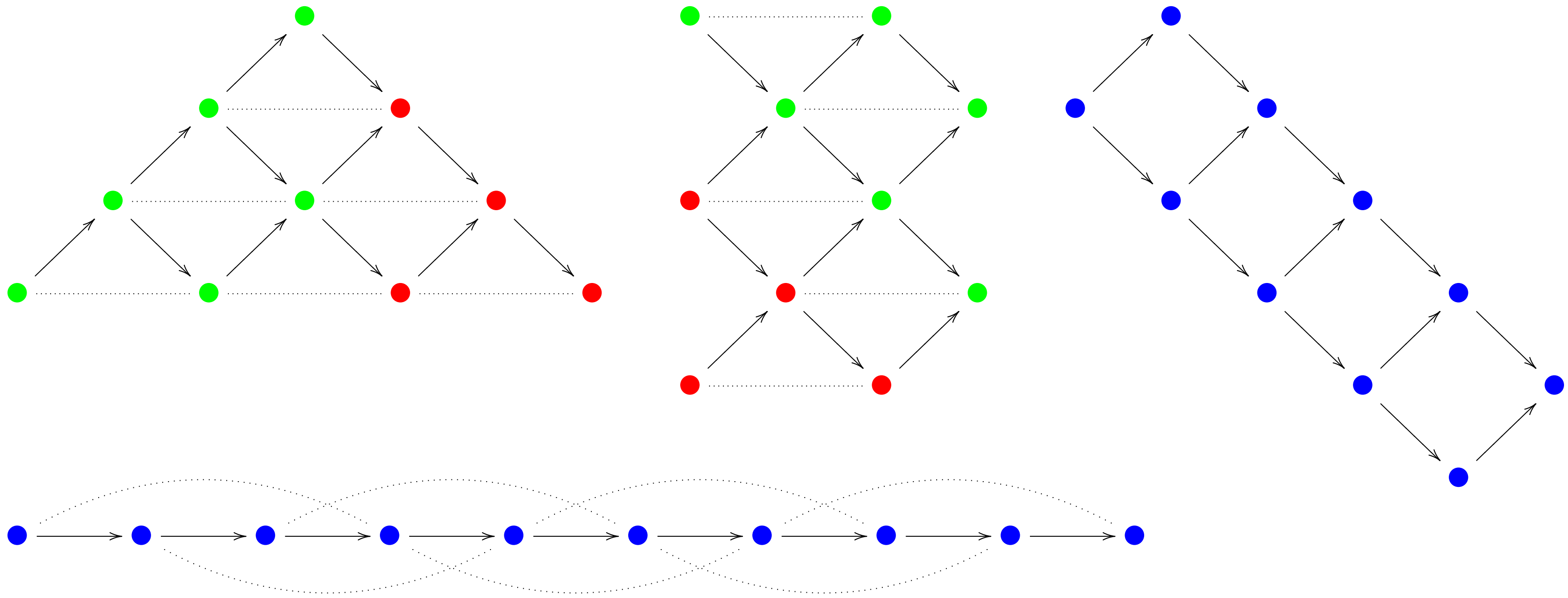
Back to triangles through repetitive algebras



Corollary.

$$\text{Auslander}(\overrightarrow{A_{2n}}) \sim \text{End}_{k\overleftrightarrow{A_{2n+1}}}\left(\bigoplus_{i=0}^{n-1} \tau^{-i} k\overleftrightarrow{A_{2n+1}}\right) \sim k(A_{2n+1} \times A_n)$$

. . . All these algebras are derived equivalent



$$\text{Auslander}(\overrightarrow{A_4}) \sim \text{End}_{k\overleftrightarrow{A_5}}\left(k\overleftrightarrow{A_5} \oplus \tau^{-1}k\overleftrightarrow{A_5}\right) \sim k(A_2 \times A_5) \sim A(10,3)$$