On Jacobian algebras from closed surfaces

Sefi Ladkani

University of Bonn

http://www.math.uni-bonn.de/people/sefil/

Motivation – QP from triangulations

Labardini associated quivers with potentials (QP) to ideal triangulations of surfaces with marked points, linking:

- cluster algebras from surfaces [Fomin-Shapiro-Thurston]
- theory of quivers with potentials [Derksen-Weyman-Zelevinsky]

For surfaces with *non-empty* boundary, the QP are *rigid* and the Jacobian algebras are *finite-dimensional* [Labardini].

Question. What happens for *empty* boundary (i.e. *closed* surfaces)?

Known cases:

- Torus with one puncture [Labardini]
- Spheres [Barot-Geiss, Trepode-Valdivieso-Diaz]

Motivation – derived equivalences

Problem. Find the mutation classes of QP such that *all* their Jacobian algebras are *derived equivalent*.

Non-example: acyclic quivers with more than 2 vertices.

Example: 3-Calabi-Yau [Keller-Yang] (infinite-dimensional).

More instances [L.]:

- Unpunctured surfaces with exactly one marked point on each boundary component (finite-dimensional).
- Once-punctured closed surfaces with "non-standard" potentials (infinite-dimensional, locally gentle).

Questions. What happens for the standard potentials? more punctures?

Results – Jacobian algebras

(S, M) – surface with marked points and empty boundary.

Theorem [L.]

- If (S, M) is not a sphere with 4 punctures, then the QP associated to any ideal triangulation of (S, M) is *not rigid* and its (completed) Jacobian algebra is *finite-dimensional* and *symmetric*.
- If (S, M) is a sphere with 4 punctures, then the same holds when the product of the scalars defining the potential is not 1.

Corollary. There is a Hom-finite triangulated 2-Calabi-Yau category $\mathcal{C}_{(S,M)}$ with a cluster-tilting object for each ideal triangulation.

Results – derived equivalences

$\mathcal{P}(Q,W)$ – the Jacobian algebra of a QP (Q,W).

Proposition [L.]. If $\mathcal{P}(Q,W)$ is *(weakly) symmetric*, then $\mathcal{P}(\mu_k(Q,W))$ is (weakly) symmetric [Herschend-Iyama] and *derived equivalent* to $\mathcal{P}(Q,W)$.

Corollary 1. *All* the Jacobian algebras associated to the triangulations of a closed surface are derived equivalent.

Corollary 2. Let A be $\frac{2n-2}{n}$-*CY* with $\mathrm{gl.dim}\, A \leq 2$. Write $T_A(\mathrm{Ext}^2_A(DA,A)) = \mathcal{P}(Q,W)$. If (Q,W) is non-degenerate, then all the Jacobian algebras in its mutation class are *symmetric* and *derived equivalent*.

Example. $A = KD_4 \otimes KD_4$ is $\frac{4}{3}$-CY $\Longrightarrow$
infinite mutation class of finite-dimensional, symmetric, derived equivalent Jacobian algebras.

Combinatorial model for the quivers

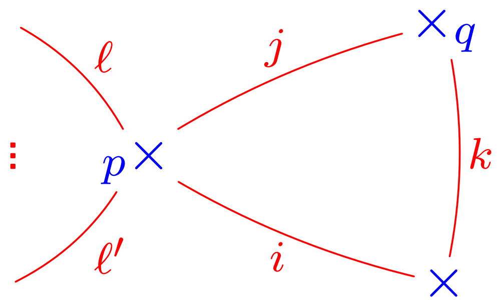
T – a fixed triangulation of (S, M) such that:
there are at least 3 *arcs of* T *incident to each puncture.*

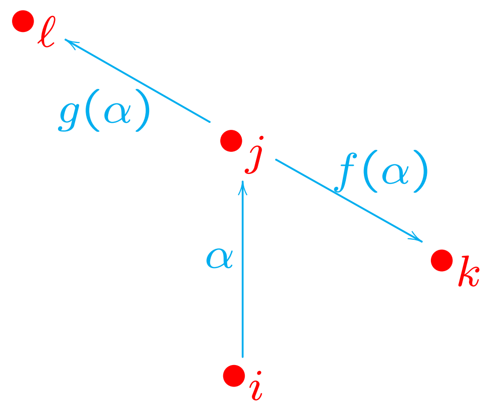
Proposition [L.]. Let (Q, W) be the QP associated to T. Then:

- Q is connected without any loops or 2-cycles.
- For any $i \in Q_0$, there are exactly two arrows in Q_1 starting at i and two arrows ending at i.
- There are invertible maps $f, g : Q_1 \to Q_1$ with the following properties:
  - For any $\alpha \in Q_1$, the set $\{f(\alpha), g(\alpha)\}$ consists of the two arrows that start at the vertex which α ends at;
  - f^3 is the identity on Q_1.

$\mathrm{PSL}_2(\mathbb{Z})$-action on the set of arrows

The definition of the maps f and g:





For an arrow $\alpha \in Q_1$, denote by $\bar{\alpha}$ the other arrow starting at the same vertex as α. In particular, $\overline{g(\alpha)} = f(\alpha)$.

Proposition [L.] $\mathrm{PSL}_2(\mathbb{Z})$ acts transitively on Q_1.

Remark. Non-trivial path in Q = arrow + word in f, g.

The potentials

Two kinds of cycles in Q: *f-cycles* and *g-cycles*.

f-cycles are 3-cycles corresponding to the triangles of T, g-cycles arise from traversing arcs around a puncture.

Proposition [L.] The potential W is given by

$$W = \sum \alpha \cdot f(\alpha) \cdot f^2(\alpha) - \sum c_\beta \beta \cdot g(\beta) \cdot \ldots \cdot g^{n_\beta - 1}(\beta)$$

where $c : Q_1 \to K^\times$ is g-invariant.

Finite-dimensionality

Let $\Lambda = \mathcal{P}(Q, W)$.

By computing cyclic derivatives of the potential we get:

Lemma. For any $\beta \in Q_1$,

$$\beta \cdot f(\beta) = c_{\bar{\beta}} \bar{\beta} \cdot g(\bar{\beta}) \cdot \ldots \cdot g^{n_{\bar{\beta}}-2}(\bar{\beta}).$$

Lemma. For any $\alpha \in Q_1$,

$$\begin{aligned} \alpha \cdot f(\alpha) \cdot f^2(\alpha) &= c_\alpha \alpha \cdot g(\alpha) \cdot g^2(\alpha) \cdot \ldots \cdot g^{n_\alpha - 1}(\alpha) \\ &= c_{\bar{\alpha}} \bar{\alpha} \cdot g(\bar{\alpha}) \cdot g^2(\bar{\alpha}) \cdot \ldots \cdot g^{n_{\bar{\alpha}}-1}(\bar{\alpha}) \\ &= \bar{\alpha} \cdot f(\bar{\alpha}) \cdot f^2(\bar{\alpha}) \end{aligned}$$

$$\alpha \cdot g(\alpha) \cdot fg(\alpha) = c_{f(\alpha)} \alpha \cdot f(\alpha) \cdot gf(\alpha) \cdot g^2 f(\alpha) \cdot \ldots \cdot g^{n_{f(\alpha)}-2} f(\alpha)$$

$$\alpha \cdot f(\alpha) \cdot gf(\alpha) = c_{\bar{\alpha}} \bar{\alpha} \cdot g(\bar{\alpha}) \cdot \ldots \cdot g^{n_{\bar{\alpha}}-3}(\bar{\alpha}) \cdot g^{n_{\bar{\alpha}}-2}(\bar{\alpha}) \cdot fg^{n_{\bar{\alpha}}-2}(\bar{\alpha})$$

Finite-dimensionality (continued)

Assume further that:
any arc of T has an endpoint with ≥ 4 arcs incident to it

(such T always exists if $(S, M) \neq$ sphere with 4 punctures)

$\Longrightarrow$ Λ has a *finite basis* consisting of the paths

$$\{e_i\}_{i \in Q_0} \cup \{\alpha \cdot g(\alpha) \cdot \ldots \cdot g^r(\alpha)\}_{\alpha \in Q_1, 0 \leq r < n_\alpha - 1} \cup \{z_i\}_{i \in Q_0}$$

where z_i is a g-cycle starting at i.

Remark. T gives rise also to a *Brauer graph algebra* (via the data of a graph + cyclic order at nodes):

- Same quiver as Λ,
- Different defining relations, but same basis.

Symmetry, non-rigidity and more

- Λ is *symmetric*:

  The isomorphism $D\Lambda \simeq \Lambda$ as Λ-Λ-bimodules follows from the "duality"

$$\alpha \cdot g(\alpha) \cdot \ldots \cdot g^{r-1}(\alpha) \longleftrightarrow c_\alpha \, g^r(\alpha) \cdot \ldots \cdot g^{n_\alpha - 1}(\alpha)$$

  for $0 \leq r \leq n_\alpha$.

- (Q, W) is *not rigid*:

  The image of any cycle z_i in $\Lambda / [\Lambda, \Lambda]$ is not zero.

- Can compute the *Cartan matrix* of Λ, its *center*, . . .