WHICH MUTATION CLASSES OF QUIVERS HAVE CONSTANT NUMBER OF ARROWS?

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Abstract. We classify the connected quivers with the property that all the quivers in their mutation class have the same number of arrows. These are the ones having at most two vertices, or the ones arising from triangulations of marked bordered oriented surfaces of two kinds: either surfaces with non-empty boundary having exactly one marked point on each boundary component and no punctures, or surfaces without boundary having exactly one puncture.

This combinatorial property has also a representation-theoretic counterpart: to each such quiver there is a naturally associated potential such that the Jacobian algebras of all the QP in its mutation class are derived equivalent.

1. Motivation and Summary of Results

Quiver mutation is a combinatorial notion introduced by Fomin and Zelevinsky [15] in their theory of cluster algebras. Let us briefly recall its definition.

A *quiver* is a directed graph, where multiple arrows between two vertices are allowed. Throughout this paper, we consider only quivers without *loops* (arrows starting and ending at the same vertex) and 2-*cycles* (i.e. pairs of arrows $i \to j$ and $j \to i$). Let Q be such quiver and let k be a vertex of Q. The *quiver mutation* of Q at k is a new quiver $\mu_k(Q)$ obtained from Q by performing the following three steps:

(i) For any pair of arrows of the form $(i \to k, k \to j)$, add a new arrow $i \to j$;
(ii) Reverse the direction of all the arrows starting or terminating at k;
(iii) Remove a maximal set of 2-cycles.

When no arrow starts at the vertex k (it is then called a *sink*) or no arrow ends at k (it is then called a *source*), mutation at k reduces to step (ii) above and coincides with the *BGP reflection* considered by Bernstein, Gelfand and Ponomarev [6]. Obviously, in this case the quivers Q and $\mu_k(Q)$ have the same number of arrows.

However, mutation in general does not preserve the number of arrows in the quivers, as can be seen for example by mutating the left quiver with two arrows at the vertex 2 obtaining the right quiver with three arrows.

It is therefore interesting to search for quivers with the property that performing an arbitrary sequence of mutations does not change the number of arrows. To formulate

Date: April 4, 2011.
This work was supported by a European Postdoctoral Institute (EPDI) fellowship.

this more precisely, recall that the *mutation class* of a quiver Q consists of all the quivers that can be obtained from Q by performing sequences of mutations.

Question. Which mutation classes have the property that all their quivers have the same number of arrows?

1.1. **Combinatorial results.** It turns out that the answer to this question is closely related with the analysis of mutation classes of quivers arising from triangulations of marked bordered oriented surfaces as introduced by Fomin, Shapiro and Thurston [14]. Recall that these consist of pairs (S, M) where S is a compact connected oriented Riemann surface and $M \subset S$ is a finite set of *marked points* containing at least one point from each connected component of the boundary of S (which might be empty). The homeomorphism type of (S, M) is governed by the following discrete data:

- the *genus* $g \geq 0$ of the surface S;
- the number $b \geq 0$ of connected components of its boundary;
- the number of marked points on each boundary component;
- the number of marked points not on the boundary (called *punctures*).

The following mutation classes will play significant role in our considerations.

Definition. Let $g, b \geq 0$ such that $(g, b) \notin \{(0, 0), (0, 1)\}$.

(a) If $b = 0$, denote by $\mathcal{Q}_{g,0}$ the mutation class consisting of the quivers arising from triangulations of a surface without boundary of genus g with one puncture.
(b) If $b > 0$, denote by $\mathcal{Q}_{g,b}$ the mutation class consisting of the quivers arising from triangulations of a surface of genus g with b boundary components and exactly one marked point on each boundary component.

A procedure to produce explicit members from the classes $\mathcal{Q}_{g,b}$ when $b > 0$ has been described in our previous work [19]. With slight modifications, it can also be used to produce explicit members from the classes $\mathcal{Q}_{g,0}$, see Section 3.2. Examples of such quivers for small values of g and b are shown in Figure 1 and Figure 2.

The next theorem provides a complete answer to our question.

Theorem 1. *For a connected quiver Q, the following statements are equivalent.*

(i) *All the quivers in the mutation class of Q have the same number of arrows.*
(ii) *Q has at most two vertices, or $Q \in \mathcal{Q}_{g,b}$ for some $g, b \geq 0$ such that $(g, b) \notin \{(0, 0), (0, 1)\}$.*

Moreover, any quiver in $\mathcal{Q}_{g,b}$ for $b > 0$ has $6(g-1) + 4b$ vertices and $12(g-1) + 7b$ arrows, whereas any quiver in $\mathcal{Q}_{g,0}$ has $6g - 3$ vertices, $12g - 6$ arrows and any of its vertices has exactly two incoming and two outgoing arrows.

From the theorem we see that mutation classes consisting of connected quivers with constant number of arrows are quite rare. In fact, for any $n \geq 3$ the number of such classes with n vertices is finite. We also deduce the following.

Corollary. *Let $n > 1$. There exists a mutation class consisting of connected quivers with n vertices and constant number of arrows if and only if $n \not\equiv 1, 5 \pmod 6$.*

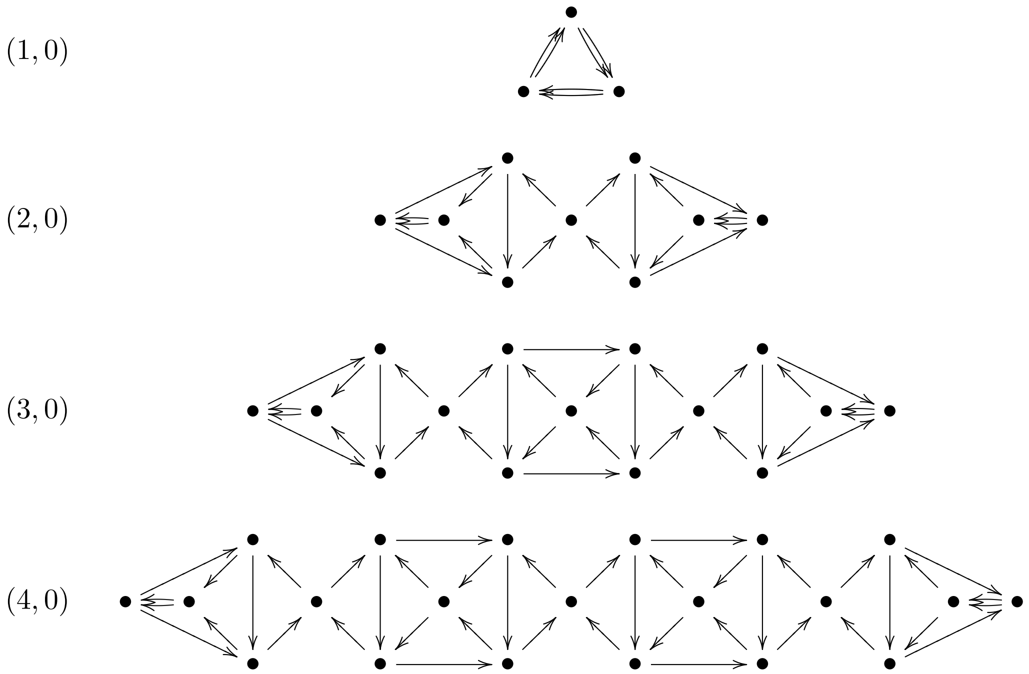


FIGURE 1. Representative quivers in $\mathcal{Q}_{g,0}$ for $g = 1, 2, 3, 4$.

1.2. **Outline of the proof.** A mutation class whose quivers have the same number of arrows must consist of a finite number of quivers. The proof of the implication (i) $\Rightarrow$ (ii) in Theorem 1 relies on the classification by Felikson, Shapiro and Tumarkin [13] of the connected quivers whose mutation classes are finite; either they arise from triangulations of marked bordered oriented surfaces, or they are mutation equivalent to one of 11 exceptional quivers, or they are acyclic with two vertices and at least three arrows between them.

Obviously, mutations of quivers with one or two vertices are just reflections preserving the number of arrows. Next, one checks that each of the 11 exceptional mutation classes contains two quivers with different numbers of arrows, see Section 2.1.

We are left to consider quivers arising from triangulations of marked bordered surfaces. We show in Section 2.2 that for such marked surface (S, M') admitting a triangulation, adding a puncture or, under mild condition, a marked point on its boundary, results in a marked surface (S, M) which has two triangulations giving rise to two quivers in the corresponding mutation class with different numbers of arrows. Therefore we need only to consider "minimal" marked bordered surfaces, and those that remain are precisely the ones giving rise to the mutation classes $\mathcal{Q}_{g,b}$ defined above.

To prove the implication (ii) $\Rightarrow$ (i) in Theorem 1 for these classes, we show in Section 3 that mutating any of their quivers at any vertex does not change the number of arrows. Note that for the classes $\mathcal{Q}_{g,b}$ with $b > 0$ the statement of the theorem has already been shown in [19, §2] by a counting argument yielding formulae for the numbers of arrows and 3-cycles in any quiver in $\mathcal{Q}_{g,b}$ which depend only on g and b.

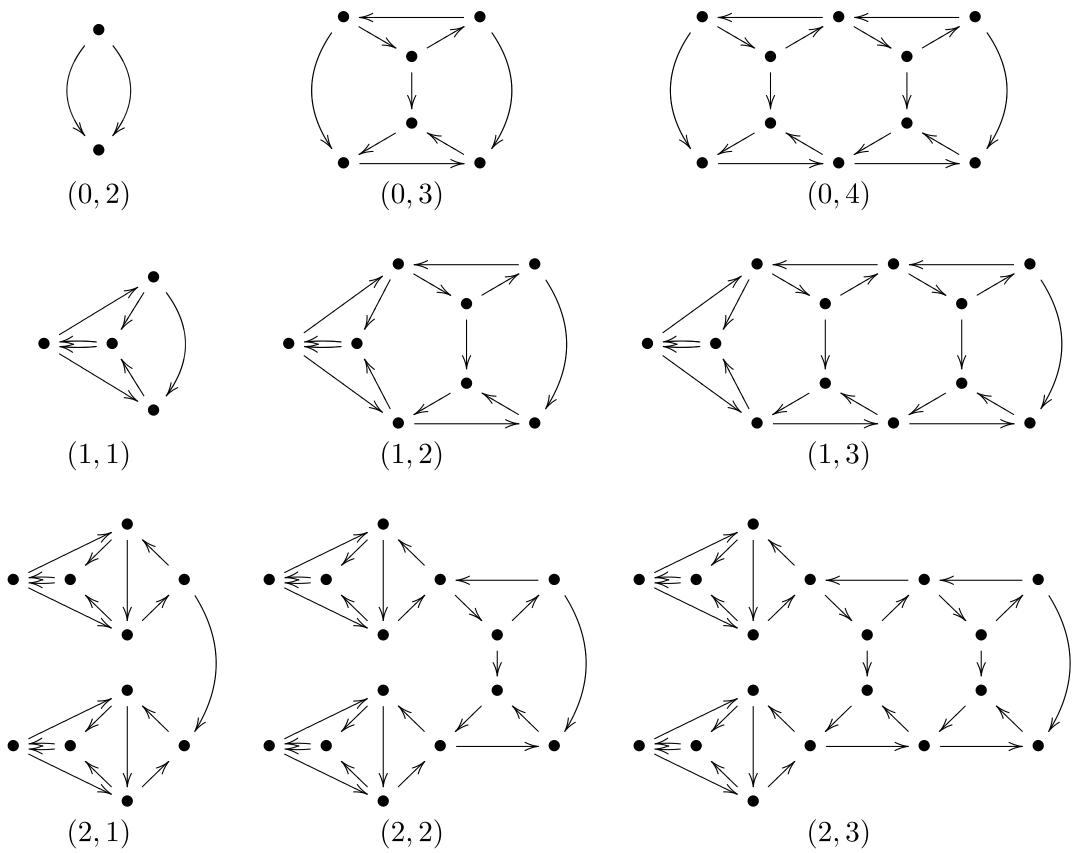


FIGURE 2. Representative quivers in each $\mathcal{Q}_{g,b}$ for small values of (g, b), $b > 0$.

In this paper we take a more general approach and characterize, for the family of quivers arising from triangulations of marked bordered surfaces without punctures, those mutations that preserve the number of arrows. This is done by analyzing the possible local "neighborhoods" at any vertex. The result for $\mathcal{Q}_{g,b}$ with $b > 0$ then follows as a special case, see Section 3.1.

Similar analysis for the remaining classes $\mathcal{Q}_{g,0}$ is done in Section 3.2, where we also explore their relations with mutation classes of quivers of Dynkin type A.

1.3. **Algebraic interpretation – derived equivalence.** By using the theory of quivers with potentials (QP) and their mutations developed by Derksen, Weyman and Zelevinsky [12] it is possible, in certain cases, to interpret mutations of QP preserving the number of arrows as derived equivalences of the corresponding Jacobian algebras. In particular, to any quiver in the mutation classes $\mathcal{Q}_{g,b}$ with $(g, b) \neq (0, 0), (0, 1)$ there is a naturally associated potential allowing each class $\mathcal{Q}_{g,b}$ to be regarded as a mutation classes of QP whose Jacobian algebras are all derived equivalent. This is elaborated in Section 4.

2. Mutation classes with varying number of arrows

In this section we prove the implication (i) $\Rightarrow$ (ii) in Theorem 1. We start with the following useful lemma.

Lemma 2.1. *Let Q be a quiver and let k be a vertex in Q with exactly one incoming arrow and exactly one outgoing arrow. Then the numbers of arrows in Q and in $\mu_k(Q)$ differ by one.*

Proof. Denote by $i \xrightarrow{\alpha} k$ the incoming arrow and by $k \xrightarrow{\beta} j$ the outgoing arrow. Apart from inverting the arrows α and β, the quiver mutation at k modifies only the arrows between i and j.

If Q has $a \geq 0$ arrows from i to j, its mutation $\mu_k(Q)$ has $a+1$ arrows from i to j, whereas if Q has $a \geq 1$ arrows from j to i, its mutation $\mu_k(Q)$ has $a-1$ arrows from j to i. In any case, the number of arrows changes by one. □

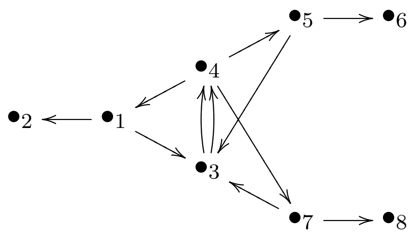
2.1. The exceptional quivers.

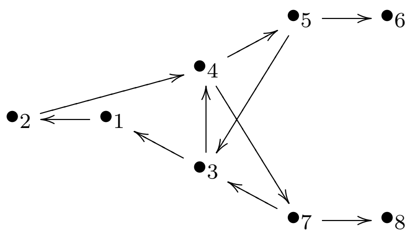
Lemma 2.2. *The mutation class of each of the* 11 *exceptional quivers contains two quivers with different numbers of arrows.*

Proof. In principle, since each mutation class is finite, the claim can be verified on a computer. For the convenience of the reader, we give a direct proof.

Since each of the 8 quivers E_6, E_7, E_8, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ contains a vertex with exactly one incoming arrow and exactly one outgoing arrow, by Lemma 2.1 we can mutate at this vertex and get a quiver with a different number of arrows.

The quiver $E_6^{(1,1)}$ is shown in the left picture below. Note that performing any single mutation on this quiver does not change the number of arrows. However, when mutating at the vertex 1 and then at 2, we get a quiver with one arrow less, as shown in the right picture.



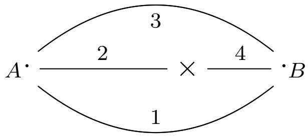


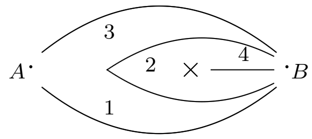
It remains to consider the quivers X_6 and X_7 introduced by Derksen and Owen [11] who also wrote down explicitly their mutation classes which consist of 5 and 2 quivers, respectively. We see that the mutation class of X_6 contains quivers with 9 and 11 arrows, whereas that of X_7 contains ones with 12 and 15 arrows. □

2.2. Quivers from marked surfaces. Consider now quivers arising from triangulations of bordered oriented surfaces with marked points. We refer the reader to [14] for the relevant definitions and constructions.

Lemma 2.3. *Let (S, M') be a marked surface which has a triangulation, and let M be obtained from M' by adding a puncture. Then (S, M) has two triangulations whose corresponding quivers have different numbers of arrows.*

Proof. Choose an arc γ of a triangulation T' of (S, M') and denote the marked points at its ends by A and B (which may coincide). We may place the new puncture $\times$ in M on γ and obtain a triangulation T of (S, M) by replacing γ with four arcs as depicted in the left picture below. By flipping the arc labeled 2, we obtain another triangulation of (S, M), part of which is depicted in the right picture.





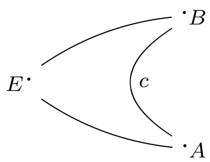
Consider the quiver Q corresponding to T and its mutation $\mu_2(Q)$ corresponding to the flip of T at the arc labeled 2. Since Q has precisely one arrow ending at 2, namely the arrow $1 \to 2$ induced from the triangle $\{1, 2, 4\}$ in T, and precisely one arrow starting at 2, namely $2 \to 3$ induced from the triangle $\{2, 3, 4\}$, by Lemma 2.1 the numbers of arrows in Q and in $\mu_2(Q)$ differ by one. $\square$

Lemma 2.4. *Let (S, M') be a marked surface with non-empty boundary which has a triangulation satisfying the following condition:*

(♠) *There is a triangle which is not self-folded such that exactly one of its sides is a boundary segment.*

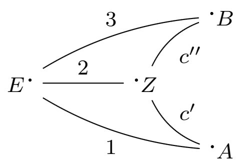
Denoting this segment by c, let M be obtained from M' by adding a marked point on c. Then (S, M) has two triangulations whose corresponding quivers have different numbers of arrows. Moreover, one of these triangulations satisfies the condition (♠).

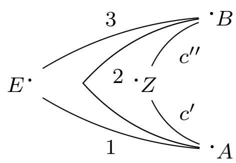
Proof. Let T' be a triangulation of (S, M') satisfying condition (♠). The triangle of T' from that condition looks like



where A and B denote the endpoints of the boundary segment c and E is the other vertex of the triangle (note that all these marked points might coincide).

Let $M = M' \cup \{Z\}$ where Z is a point on the segment c, splitting it into two segments c' and c''. We obtain from T' a triangulation T of (S, M) by adding the arc connecting E and Z as in the left picture below. The triangle consisting of the arcs labeled 1, 2 and the segment c' shows that T satisfies condition (♠). By flipping the arc labeled 2, we obtain another triangulation of (S, M), part of which is depicted in the right picture.





Consider the quiver Q corresponding to T and its mutation $\mu_2(Q)$ corresponding to the flip of T at the arc labeled 2. Since Q has precisely one arrow ending at 2, namely the arrow $1 \to 2$ induced from the triangle $\{1, 2, c'\}$ in T, and precisely one arrow starting at 2, namely $2 \to 3$ induced from the triangle $\{2, 3, c''\}$, by Lemma 2.1 the numbers of arrows in Q and in $\mu_2(Q)$ differ by one. □

Lemma 2.5. *Let (S, M) be a marked surface without boundary whose triangulations give rise to a mutation class consisting of quivers with the same number of arrows. Then (S, M) must be one of the following:*

(a) *A sphere with 3 punctures. The corresponding quiver is then a disjoint union of three vertices;*
(b) *A (closed) surface of genus $g > 0$ with one puncture. The corresponding mutation class is $\mathcal{Q}_{g,0}$.*

Proof. It follows from Lemma 2.3 that for any puncture P of M, the marked surface $(S, M \setminus \{P\})$ does not have any triangulation. Hence M must be minimal with respect to the property that (S, M) still has a triangulation. □

Lemma 2.6. *Let (S, M) be a marked surface with non-empty boundary whose triangulations give rise to a mutation class consisting of quivers with the same number of arrows. Then (S, M) must be one of the following:*

(a) *A disc with one puncture and at most two marked points on its boundary. The corresponding quivers are A_1 and A_2;*
(b) *A disc with no punctures and four or five marked points on its boundary. The corresponding quivers are A_1 and A_2;*
(c) *A surface of genus g with $b > 0$ boundary components such that $(g, b) \neq (0, 1)$ with exactly one marked point on each boundary component and no punctures. The corresponding mutation class is $\mathcal{Q}_{g,b}$.*

Proof. Applying Lemma 2.3 we deduce that the number b of boundary components of S, its genus g and the number of punctures in M must be one of the following:

- $b = 1$, $g = 0$ (disc) with one puncture;
- $b = 1$, $g = 0$ (disc) with no punctures;
- $g = 0$, $b \geq 2$ with no punctures;
- $g > 0$, $b \geq 1$ with no punctures.

The following bordered marked surfaces of genus g with $b > 0$ boundary components admit triangulations satisfying the condition (♠):

- $b = 1$, $g = 0$ (disc) with one puncture and two marked points on its boundary;
- $b = 1$, $g = 0$ (disc) with no punctures and five marked points on its boundary;
- $(g, b) \neq (0, 1)$ with one marked point on each boundary component.

By applying Lemma 2.4 we see that the numbers of marked points on each boundary component of S cannot exceed the ones in the list above, and the result follows. □

3. On the mutation classes $\mathcal{Q}_{g,b}$

In order to show the implication (ii) ⇒ (i) in Theorem 1, it remains to show that each of the mutation classes $\mathcal{Q}_{g,b}$ for $(g, b) \notin \{(0, 0), (0, 1)\}$ consists of quivers with the same number of arrows.

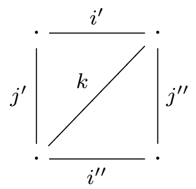
3.1. **Mutations of quivers from unpunctured marked surfaces.** For the classes $\mathcal{Q}_{g,b}$ with $b > 0$ this has already been shown in our previous work [19], but we give here a different proof by characterizing the mutations of quivers arising from triangulations of marked bordered surfaces without punctures that preserve the number of arrows. In fact, we show that for such quivers the converse of Lemma 2.1 holds.

For a vertex k in a quiver Q, recall that its *in-degree* is the number of arrows ending at k. Similarly, its *out-degree* is the number of arrows starting at k. The *neighborhood* of k is the full subquiver of Q on the set of vertices consisting of k, the vertices i having arrows $i \to k$ and the vertices j having arrows $k \to j$. Mutation at k does not change any arrows outside the neighborhood of k. Thus, when assessing its effect on the number of arrows, it is enough to consider its effect on the neighborhood of k.

Proposition 3.1. *Let Q be a quiver arising from a triangulation of a marked bordered surface without punctures. For a vertex k of Q, the following conditions are equivalent:*

(i) *k has in-degree 1 and out-degree 1.*
(ii) *The quivers Q and $\mu_k(Q)$ do not have the same number of arrows.*

Proof. In the triangulation corresponding to Q, the arc corresponding to the vertex k is a side of two triangles whose other sides are denoted i', j' and i'', j'' as shown below:



Some of these sides may be boundary segments. In addition, it may happen that the sides denoted i' and i'' are in fact the same arc in the triangulation, and similarly for j' and j''.

The proof goes by examining all the possible cases and computing the corresponding neighborhoods of k and their mutations. The details of this verification are given in a concise form in Table 1 for the cases where at least one of the sides is a boundary segment and in Table 2 for the remaining cases where all the sides are arcs.

Each row in these tables represents one such case and its mutation. The left two columns show the triangles whose side is the arc k together with (part of) the corresponding neighborhood of k. The right two columns show what happens under mutation at k, which corresponds to a flip of the arc k. Note that mutation of each of the cases (4a) and (4c) leads to the same case (up to relabeling of vertices/arcs), hence there are actually only four different cases (and not six) in Table 2.

In the pictures, we use the following conventions. When drawing the triangulation, a line of the form —— denotes an arc and a line of the form ┈┈ denotes a boundary segment. As for the corresponding quiver, we draw only the arrows that must be present in the neighborhood of k, in particular all the arrows starting or terminating at k, and complement this by writing down some extra conditions that must be also satisfied. These are given in terms of numbers of arrows between vertices, where we denote by a_{ij} the number of arrows starting at the vertex i and ending at j.

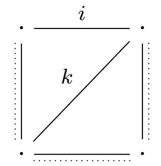
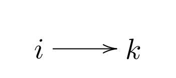
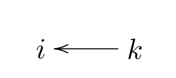
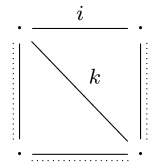
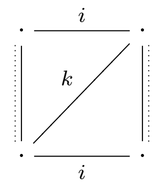
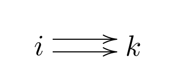
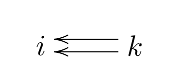
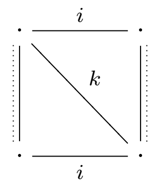
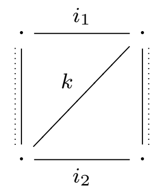
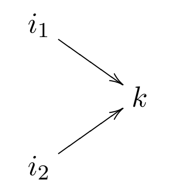
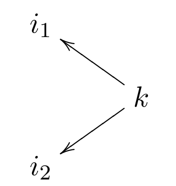
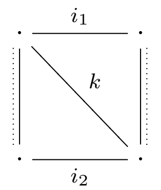
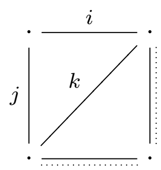
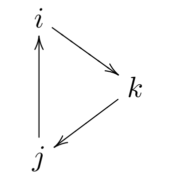
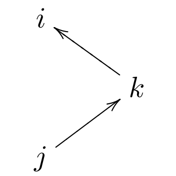
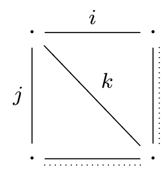
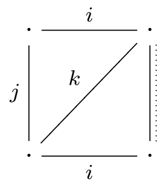
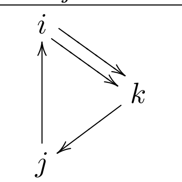
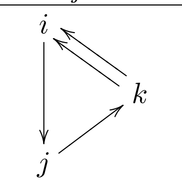
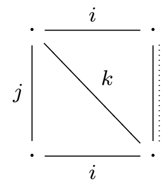
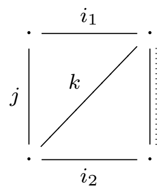
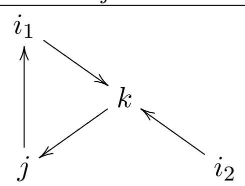
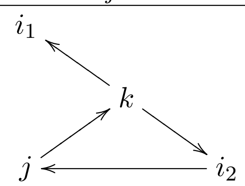
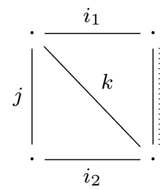
1				
$2a$				
$2b$				
$2c$		 $a_{ji} \geq 1$	 $a_{ij} = 0$	
$3a$		 $a_{ji} = 1$	 $a_{ij} = 1$	
$3b$		 $a_{ji_1} \geq 1,\ a_{ji_2} = 0$	 $a_{i_1j} = 0,\ a_{i_2j} \geq 1$	

TABLE 1. Neighborhoods of k when at least one side is a boundary segment.

Most of these constraints follow from the fact that the marked points lie on the boundary and are not punctures. In addition, in case (3a) it cannot happen that $a_{ji} = 2$ since otherwise the neighborhood of i would fall into case (4a), but there is only one arrow $k \to j$. In case (4b) we have $a_{ji_1} \geq 1$ and $a_{ji_2} \geq 1$, but these are actually equalities since otherwise the vertex j would have too many outgoing arrows. Finally, note that case (4a) actually never occurs for an unpunctured surface, as it leads to a triangulation of the torus with one puncture. $\square$

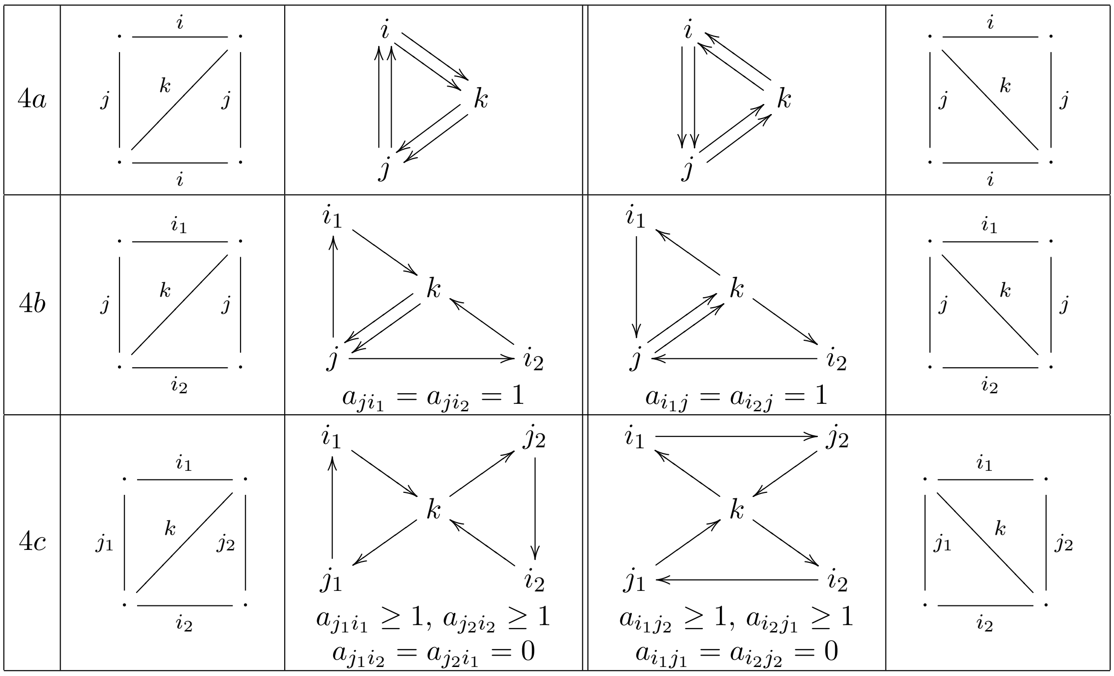


TABLE 2. Neighborhoods of k when all sides are arcs.

Remark 3.2. In order to make the presentation concise, we have not explicitly written down all the possible neighborhoods of k, but rather shown only the arrows that must be present together with additional constraints on numbers of arrows which are altogether enough to guarantee that the mutation at k does not change the total number of arrows.

Not every quiver satisfying these constraints is actually realized as a neighborhood of a vertex k in a quiver arising from a triangulation. For example, we have not implied any restrictions on quantities like $a_{i_1i_2}$ and $a_{i_2i_1}$ which are not affected by mutation at k but are obviously bounded by 2 for quivers arising from triangulations.

Remark 3.3. The statement of Proposition 3.1 does not hold for quivers in general, not even for those arising from triangulations of marked surfaces with punctures. For example, the left quiver below (an orientation of the Dynkin diagram D_4) arises from a triangulation of the disc with one puncture and four marked points on its boundary [14, §6]. The vertex 1 has in-degree 2, but the corresponding mutation, given by the right quiver, has two more arrows.

Lemma 3.4. *Let $Q \in \mathcal{Q}_{g,b}$ for $b > 0$. Then there are no vertices of Q with in-degree* 1 *and out-degree* 1.

Proof. Indeed, the existence of such vertex in a quiver arising from a triangulation of an unpunctured bordered marked surface corresponds to case (2c) in Table 1, implying that there is a boundary component containing at least two marked points. □

Corollary 3.5. *The quivers in a class $\mathcal{Q}_{g,b}$ when $b > 0$ have the same number of arrows.*

Proof. Combine Proposition 3.1 and Lemma 3.4. □

3.2. **Quivers from once punctured closed surfaces.** We are left with the mutation classes $\mathcal{Q}_{g,0}$ for $g > 0$.

Proposition 3.6. *Let Q be a quiver in $\mathcal{Q}_{g,0}$ and let k be a vertex of Q. Then the neighborhood of k is one of the four appearing in Table 2. In particular, its in-degree and out-degree are both* 2.

Proof. From [14, Prop. 2.10] we see that any triangulation of a closed surface of genus $g > 0$ with one puncture consists of $6g - 3$ arcs. Since $6g - 3 > 2$ and all the arcs are incident to the single puncture, there cannot be any self-folded triangles.

In the triangulation corresponding to Q, the arc corresponding to the vertex k is a side of two triangles whose other sides are the arcs i', j' and i'', j'' as shown in the left picture

where $\times$ denotes the puncture. The sets $\{i', i''\}$ and $\{j', j''\}$ are disjoint since otherwise the triangulation or its flip at k shown in the right picture would contain self-folded triangles. Hence we are in one of the situations depicted in Table 2, and we only need to verify the constraints on the neighborhood of k of the corresponding quivers.

Indeed, when $g = 1$ (the once punctured torus) we get the quiver as in case (4a). Otherwise, the fact that there are more than three arcs incident to the puncture implies that the quivers and constraints on their numbers of arrows are as shown in Table 2.

In case (4c), for example, in the counterclockwise order around the puncture, the arc j_1 does not immediately follow i_1 (top left), i_2 does not follow j_1 (bottom left), j_2 does not follow i_2 (bottom right) and i_1 does not follow j_2 (top right), hence there are arrows $j_1 \to i_1$ and $j_2 \to i_2$ coming from the triangles containing k but no arrows $j_1 \to i_2$ or $j_2 \to i_1$, leading to the quiver with constraints as in Table 2. □

In particular we see that the number of arrows of a quiver in a class $\mathcal{Q}_{g,0}$ is twice the number of its vertices. This implies the following corollary, thus completing the proof of Theorem 1.

Corollary 3.7. *Any quiver in $\mathcal{Q}_{g,0}$ has $12g - 6$ arrows.*

Next we provide explicit members from these classes in a similar way to our treatment in [19, §3]. Draw the fundamental polygon with $4g$ sides labeled $1, 2, 1, 2, \ldots, 2g - 1, 2g, 2g - 1, 2g$ corresponding to a surface of genus g, and identify the puncture with its

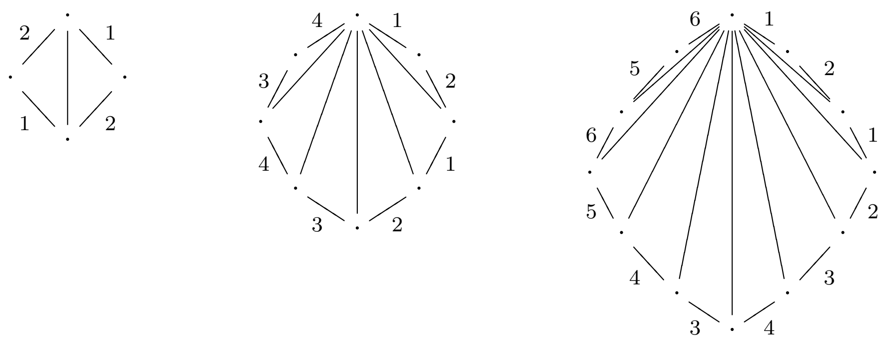


FIGURE 3. Triangulations of closed surfaces of genus g with one puncture, for $g = 1, 2, 3$. Arcs having the same label are identified.

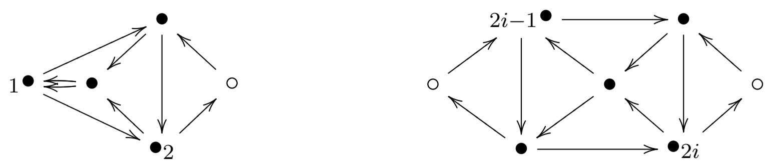


FIGURE 4. Two building blocks for quivers in $\mathcal{Q}_{g,0}$ ($g > 1$) arising from triangulations as in Figure 3. Gluing points are marked with $\circ$.

vertices (recall that they are all being identified on the surface). Any triangulation of the $4g$-gon gives rise to a triangulation of the punctured (closed) surface of genus g.

By taking triangulations as in Figure 3, one gets explicit such quivers which are shown in Figure 1. They are built, for $g > 1$, by gluing two kinds of building blocks given in Figure 4. Each block corresponds to a pair of consecutive labels $2i - 1, 2i$ of sides in the $4g$-gon and is obtained by considering all the triangles incident to these sides. The left block arises from the initial and terminal pairs of labels $\{1, 2\}$ and $\{2g - 1, 2g\}$ whereas the right one arises from all other pairs $\{2i - 1, 2i\}$.

The close connection between triangulations of the $4g$-gon and quivers in $\mathcal{Q}_{g,0}$ can be expressed more precisely in terms of mutation classes of the Dynkin diagrams A_n. Let $n \geq 1$ and denote by A_n the quiver with n vertices labeled $1, 2, \dots, n$ and arrows $i \to i + 1$ for $1 \leq i < n$. The quivers in the mutation class of A_n are those arising from triangulations of the $(n+3)$-gon (disc with no punctures and $n+3$ marked points on its boundary), see [9]. They have been explicitly described in [8].

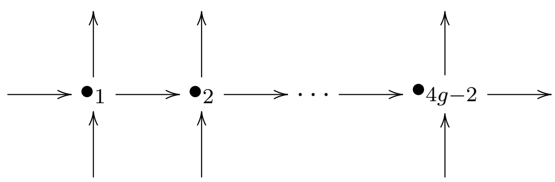
Proposition 3.8. *Let $g, n > 0$. Then a quiver in the mutation class of A_n is a full subquiver of some quiver in the class $\mathcal{Q}_{g,0}$ if and only if $n \leq 4g - 3$.*

Proof. We show first that any quiver Q' in the mutation class of A_{4g-3} is a full subquiver of some quiver in $\mathcal{Q}_{g,0}$. Indeed, by taking the fundamental $4g$-gon of a genus g surface we see that any triangulation of the $4g$-gon corresponding to Q' yields a triangulation of the closed genus g surface with one puncture, giving a quiver $Q \in \mathcal{Q}_{g,0}$ containing Q'

as a full subquiver. For example, for each of the quivers in Figure 1, its full subquiver on the vertices not corresponding to the sides of the $4g$-gon is the quiver A_{4g-3}.

Conversely, we show that no quiver in the mutation class of A_{4g-2} is a full subquiver of a quiver in $\mathcal{Q}_{g,0}$. Observe that if Q' is a full subquiver of Q, then by performing a mutation at a vertex k of Q' we see that $\mu_k(Q')$ is a full subquiver of $\mu_k(Q)$. Hence it is enough to prove the claim for the quiver A_{4g-2} itself.

Assume to the contrary that A_{4g-2} is a full subquiver on the vertices $1, 2, \dots, 4g-2$ of some quiver $Q \in \mathcal{Q}_{g,0}$. Since each vertex of Q has two incoming and two outgoing arrows, we deduce that Q must contain the following arrows:



Counting them, we get $3(4g-2)+1 = 12g-5$, contradicting the fact that Q has only $12g-6$ arrows. $\square$

4. Interpretation via derived equivalences

In this section we explain how to interpret, in certain cases, mutations of quivers with potentials preserving the number of arrows as derived equivalences of the corresponding Jacobian algebras. This interpretation which naturally holds for reflections, is valid also for mutations of quivers arising from triangulations of unpunctured surfaces as well as for the quivers in the classes $\mathcal{Q}_{g,0}$.

Throughout this section, we fix a field K. Recall that two K-algebras are called *derived equivalent* if their module categories have equivalent derived categories. Derived equivalent algebras share many homological properties, and we refer the reader to the survey [16] for further details on tilting theory and derived equivalence.

Recall that the *path algebra* KQ of a quiver Q has a basis consisting of the paths in Q, and the product of any two paths is their concatenation, if defined, and zero otherwise. A *quiver with potential* (QP) is a pair (Q, W) where Q is a quiver (under our assumption throughout the paper, it does not have loops and 2-cycles) and W is a *potential*, which we assume to be polynomial, i.e. a linear combination of cycles in KQ. In [12] the authors have defined the notion of mutation of QP at a vertex. Under certain conditions, it extends the notion of mutation of quivers. The (non-completed) *Jacobian algebra* of (Q, W), denoted by $\mathcal{P}(Q, W)$, is the quotient of KQ by the ideal generated by the cyclic derivatives of W with respect to all the arrows, see [12].

4.1. **Reflections.** We have already mentioned that *reflection*, that is, mutation at a sink or a source, preserves the number of arrows in the quiver. Reflection has also a well-known representation theoretic interpretation as derived equivalence. Namely, if (Q, W) is any QP and k is a sink or a source of Q, then the Jacobian algebras $\mathcal{P}(Q, W)$ and $\mathcal{P}(\mu_k(Q, W))$ are derived equivalent. Indeed, one takes the left complex (when k is sink) or the right one (when it is a source) of finitely generated right projective

$\mathcal{P}(Q,W)$-modules

$$(\star) \qquad \big(P_k \xrightarrow{f} \bigoplus_{j\to k} P_j\big) \oplus \big(\bigoplus_{i\neq k} P_i\big) \qquad\qquad \big(\bigoplus_{k\to j} P_j \xrightarrow{g} P_k\big) \oplus \big(\bigoplus_{i\neq k} P_i\big)$$

where P_i denotes the projective module corresponding to i spanned by all paths starting at i, the map f (respectively, g) is induced by all the arrows ending (respectively, starting) at k, and the terms P_i for $i \neq k$ lie in degree 0. This is a tilting complex over $\mathcal{P}(Q,W)$ whose endomorphism algebra is isomorphic to $\mathcal{P}(\mu_k(Q,W))$, so by Rickard's theorem [20] the two algebras are derived equivalent. In the finite-dimensional case this is an instance of APR-tilt [2] generalizing the BGP reflections [6] between path algebras of quivers without oriented cycles.

4.2. **Quivers from unpunctured surfaces.** For quivers arising from triangulations of marked bordered surfaces, potentials have been defined by Labardini-Fragoso [18] in such a way that flips of triangulations correspond to mutations of the associated QP. When the surface has no punctures, these potentials are sums of the oriented 3-cycles in the quiver which are induced by the internal triangles of the triangulation, and the corresponding Jacobian algebras are the finite-dimensional gentle algebras introduced by Assem, Brüstle, Charbonneau-Jodoin and Plamondon [1].

In general, the number of arrows in the quiver of any gentle algebra is invariant under derived equivalence [3], but for the gentle algebras arising from triangulations we can actually say more; by combining Proposition 3.1 and our previous work [19, §2] we deduce that for any single mutation of a QP arising from a triangulation of a marked unpunctured surface, the condition that the number of arrows is preserved is equivalent to the derived equivalence of the Jacobian algebras.

Theorem 2. *Let (Q,W) be a QP arising from a triangulation of marked bordered surface without punctures. For any vertex k of Q, the following conditions are equivalent:*

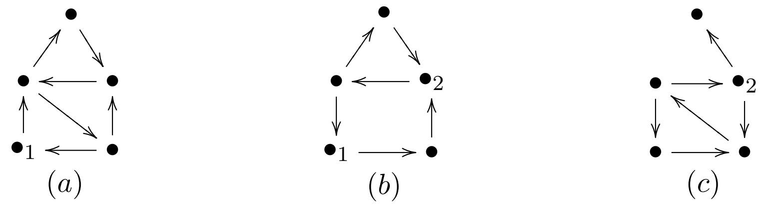
(i) *The numbers of incoming and outgoing arrows at k are not both equal to 1;*
(ii) *Q and its mutation $\mu_k(Q)$ have the same number of arrows;*
(iii) *The algebras $\mathcal{P}(Q,W)$ and $\mathcal{P}(\mu_k(Q,W))$ are derived equivalent.*

In particular, using Theorem 1 we get another proof of the following result.

Corollary 4.1 ([19])**.** *Each of the classes $\mathcal{Q}_{g,b}$ when $b > 0$ can be regarded as mutation class of QP whose Jacobian algebras are all derived equivalent.*

Remark 4.2. Combining Theorem 2 with the recent computation by David-Roesler and Schiffler in [10] of the Avella-Alaminus-Geiss derived invariants [3] allows for a description of the derived equivalence classes of the Jacobian algebras of QP arising from triangulations of marked unpunctured surfaces in terms of the properties of the corresponding triangulations, generalizing the derived equivalence classifications of cluster-tilted algebras of Dynkin type A [8] and affine type $\tilde{A}$ [4]. This is a subject of further investigations.

Remark 4.3. The statement of Theorem 2 does not hold for QP in general, not even for those arising from triangulations of marked surfaces with some punctures. For example, the following picture shows certain quivers arising from triangulations of the once-punctured pentagon (i.e. a disc with one puncture and five marked points on its boundary). The corresponding Jacobian algebras are cluster-tilted algebras of type D_5 [21].



The quivers in (a) and (b) are related by mutation at the vertex 1. They have different numbers of arrows but the corresponding algebras are derived equivalent. The quivers in (b) and (c) are related by mutation at the vertex 2. They have the same number of arrows but the corresponding algebras are not derived equivalent, see [5].

Remark 4.4. Furthermore, there are mutation classes consisting of finitely many QP whose Jacobian algebras are finite-dimensional and derived equivalent, but the quivers themselves have varying numbers of arrows. Examples are the mutation classes of the exceptional quivers $E_6^{(1,1)}$ and X_6, see [19].

4.3. **The classes $\mathcal{Q}_{g,0}$.** For a quiver Q in the classes $\mathcal{Q}_{g,0}$ arising from triangulations of closed surfaces with one puncture, the potential defined in [18] is the sum of two terms $W_\Delta + x_p W_p$ for some $0 \neq x_p \in K$. The term W_Δ is the sum of the oriented 3-cycles corresponding to the triangles comprising the triangulation, just as in the unpunctured case. The term W_p is the oriented cycle induced by traversing the arcs of the triangulation in a counter-clockwise order at the puncture p. Thus, any arrow in Q appears in the cycle W_p exactly once.

Since there are never self-folded triangles in a triangulation of a once punctured closed surface (cf. Section 3.2), the argument in [18] showing that flips of triangulations correspond to mutations of the associated QP is applicable also when we set $x_P = 0$, see Cases 1 and 2 in the proof of [18, Theorem 30]. Hence, the association of the potential $W = W_\Delta$ to the quiver Q is compatible with quiver mutations, a fact which can be also verified directly by considering the local neighborhoods in Table 2. The potential W is thus non-degenerate but not rigid. In the special case of $g = 1$, we recover Example 8.6 of [12]. The Jacobian algebra of (Q, W) satisfies all the conditions in the definition of a gentle algebra, except that it is *infinite*-dimensional.

Proposition 4.5. *Let $Q \in \mathcal{Q}_{g,0}$ for some $g > 0$ and let W be the associated potential. Then, for any vertex k of Q the two complexes given by $(\star)$ are tilting complexes over $\mathcal{P}(Q, W)$ and their endomorphism algebras are both isomorphic to $\mathcal{P}(\mu_k(Q, W))$.*

Proof. For $g = 1$, this can be checked directly. For $g > 1$ one considers the situation "locally" at the neighborhood of k and reduces to the finite-dimensional case which was treated in [19, §2]. □

Remark 4.6. The proposition implies that each class $\mathcal{Q}_{g,0}$ with the above associated potentials satisfies a stronger version of condition (δ_3) that was stated in [19]. In particular, $\mathcal{Q}_{g,0}$ can be regarded as mutation class of QP whose Jacobian algebras are all derived equivalent.

Remark 4.7. The statement of the proposition does not hold for the classes $\mathcal{Q}_{g,b}$ when $b > 0$ despite the fact that all the Jacobian algebras are derived equivalent. Indeed, in any quiver $Q \in \mathcal{Q}_{g,b}$ with $b > 0$ there is at least one vertex k whose in-degree and out-degree are not both equal to 2, and then one of the complexes in $(\star)$ is not a tilting complex.

Remark 4.8. By work of Keller and Yang [17, §6], the statement of the proposition holds for any quiver with potential (Q, W) whose Ginzburg dg-algebra has its cohomology concentrated in degree zero. This assumption implies that the Jacobian algebra $\mathcal{P}(Q, W)$ is 3-Calabi-Yau.

However, the Jacobian algebra of a quiver $Q \in \mathcal{Q}_{g,0}$ with the above associated potential is *not* 3-Calabi-Yau. Indeed, since the potential W we associate to Q is a sum of cycles of the same length, the Jacobian algebra $\mathcal{P}(Q, W)$ is naturally graded. One can thus check that it is not 3-Calabi-Yau by computing its matrix Hilbert series and using [7, Theorem 4.6].

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