Fundamental Notions in Algebra – Exc. No. 11

1. Let $\rho : G \to \mathrm{Aut}_k(V)$ be a finite-dimensional irreducible representation of a group G over an algebraically closed field k.

   (a) Show that $\rho(G) \subset \mathrm{Aut}_k(V)$ spans $\mathrm{End}_k(V)$ as a k-vector space.

   (b) Assume that G is abelian. Show that V is one-dimensional.

   (c) **Definition:** An element $a \in \mathrm{Aut}_k(V)$ is called *unipotent*, if $a - 1 \in \mathrm{End}_k(V)$ is nilpotent.

   Assume that $\rho(g)$ is unipotent for each $g \in G$. Show that ρ is a trivial one-dimensional representation.

   Hint:

   i. Show first that if endomorphism $A \in \mathrm{End}_k(V)$ satisfies $\mathrm{Tr}(AB) = 0$ for all $B \in \mathrm{End}_k(V)$, then $A = 0$.

   ii. Show that $\mathrm{Tr}((\rho(g) - 1)\rho(h)) = 0$ for all $g, h \in G$.

   iii. Deduce that $\rho(g) = 1$ for all $g \in G$.

2. Let V be a finite-dimensional vector space over an algebraically closed field k, and let G be a subgroup of $\mathrm{Aut}_k(V)$ such that each $g \in G$ is unipotent.

   (a) Show that there exists a basis $e_1, \dots, e_n$ of V such that with respect to it each $g \in G$ is upper-triangular.

   Hint:

   i. Show that there exists a non-zero vector $e_1 \in V$ such that $g(e_1) = e_1$ for all $g \in G$.

   ii. Show that for every G-invariant subspace $W \subset V$ and each $g \in G$, the induced automorphisms $g|_W \in \mathrm{Aut}_k(W)$ and $g|_{V/W} \in \mathrm{Aut}_k(V/W)$ are unipotent.

   iii. Prove the assertion by induction on the dimension of V.

   (b) Show that the group G is nilpotent.

3. Let $\rho_1 : G_1 \to \mathrm{Aut}_k(V_1)$ and $\rho_2 : G_2 \to \mathrm{Aut}_k(V_2)$ be two finite-dimensional representations of groups G_1 and G_2 over an algebraically closed field k.

   (a) Show that if ρ_1 and ρ_2 are irreducible, then the exterior product representation $\rho_1 \boxtimes \rho_2 : G_1 \times G_2 \to \mathrm{Aut}(V_1 \otimes_K V_2)$ is irreducible as well.

   (b) Conversely, for every finite-dimensional irreducible representation $\rho : G_1 \times G_2 \to \mathrm{Aut}_k(V)$ is of the form $\rho_1 \boxtimes \rho_2$ for certain irreducible representations $\rho_1 : G_1 \to \mathrm{Aut}_k(V_1)$ and $\rho_2 : G_2 \to \mathrm{Aut}_k(V_2)$.

   Hint: Show first that the restriction $\rho|_{G_1} : G_1 \to \mathrm{Aut}_k(V)$ decomposes as a direct sum $\rho|_{G_1} \cong \oplus\rho_1$ of several copies of a certain irreducible representation ρ_1 of G_1.

   (c) Show that the character of $\rho := \rho_2 \boxtimes \rho_2$ satisfies $\chi_\rho(g_1, g_2) = \chi_{\rho_1}(g_1) \cdot \chi_{\rho_2}(g_2)$ for all $g_1 \in G_1, g_2 \in G_2$.

   (d) Show that assertions (a) and (b) are false if k is not algebraically closed.